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MINIMAL GENERICS FROM SUBVARIETIES OF THE CLONE EXTENSION OF THE VARIETY OF BOOLEAN ALGEBRAS

BY

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Abstract. Let τ be a type of algebras without nullary fundamental operation symbols. We call an identity $\varphi \approx \psi$ of type τ clone compatible if φ and ψ are the same variable or the sets of fundamental operation symbols in φ and ψ are nonempty and identical. For a variety $\mathcal{V}$ of type τ we denote by $\mathcal{V}^c$ the variety of type τ defined by all clone compatible identities from $\mathrm{Id}(\mathcal{V})$. We call $\mathcal{V}^c$ the clone extension of $\mathcal{V}$. In this paper we describe algebras and minimal generics of all subvarieties of $\mathcal{B}^c$, where $\mathcal{B}$ is the variety of Boolean algebras.

1. Preliminaries. Let $\tau : F \to \mathbb{N}$ be a type of algebras, where F is the set of fundamental operation symbols and $\mathbb{N}$ is the set of positive integers. For a term φ of type τ, we denote by $\mathrm{Var}(\varphi)$ the set of variables occurring in φ and by $F(\varphi)$ the set of fundamental operation symbols occurring in φ. For a variety $\mathcal{V}$ of type τ we denote by $\mathrm{Id}(\mathcal{V})$ the set of all identities of type τ satisfied in every algebra from $\mathcal{V}$. If Σ is a set of identities of type τ we denote by $\mathrm{Mod}(\Sigma)$ the class of all algebras of type τ satisfying every identity from Σ. We shall use variables $x, y, z, u, v, x_1, \ldots, x_k, \ldots$, where $k < \omega$. An identity $\varphi \approx \psi$ of type τ is called *clone compatible* if φ and ψ are the same variable or $F(\varphi) = F(\psi) \neq \emptyset$. For a variety $\mathcal{V}$ of type τ we denote by $\mathcal{V}^c$ the variety of type τ defined by all clone compatible identities from $\mathrm{Id}(\mathcal{V})$. We call $\mathcal{V}^c$ the *clone extension of* $\mathcal{V}$ (see [2]–[9]). In [2], [4] and [6] some representation theorems for algebras from $\mathcal{V}^c$ were presented.

Let $\mathfrak{A} = (A; F^{\mathfrak{A}})$ be an algebra of type τ. If $f^{\mathfrak{A}}$ is a fundamental operation from $F^{\mathfrak{A}}$ we shall often omit the upper index $\mathfrak{A}$ in $f^{\mathfrak{A}}$ when it is clear that we consider an operation in $\mathfrak{A}$. An endomorphism $r : A \to A$ of $\mathfrak{A}$ is called a *splitting retraction* of $\mathfrak{A}$ if it is idempotent ($r \circ r = r$) and for all $f \in F$, $a_1, \ldots, a_{\tau(f)} \in A$ and $k = 1, \ldots, \tau(f)$, we have

$$r(f^{\mathfrak{A}}(a_1, \ldots, a_{\tau(f)})) = f^{\mathfrak{A}}(a_1, \ldots, a_{k-1}, r(a_k), a_{k+1}, \ldots, a_{\tau(f)}).$$

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An algebra $\mathfrak{A}$ is called a *generic* of a variety $\mathcal{V}$ if $\mathrm{HSP}(\mathfrak{A}) = \mathcal{V}$ (see [1, Appendix 4]). We call a generic $\mathfrak{A} = (A; F^{\mathfrak{A}})$ of $\mathcal{V}$ a *minimal* generic of $\mathcal{V}$ if for every generic $\mathfrak{B} = (B; F^{\mathfrak{B}})$ of $\mathcal{V}$ we have $|B| \geq |A|$. Let $\mathfrak{A} = (A; F^{\mathfrak{A}})$ be a minimal generic of $\mathcal{V}$. We put $g(\mathcal{V}) = |A|$. In the following we restrict our considerations to the type $\tau_b : \{+, \cdot, '\} \to \mathbb{N}$ where $\tau_b(+) = \tau_b(\cdot) = 2$ and $\tau_b(') = 1$. We denote by $\mathcal{B}$ the variety of Boolean algebras of type τ_b.

Let us consider the following six algebras:

- $\mathfrak{A}_1 = (\{a_1, b_1\}; +, \cdot, ')$ where for $x, y \in \{a_1, b_1\}$ we have
$$x + y = \begin{cases} x & \text{if } x = y, \\ b_1 & \text{otherwise}, \end{cases} \qquad x \cdot y = \begin{cases} x & \text{if } x = y, \\ a_1 & \text{otherwise}, \end{cases}$$
$$a_1' = b_1, \quad b_1' = a_1;$$
- $\mathfrak{A}_2 = (\{a_2, b_2\}; +, \cdot, ')$ where
$$x + y = \begin{cases} x & \text{if } x = y, \\ b_2 & \text{otherwise}, \end{cases}$$
$$x \cdot y = x' = b_2 \quad \text{for every } x, y \in \{a_2, b_2\};$$
- $\mathfrak{A}_3 = (\{a_3, b_3\}; +, \cdot, ')$ where
$$x \cdot y = \begin{cases} x & \text{if } x = y, \\ b_3 & \text{otherwise}, \end{cases}$$
$$x + y = x' = b_3 \quad \text{for every } x, y \in \{a_3, b_3\};$$
- $\mathfrak{A}_4 = (\{a_4, b_4\}; +, \cdot, ')$ where
$$x + y = x \cdot y = x' = b_4 \quad \text{for every } x, y \in \{a_4, b_4\};$$
- $\mathfrak{A}_5 = (\{a_5, b_5\}; +, \cdot, ')$ where
$$x' = x, \; x + y = x \cdot y = b_5 \quad \text{for every } x, y \in \{a_5, b_5\};$$
- $\mathfrak{A}_6 = (\{a_6, b_6, c_6\}; +, \cdot, ')$ where
$$a_6' = c_6, \quad c_6' = a_6, \quad b_6' = b_6,$$
$$x + y = x \cdot y = b_6 \quad \text{for every } x, y \in \{a_6, b_6, c_6\}.$$

We see that no two of these algebras are isomorphic and $\mathfrak{A}_1$ is a 2-element Boolean algebra.

It follows from [3, Theorem 2.10 and remarks on p. 190] that

(1.i) An algebra $\mathfrak{A}$ of type τ_b belongs to $\mathcal{B}^c$ and is subdirectly irreducible iff $\mathfrak{A}$ is isomorphic to one of the algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_6$.

Define $\mathrm{Ir}(\mathcal{B}^c) = \{\mathfrak{A}_1, \ldots, \mathfrak{A}_6\}$. If $\mathcal{V}$ is a subvariety of $\mathcal{B}^c$ and an algebra $\mathfrak{B}$ belongs to $\mathcal{V}$ and is subdirectly irreducible then by (1.i) it has to be isomorphic to some algebra from $\mathrm{Ir}(\mathcal{B}^c)$. Since by Birkhoff's theorem (see

[1, Theorem 20.3]) every variety is uniquely determined by its subdirectly irreducible algebras, by (1.i) we have

(1.ii) Every subvariety $\mathcal{V}$ of $\mathcal{B}^c$ is uniquely determined by the set $\mathrm{Ir}(\mathcal{V}) = \mathcal{V} \cap \mathrm{Ir}(\mathcal{B}^c)$, namely $\mathcal{V} = \mathrm{HSP}(\mathrm{Ir}(\mathcal{V}))$.

If $\mathcal{V}$ is a subvariety of $\mathcal{B}^c$ and $S = \mathcal{V} \cap \mathrm{Ir}(\mathcal{B}^c)$ we shall write $\mathcal{V} = \mathcal{V}(S)$. So one wishes to determine which subsets of $\mathrm{Ir}(\mathcal{B}^c)$ are of the form $\mathrm{Ir}(\mathcal{V})$ for some $\mathcal{V} \in L(\mathcal{B}^c)$, where $L(\mathcal{B}^c)$ is the lattice of subvarieties of $\mathcal{B}^c$.

It was shown in [5] that

(1.iii) The family $\boldsymbol{S}$ of all sets $\mathrm{Ir}(\mathcal{V})$ with $\mathcal{V} \in L(\mathcal{B}^c)$ consists of the following 29 sets: $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_6\}$, $\{\mathfrak{A}_2, \ldots, \mathfrak{A}_6\}$, $\{\mathfrak{A}_2, \ldots, \mathfrak{A}_5\}$, $\{\mathfrak{A}_3, \ldots, \mathfrak{A}_6\}$, $\{\mathfrak{A}_1, \mathfrak{A}_3, \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6\}$, $\{\mathfrak{A}_3, \mathfrak{A}_4, \mathfrak{A}_5\}$, $\{\mathfrak{A}_2, \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6\}$, $\{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6\}$, $\{\mathfrak{A}_2, \mathfrak{A}_4, \mathfrak{A}_5\}$, $\{\mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4\}$, $\{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4\}$, $\{\mathfrak{A}_4\}$, $\{\mathfrak{A}_1, \mathfrak{A}_4\}$, $\{\mathfrak{A}_2\}$, $\{\mathfrak{A}_1, \mathfrak{A}_2\}$, $\{\mathfrak{A}_2, \mathfrak{A}_4\}$, $\{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_4\}$, $\{\mathfrak{A}_3\}$, $\{\mathfrak{A}_1, \mathfrak{A}_3\}$, $\{\mathfrak{A}_3, \mathfrak{A}_4\}$, $\{\mathfrak{A}_1, \mathfrak{A}_3, \mathfrak{A}_4\}$, $\{\mathfrak{A}_5, \mathfrak{A}_6\}$, $\{\mathfrak{A}_1, \mathfrak{A}_5, \mathfrak{A}_6\}$, $\{\mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6\}$, $\{\mathfrak{A}_1, \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6\}$, $\{\mathfrak{A}_5\}$, $\{\mathfrak{A}_4, \mathfrak{A}_5\}$, $\emptyset$, $\{\mathfrak{A}_1\}$. Moreover, the lattice $L(\mathcal{B}^c)$ is isomorphic to $(\boldsymbol{S}; \subseteq)$.

Also in [5, p. 164] we showed that

(1.iv) $\mathfrak{A}_5 \in \mathrm{HSP}(\{\mathfrak{A}_6\})$.

(1.v) If $i, j \in \{2, 3, 5, 6\}$, $i \neq j$ and $\{i, j\} \neq \{5, 6\}$, then $\mathfrak{A}_4 \in \mathrm{HSP}(\{\mathfrak{A}_i, \mathfrak{A}_j\})$.

(1.vi) $\mathfrak{A}_6 \in \mathrm{HSP}(\{\mathfrak{A}_1, \mathfrak{A}_5\})$.

By (1.i) we have

(1.vii) $\mathcal{B}^c = \mathcal{V}(\{\mathfrak{A}_1, \ldots, \mathfrak{A}_6\})$.

For an arbitrary variety $\mathcal{V}$ let $\mathrm{CL}(\mathcal{V})$ denote the set of all clone compatible identities from $\mathrm{Id}(\mathcal{V})$. The set $\mathrm{CL}(\mathcal{V})$ need not be an equational theory. It is if $\mathcal{V}$ is the variety of distributive lattices (see [8]). This is also the case for every variety $\mathcal{V}$ of groupoids. However, $\mathrm{CL}(\mathcal{B})$ is not an equational theory. In fact, the identity $x + x \cdot y \approx x + x \cdot z$ is clone compatible but its consequence $x + x \cdot y \approx x + x \cdot y'$ is not; here we adopt the convention that $\cdot$ binds stronger than $+$ and we omit suitable parentheses.

In [9] we described forms of identities and we constructed equational bases of all subvarieties of $\mathcal{B}^c$.

2. Representations and minimal generics. By Birkhoff's subdirect irreducibility theorem and (1.i)–(1.iii) we already have:

If an algebra $\mathfrak{A}$ belongs to $\mathcal{V}(S)$, where $S \in \boldsymbol{S}$, then $\mathfrak{A}$ is isomorphic to a subdirect product of some algebras from S.

To get a more illustrative representation of algebras from subvarieties of $\mathcal{B}^c$ we need Theorem 1 below, which is in fact an application of more general theorems (see [2, Section 3], [4, Section 2], [6, Section 3]) to the variety $\mathcal{B}^c$. However, in Theorem 1 we give more details specifically for the variety $\mathcal{B}^c$.

We put

$$\begin{aligned} q_{(+)}(x) &= x + x, \\ q_{(\cdot)}(x) &= x \cdot x, \\ q_{(')}(x) &= x', \\ q_{('')}(x) &= (x')', \\ q_b(x) &= q_{(+)}(q_{(\cdot)}(q_{('')}(x))). \end{aligned}$$

THEOREM 1. *If an algebra* $\mathfrak{A} = (A; +, \cdot, ')$ *belongs to* $\mathcal{B}^c$, *then the following conditions hold.*

(2.i) *Each of the mappings* $q^{\mathfrak{A}}_{(+)}$, $q^{\mathfrak{A}}_{(\cdot)}$, $q^{\mathfrak{A}}_{('')}$, $q^{\mathfrak{A}}_b$ *is a splitting retraction of* $\mathfrak{A}$ *and any two of them commute.*

(2.ii) *Put* $A_{(+)} = q^{\mathfrak{A}}_{(+)}(A)$, $A_{(\cdot)} = q^{\mathfrak{A}}_{(\cdot)}(A)$, $A_{('')} = q^{\mathfrak{A}}_{('')}(A)$, $A_b = q^{\mathfrak{A}}_b(A)$. *Then* $q^{\mathfrak{A}}_{(+)}$ *is the identity on* $A_{(+)}$, $q^{\mathfrak{A}}_{(\cdot)}$ *is the identity on* $A_{(\cdot)}$, $q^{\mathfrak{A}}_{('')}$ *is the identity on* $A_{('')}$ *and* $q^{\mathfrak{A}}_b$ *is the identity on* A_b.

(2.iii) *If* $a \in A$, *then* $q^{\mathfrak{A}}_{\alpha_1}(q^{\mathfrak{A}}_{\alpha_2}(\ldots(q^{\mathfrak{A}}_{\alpha_n}(a))\ldots)) = q^{\mathfrak{A}}_b(a)$ *for every* $\alpha_1, \ldots, \alpha_n$ *in* $\{(+), (\cdot), ('')\}$ *with* $|\{\alpha_1, \ldots, \alpha_n\}| > 1$.

(2.iv) $A_{(+)} \cap A_{(\cdot)} = A_{(+)} \cap A_{('')} = A_{(\cdot)} \cap A_{('')} = A_b$.

(2.v) *The algebra* $\mathfrak{A}_{(+)} = (A_{(+)}; +|A_{(+)})$ *is a* $+$*-semilattice, the algebra* $\mathfrak{A}_{(\cdot)} = (A_{(\cdot)}; \cdot|A_{(\cdot)})$ *is a* $\cdot$*-semilattice, the algebra* $\mathfrak{A}_{('')} = (A_{('')}; '|A_{('')})$ *is an algebra with involution, i.e. it satisfies* $(x')' = x$, *and the algebra* $\mathfrak{A}_b = (A_b; \{+, \cdot, '\}|A_b)$ *belongs to* $\mathcal{B}$.

(2.vi) *If* $a, b \in A$, *then* $a + b = q^{\mathfrak{A}}_{(+)}(a) + q^{\mathfrak{A}}_{(+)}(b)$, $a \cdot b = q^{\mathfrak{A}}_{(\cdot)}(a) \cdot q^{\mathfrak{A}}_{(\cdot)}(b)$ *and* $a' = (q^{\mathfrak{A}}_{('')}(a))'$.

The construction used in Theorem 1 was called a *clone extension of an algebra* $\mathfrak{A}$ in [2] and [4], and a *clone network over a network of splitting retractions* in [6].

EXAMPLE 1. Let $a \in A_{(+)}$ and $b \in A_{(\cdot)}$. Then by (2.vi), (2.ii), (2.iii), (2.i) we have:

$$\begin{aligned} a + b &= q^{\mathfrak{A}}_{(+)}(a) + q^{\mathfrak{A}}_{(+)}(b) = q^{\mathfrak{A}}_{(+)}(a) + q^{\mathfrak{A}}_{(+)}(q^{\mathfrak{A}}_{(\cdot)}(b)) \\ &= q^{\mathfrak{A}}_{(+)}(a) + q^{\mathfrak{A}}_b(b) = q^{\mathfrak{A}}_b(a) + q^{\mathfrak{A}}_b(b). \end{aligned}$$

We also have $a' = (q^{\mathfrak{A}}_{('')}(a))' = (q^{\mathfrak{A}}_{('')}(q^{\mathfrak{A}}_{(+)}(a)))' = (q^{\mathfrak{A}}_b(a))'$.

(2.vii) $g(\mathcal{B}^c) = 6$. Moreover, the subdirect product

$$\begin{aligned} \mathfrak{A}(1,2,3,5) = (\{&\langle a_1, a_2, b_3, b_5\rangle, \langle a_1, b_2, a_3, b_5\rangle, \langle a_1, b_2, b_3, b_5\rangle, \\ &\langle b_1, b_2, b_3, b_5\rangle, \langle a_1, b_2, b_3, a_5\rangle, \langle b_1, b_2, b_3, a_5\rangle\}; +, \cdot, ') \end{aligned}$$

of the direct product $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_5$ is a minimal generic of $\mathcal{B}^c$.

Proof. The first statement of (2.vii) holds by Theorem 4 from [4]. By (1.v) and (1.vi) we have $\{\mathfrak{A}_1, \dots, \mathfrak{A}_6\} \subseteq \mathrm{HSP}(\mathfrak{A}(1,2,3,5))$, so by (1.vii) we have $\mathcal{B}^c \subseteq \mathrm{HSP}(\mathfrak{A}(1,2,3,5))$. But $\mathfrak{A}(1,2,3,5) \in \mathcal{B}^c$ by (1.i), so $\mathrm{HSP}(\mathfrak{A}(1,2,3,5)) = \mathcal{B}^c$. ∎

To find minimal generics of proper subvarieties of $\mathcal{B}^c$ we need some lemmas.

From now on we assume that $\mathfrak{A} = (A; +, \cdot, ')$ *belongs to* $\mathcal{B}^c$ *so it is of the form described in Theorem* 1.

Let us record the following obvious observation. If e is an identity of type τ_b and $\mathfrak{A}$ is a generic of $\mathcal{V}(S)$, $S \in \boldsymbol{S}$, then $e \in \mathrm{Id}(\mathfrak{A})$ iff $e \in \mathrm{Id}(\mathcal{V}(S))$ iff for every $\mathfrak{A}_k \in S$ we have $e \in \mathrm{Id}(\mathfrak{A}_k)$. So $e \notin \mathrm{Id}(\mathfrak{A})$ iff there is $\mathfrak{A}_k \in S$ with $e \notin \mathrm{Id}(\mathfrak{A}_k)$. This observation will be useful in the proofs of some of the corollaries below.

LEMMA 1. $|A_b| = 1$ *iff* $\mathfrak{A}$ *satisfies*

$$q_b(x) \approx q_b(y). \tag{1}$$

Proof. ⇒ Follows from the fact that for every $a, b \in A$ we have $q_b^{\mathfrak{A}}(a), q_b^{\mathfrak{A}}(b) \in A_b$ by (2.ii).

⇐ If $a, b \in A_b$, then by (2.ii) and (1) we have $a = q_b^{\mathfrak{A}}(a) = q_b^{\mathfrak{A}}(b) = b$. ∎

LEMMA 2. $A_{(+)} \setminus A_b = \emptyset$, *i.e.* $A_{(+)} = A_b$, *iff* $\mathfrak{A}$ *satisfies*

$$q_{(+)}(x) \approx q_b(x). \tag{2}$$

Proof. ⇒ By (2.iv) we have $A_b \subseteq A_{(+)}$, so $A_{(+)} \setminus A_b = \emptyset$ iff $A_{(+)} = A_b$. So if $A_{(+)} = A_b$, then for $a \in A$ we have $q_{(+)}(a) \in A_b$. Then by (2.ii) and (2.iii) we have $q_{(+)}^{\mathfrak{A}}(a) = q_b^{\mathfrak{A}}(q_{(+)}^{\mathfrak{A}}(a)) = q_b^{\mathfrak{A}}(a)$.

⇐ Obvious. ∎

The proofs of the next two lemmas are analogous to that of Lemma 2. It is enough to replace $(+)$ by $(\cdot)$ and $(+)$ by $('')$, respectively.

LEMMA 3. $A_{(\cdot)} = A_b$ *iff* $\mathfrak{A}$ *satisfies*

$$q_{(\cdot)}(x) \approx q_b(x). \tag{3}$$

LEMMA 4. $A_{('')} = A_b$ *iff* $\mathfrak{A}$ *satisfies*

$$q_{('')}(x) \approx q_b(x). \tag{4}$$

COROLLARY 1. *If* $S \in \boldsymbol{S}$, $\mathfrak{A} \in \mathcal{V}(S)$ *and* $\mathfrak{A}_1 \notin S$, *then* $|A_b| = 1$.

Proof. If $k \neq 1$ and $\mathfrak{A}_k \in \mathrm{Ir}(\mathcal{B}^c)$, then $\mathfrak{A}_k$ satisfies (1). By (1.ii) we have $\mathcal{V}(S) = \mathrm{HSP}(S)$, so $\mathcal{V}(S)$ satisfies (1) and consequently $\mathfrak{A}$ satisfies (1). Now by Lemma 1, A_b from $\mathfrak{A}$ is 1-element. ∎

COROLLARY 1′. *If* $S \in \boldsymbol{S}$ *and* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(S)$, *then* $|A_b| = 1$ *iff* $\mathfrak{A}_1 \notin S$.

Proof. $\Leftarrow$ Follows from Corollary 1.

$\Rightarrow$ If $\mathfrak{A}_1 \in S$, then $\mathcal{V}(S)$ does not satisfy (1) since $\mathfrak{A}_1$ does not. So $\mathfrak{A}$ does not satisfy (1). Now by Lemma 1 we get $|A_b| > 1$. ∎

COROLLARY 2. *If $S \in \boldsymbol{S}$, $\mathfrak{A} \in \mathcal{V}(S)$ and $\mathfrak{A}_2 \notin S$, then $A_{(+)} = A_b$.*

Proof. If $k \neq 2$ and $A_k \in \mathrm{Ir}(\mathcal{B}^c)$, then A_k satisfies (2). By (1.ii) we have $\mathcal{V}(S) = \mathrm{HSP}(S)$, so $\mathcal{V}(S)$ satisfies (2) and consequently $\mathfrak{A}$ does. Now by Lemma 2 we have $A_{(+)} = A_b$. ∎

COROLLARY 2′. *If $S \in \boldsymbol{S}$ and $\mathfrak{A}$ is a generic of $\mathcal{V}(S)$, then $A_{(+)} = A_b$ iff $\mathfrak{A}_2 \notin S$.*

Proof. $\Leftarrow$ Follows from Corollary 2.

$\Rightarrow$ If $\mathfrak{A}_2 \in S$ then $\mathcal{V}(S)$ does not satisfy (2) since $\mathfrak{A}_2$ does not. So $\mathfrak{A}$ does not satisfy (2) and by Lemma 2 we get $A_{(+)} \neq A_b$. ∎

COROLLARY 3. *If $S \in \boldsymbol{S}$, $\mathfrak{A} \in \mathcal{V}(S)$ and $\mathfrak{A}_3 \notin S$, then $A_{(\cdot)} = A_b$.*

The proof is analogous to that of Corollary 2. It is enough to replace (2) by (3) and $(+)$ by $(\cdot)$.

COROLLARY 3′. *If $S \in \boldsymbol{S}$ and $\mathfrak{A}$ is a generic of $\mathcal{V}(S)$, then $A_{(\cdot)} = A_b$ iff $\mathfrak{A}_3 \notin S$.*

The proof is analogous to that of Corollary 2′. It is enough to replace (2) by (3) and $(+)$ by $(\cdot)$.

COROLLARY 4. *If $S \in \boldsymbol{S}$, $\mathfrak{A} \in \mathcal{V}(S)$ and $\mathfrak{A}_5 \notin S$, then $A_{('')} = A_b$.*

Proof. If $\mathfrak{A}_5 \notin S$ then by (1.iv), $\mathfrak{A}_6 \notin S$. If $k \notin \{5,6\}$ and $\mathfrak{A}_k \in \mathrm{Ir}(\mathcal{B}^c)$, then $\mathfrak{A}_k$ satisfies (4). So $\mathcal{V}(S)$ satisfies (4) and $\mathfrak{A}$ satisfies (4). Now by Lemma 4 we get $A_{('')} = A_b$. ∎

COROLLARY 4′. *If $S \in \boldsymbol{S}$ and $\mathfrak{A}$ is a generic of $\mathcal{V}(S)$, then $A_{('')} = A_b$ iff $\mathfrak{A}_5 \notin S$.*

Proof. $\Leftarrow$ Follows from Corollary 4.

$\Rightarrow$ If $\mathfrak{A}_5 \in S$ then $\mathcal{V}(S)$ does not satisfy (4) since $\mathfrak{A}_5$ does not. So $\mathfrak{A}$ does not satisfy (4) and by Lemma 4 we get $A_{('')} \neq A_b$. ∎

LEMMA 5. *$\mathfrak{A}$ satisfies*

$$q_{('')}(x) \approx q_{(')}(x) \tag{5}$$

iff for every $a \in A_{('')}$ we have $a' = a$.

Proof. $\Rightarrow$ Let $a \in A_{('')}$. Then by (2.ii) and (5) we have $a = q^{\mathfrak{A}}_{('')}(a) = q^{\mathfrak{A}}_{(')}(a)$.

$\Leftarrow$ Let $a \in A$. Then $q^{\mathfrak{A}}_{('')}(a) \in A_{('')}$ by (2.ii). So by (2.vi) and the assumption we have $q^{\mathfrak{A}}_{(')}(a) = (q^{\mathfrak{A}}_{('')}(a))' = q^{\mathfrak{A}}_{('')}(a)$. ∎

LEMMA 6. *If* $\mathfrak{A}$ *does not satisfy* (5) *and* $|A_b| = 1$, *then* $|A_{('')} \setminus A_b| \geq 2$.

Proof. If $\mathfrak{A}$ does not satisfy (5) then by Lemma 5 there exists $a \in A_{('')}$ with $a \neq a'$. It cannot be the case that $a \in A_b$ since by assumption $\mathfrak{A}_b$ is a 1-element algebra. Consequently, $a \in A_{('')} \setminus A_b$. By (2.v) we have $a' \in A_{('')}$. We cannot have $a' \in A_b$ since then, by (2.v), $a = (a')' \in A_b$, which contradicts the assumption that $|A_b| = 1$. Thus $a' \in A_{('')} \setminus A_b$. ∎

LEMMA 7. *If* $|A_b| \geq 2$ *and* $a \in A_{('')} \setminus A_b$, *then* $a' \neq a$ *and* $a' \in A_{('')} \setminus A_b$. *So* $|A_{('')} \setminus A_b| \geq 2$.

Proof. By (2.vi) and (2.v) we have $a' \in A_{('')}$. Since $\mathfrak{A}_b$ is a nontrivial Boolean algebra (see (2.v)), for $b \in A_b$ we must have $b' \neq b$. Therefore since $q_b^{\mathfrak{A}}$ is an endomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_b$, we have $a' \neq a$. Moreover, $a' \notin A_b$ since otherwise $a = (a')' \in A_b$ contrary to the assumptions. Thus $a' \in A_{('')} \setminus A_b$. ∎

LEMMA 8. *If* $\mathfrak{A}_6 \in S$, $S \in \boldsymbol{S}$ *and* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(S)$, *then* $|A_{('')} \setminus A_b| \geq 2$.

Proof. $\mathfrak{A}$ does not satisfy (5) since $\mathfrak{A}_6$ does not. So if $|A_b| = 1$ we get the statement by Lemma 6. Since $\mathfrak{A}_6 \in S$ and $\mathfrak{A}_6$ does not satisfy (4), it follows that $\mathfrak{A}$ does not satisfy (4) and by Lemma 4 we get $A_{('')} \setminus A_b \neq \emptyset$. Hence, if $|A_b| > 1$, we get the statement by Lemma 7. ∎

If a set S belongs to $\boldsymbol{S}$ (see (1.iii)), then we shall write $\mathcal{V}(i_1, \ldots, i_k)$ instead of $\mathcal{V}(S)$, where $i_1, \ldots, i_k$ is the sequence of different indices of all algebras from S written in increasing order. For example $\mathcal{V}(2, 4)$ stands for $\mathcal{V}(\{\mathfrak{A}_2, \mathfrak{A}_4\})$.

THEOREM 2. *We have*

(2.1) *If* $\mathfrak{A} \in \mathcal{V}(2, \ldots, 6)$, *then* $|A_b| = 1$.

(2.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(2, \ldots, 6)$, *then* $A_{(+)} \setminus A_b \neq \emptyset \neq A_{(\cdot)} \setminus A_b$ *and* $|A_{('')} \setminus A_b| \geq 2$.

(2.3) *The subdirect product*

$$\mathfrak{A}(2,3,6) = (\{\langle a_2, b_3, b_6\rangle, \langle b_2, a_3, b_6\rangle, \langle a_2, b_3, a_6\rangle, \langle b_2, b_3, c_6\rangle, \langle b_2, b_3, b_6\rangle\}; +, \cdot, ')$$

of the direct product $\mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_6$ *is a minimal generic of* $\mathcal{V}(2, \ldots, 6)$, *i.e.* $g(\mathcal{V}(2, \ldots, 6)) = 5$.

Proof. (2.1) holds by Corollary 1; (2.2) holds by Corollaries 2′, 3′ and Lemma 8. It remains to prove (2.3). By (1.v) and (1.iv) we get $\mathfrak{A}_2, \ldots, \mathfrak{A}_6 \in \mathrm{HSP}(\mathfrak{A}(2,3,6))$. Therefore $\mathcal{V}(2, \ldots, 6) \subseteq \mathrm{HSP}(\mathfrak{A}(2,3,6))$ by (1.ii). Since $\mathfrak{A}(2,3,6) \in \mathcal{V}(2, \ldots, 6)$, it follows that $\mathcal{V}(2, \ldots, 6) = \mathrm{HSP}(\mathfrak{A}(2,3,6))$. Thus $\mathfrak{A}(2,3,6)$ is a generic of $\mathcal{V}(2, \ldots, 6)$ and by (2.2) and (2.iv) it is a minimal generic of $\mathcal{V}(2, \ldots, 6)$ since it contains five elements. ∎

THEOREM 3. *We have*

(3.1) *If* $\mathfrak{A} \in \mathcal{V}(2,\ldots,5)$, *then* $|A_b| = 1$.

(3.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(2,\ldots,5)$, *then* $A_{(+)} \setminus A_b \neq \emptyset \neq A_{(\cdot)} \setminus A_b$ *and* $A_{('')} \setminus A_b \neq \emptyset$.

(3.3) *The subdirect product*

$$\mathfrak{A}(2,3,5) = (\{\langle a_2,b_3,b_5\rangle, \langle b_2,a_3,b_5\rangle, \langle b_2,b_3,a_5\rangle, \langle b_2,b_3,b_5\rangle\}; +,\cdot,')$$

of $\mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_5$ *is a minimal generic of* $\mathcal{V}(2,\ldots,5)$, *and consequently*, $g(\mathcal{V}(2,\ldots,5)) = 4$.

Proof. (3.1) holds by Corollary 1; (3.2) holds by Corollaries 2′, 3′ and 4′; (3.3) holds by (1.v) for $\{i,j\} = \{3,5\}$. Thus $\mathcal{V}(2,\ldots,5) = \mathrm{HSP}(\mathfrak{A}(2,3,5))$ and we use the statement of (3.2). ∎

THEOREM 4. *We have*

(4.1) *If* $\mathfrak{A} \in \mathcal{V}(3,\ldots,6)$, *then* $|A_b| = 1$ *and* $A_{(+)} = A_b$.

(4.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(3,\ldots,6)$, *then* $A_{(\cdot)} \setminus A_b \neq \emptyset$ *and* $|A_{('')} \setminus A_b| \geq 2$.

(4.3) *The subdirect product*

$$\mathfrak{A}(3,6) = (\{\langle a_3,b_6\rangle, \langle b_3,a_6\rangle, \langle b_3,c_6\rangle, \langle b_3,b_6\rangle\}; +,\cdot,')$$

of $\mathfrak{A}_3 \times \mathfrak{A}_6$ *is a minimal generic of* $\mathcal{V}(3,\ldots,6)$, *and consequently*, $g(\mathcal{V}(3,\ldots,6)) = 4$.

Proof. (4.1) holds by Corollaries 1 and 2; (4.2) holds by Corollary 3′ and Lemma 8; (4.3) holds by (1.iv) and (1.v). ∎

THEOREM 5. *We have*

(5.1) *If* $\mathfrak{A} \in \mathcal{V}(1,3,4,5,6)$, *then* $A_{(+)} \setminus A_b = \emptyset$.

(5.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,3,4,5,6)$, *then* $|A_b| \geq 2$, $A_{(\cdot)} \setminus A_b \neq \emptyset$ *and* $|A_{('')} \setminus A_b| \geq 2$.

(5.3) *The subdirect product*

$$\mathfrak{A}(1,3,5) = (\{\langle a_1,b_3,b_5\rangle, \langle b_1,b_3,b_5\rangle, \langle a_1,a_3,b_5\rangle, \langle a_1,b_3,a_5\rangle, \langle b_1,b_3,a_5\rangle\}; +,\cdot,')$$

of $\mathfrak{A}_1 \times \mathfrak{A}_3 \times \mathfrak{A}_5$ *is a minimal generic of* $\mathcal{V}(1,3,4,5,6)$. *Consequently*, $g(\mathcal{V}(1,3,4,5,6)) = 5$.

Proof. (5.1) holds by Corollary 2; (5.2) holds by Corollaries 1′, 3′ and Lemma 8; (5.3) holds by (1.v) and (1.vi). ∎

THEOREM 6. *We have*

(6.1) *If* $\mathfrak{A} \in \mathcal{V}(3,4,5)$, *then* $|A_b| = 1$ *and* $A_{(+)} \setminus A_b = \emptyset$.

(6.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(3,4,5)$, *then* $A_{(\cdot)} \setminus A_b \neq \emptyset \neq A_{('')} \setminus A_b$.

(6.3) *The subdirect product*

$$\mathfrak{A}(3,5) = (\{\langle a_3, b_5\rangle, \langle b_3, b_5\rangle, \langle b_3, a_5\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_3 \times \mathfrak{A}_5$ *is a minimal generic of* $\mathcal{V}(3,4,5)$. *So* $g(\mathcal{V}(3,4,5)) = 3$.

Proof. (6.1) holds by Corollaries 1 and 2; (6.2) holds by Corollaries 3′ and 4′; (6.3) holds by (1.v). ∎

The proofs of the next three theorems are analogous to those of Theorems 4–6.

Theorem 7. *We have*

(7.1) *If* $\mathfrak{A} \in \mathcal{V}(2,4,5,6)$, *then* $|A_b| = 1$ *and* $A_{(\cdot)} = A_b$.
(7.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(2,4,5,6)$, *then* $A_{(+)} \setminus A_b \neq \emptyset$ *and* $|A_{('')} \setminus A_b| \geq 2$.
(7.3) *The subdirect product*

$$\mathfrak{A}(2,6) = (\{\langle a_2, b_6\rangle, \langle b_2, a_6\rangle, \langle b_2, c_6\rangle, \langle b_2, b_6\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_2 \times \mathfrak{A}_6$ *is a minimal generic of* $\mathcal{V}(2,4,5,6)$. *Consequently*, $g(\mathcal{V}(2,4,5,6)) = 4$.

Theorem 8. *We have*

(8.1) *If* $\mathfrak{A} \in \mathcal{V}(1,2,4,5,6)$, *then* $A_{(\cdot)} \setminus A_b = \emptyset$.
(8.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,2,4,5,6)$, *then* $|A_b| \geq 2$, $A_{(+)} \setminus A_b \neq \emptyset$ *and* $|A_{('')} \setminus A_b| \geq 2$.
(8.3) *The subdirect product*

$$\mathfrak{A}(1,2,5) = (\{\langle a_1, b_2, b_5\rangle, \langle b_1, b_2, b_5\rangle, \langle a_1, a_2, b_5\rangle, \langle a_1, b_2, a_5\rangle, \langle b_1, b_2, a_5\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_5$ *is a minimal generic of* $\mathcal{V}(1,2,4,5,6)$. *Consequently*, $g(\mathcal{V}(1,2,4,5,6)) = 5$.

Theorem 9. *We have*

(9.1) *If* $\mathfrak{A} \in \mathcal{V}(2,4,5)$, *then* $|A_b| = 1$ *and* $A_{(\cdot)} \setminus A_b = \emptyset$.
(9.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(2,4,5)$, *then* $A_{(+)} \setminus A_b \neq \emptyset \neq A_{('')} \setminus A_b$.
(9.3) *The subdirect product*

$$\mathfrak{A}(2,5) = (\{\langle a_2, b_5\rangle, \langle b_2, b_5\rangle, \langle b_2, a_5\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_2 \times \mathfrak{A}_5$ *is a minimal generic of* $\mathcal{V}(2,4,5)$. *So* $g(\mathcal{V}(2,4,5)) = 3$.

Theorem 10. *We have*

(10.1) *If* $\mathfrak{A} \in \mathcal{V}(2,3,4)$, *then* $|A_b| = 1$ *and* $A_{('')} \setminus A_b = \emptyset$.
(10.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(2,3,4)$, *then* $A_{(+)} \setminus A_b \neq \emptyset \neq A_{(\cdot)} \setminus A_b$.

(10.3) *The subdirect product*

$$\mathfrak{A}(2,3) = (\{\langle a_2, b_3\rangle, \langle b_2, a_3\rangle, \langle b_2, b_3\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_2 \times \mathfrak{A}_3$ *is a minimal generic of* $\mathcal{V}(2,3,4)$*. So* $g(\mathcal{V}(2,3,4)) = 3$.

Proof. (10.1) holds by Corollaries 1 and 4; (10.2) holds by Corollaries $2'$ and $3'$; (10.3) holds by (1.v) and (10.2). ■

THEOREM 11. *We have*

(11.1) *If* $\mathfrak{A} \in \mathcal{V}(1,2,3,4)$, *then* $A_{('')} \setminus A_b = \emptyset$.

(11.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,2,3,4)$, *then* $|A_b| \geq 2$ *and* $A_{(+)} \setminus A_b \neq \emptyset \neq A_{(\cdot)} \setminus A_b$.

(11.3) *The subdirect product*

$$\mathfrak{A}(1,2,3) = (\{\langle a_1, b_2, b_3\rangle, \langle b_1, b_2, b_3\rangle, \langle a_1, a_2, b_3\rangle, \langle a_1, b_2, a_3\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3$ *is a minimal generic of* $\mathcal{V}(1,2,3,4)$*. Consequently,* $g(\mathcal{V}(1,2,3,4)) = 4$.

Proof. (11.1) holds by Corollary 4; (11.2) holds by Corollaries $1'$, $2'$ and $3'$; (11.3) holds by (1.v). ■

LEMMA 9. *If* $S \in \boldsymbol{S}$, $\mathfrak{A}$ *is a generic of* $\mathcal{V}(S)$ *and* $\mathfrak{A}_4 \in \mathcal{V}(S)$, *then* $\mathfrak{A}$ *satisfies none of the identities* $q_{(+)}(x) \approx x$, $q_{(\cdot)}(x) \approx x$, $q_{('')}(x) \approx x$, $q_{(')}(x) \approx x$, $q_b(x) \approx x$.

In fact, $\mathfrak{A}_4$ satisfies none of these identities, so neither does $\mathfrak{A}$.

LEMMA 10. $A = A_b$ *iff* $\mathfrak{A}$ *satisfies*

$$q_b(x) \approx x. \tag{6}$$

Proof. $\Rightarrow$ If $a \in A$, then $a \in A_b$, so $q_b(a) = a$ by (2.ii).

$\Leftarrow$ If (6) holds, then for every $a \in A$ we have $a \in A_b$ by (2.ii), so $A \subseteq A_b$ and $A = A_b$. ■

Similarly, we prove that

LEMMA 11. $A = A_{(+)}$ *iff* $\mathfrak{A}$ *satisfies*

$$q_{(+)}(x) \approx x. \tag{7}$$

LEMMA 12. $A = A_{(\cdot)}$ *iff* $\mathfrak{A}$ *satisfies*

$$q_{(\cdot)}(x) \approx x. \tag{8}$$

LEMMA 13. $A = A_{('')}$ *iff* $\mathfrak{A}$ *satisfies*

$$q_{('')}(x) \approx x. \tag{9}$$

LEMMA 14. *If* $\mathfrak{A}$ *satisfies* (5), *then* $A = A_{('')}$ *iff* $\mathfrak{A}$ *satisfies*

$$q_{(')}(x) \approx x. \tag{10}$$

This follows at once from Lemma 13.

THEOREM 12. *We have*

(12.1) *If* $\mathfrak{A} \in \mathcal{V}(4)$, *then* $|A_b| = 1$ *and* $A_{(+)} = A_{(\cdot)} = A_{('')} = A_b$.
(12.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(4)$, *then* $A \setminus A_b \neq \emptyset$.
(12.3) *The algebra* $\mathfrak{A}_4$ *is a minimal generic of* $\mathcal{V}(4)$. *So* $g(\mathcal{V}(4)) = 2$.

Proof. (12.1) holds by Corollaries 1–4; (12.2) holds by Lemmas 10 and 9. In fact, $\mathfrak{A}$ does not satisfy (6) since $\mathfrak{A}_4$ does not. (12.3) holds by (1.ii). ∎

THEOREM 13. *We have*

(13.1) *If* $\mathfrak{A} \in \mathcal{V}(1,4)$, *then* $A_{(+)} = A_{(\cdot)} = A_{('')} = A_b$.
(13.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,4)$, *then* $|A_b| \geq 2$ *and* $A \setminus A_b \neq \emptyset$.
(13.3) *The subdirect product*

$$\mathfrak{A}(1,4) = (\{\langle a_1, b_4\rangle, \langle b_1, a_4\rangle, \langle b_1, b_4\rangle\};;+,\cdot,')$$

of $\mathfrak{A}_1 \times \mathfrak{A}_4$ *is a minimal generic of* $\mathcal{V}(1,4)$. *So* $g(\mathcal{V}(1,4)) = 3$.

Proof. (13.1) holds by Corollaries 2–4; (13.2) holds by Corollary 1′, Lemmas 10 and 9; (13.3) holds by (1.ii). ∎

THEOREM 14. *We have*

(14.1) *If* $\mathfrak{A} \in \mathcal{V}(2)$, *then* $|A_b| = 1$ *and* $A_{(\cdot)} = A_{('')} = A_b$ *and* $A = A_{(+)}$.
(14.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(2)$, *then* $A_{(+)} \setminus A_b \neq \emptyset$.
(14.3) *The algebra* $\mathfrak{A}_2$ *is a minimal generic of* $\mathcal{V}(2)$. *So* $g(\mathcal{V}(2)) = 2$.

Proof. (14.1) holds by Corollaries 1, 3, 4 and Lemma 11. In fact, $\mathcal{V}(2)$ satisfies (7) since $\mathfrak{A}_2$ does. (14.2) holds by Corollary 2′, and (14.3) is obvious. ∎

THEOREM 15. *We have*

(15.1) *If* $\mathfrak{A} \in \mathcal{V}(1,2)$, *then* $A_{(\cdot)} = A_{('')} = A_b$ *and* $A = A_{(+)}$.
(15.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,2)$, *then* $|A_b| \geq 2$ *and* $A_{(+)} \setminus A_b \neq \emptyset$.
(15.3) *The subdirect product*

$$\mathfrak{A}(1,2) = (\{\langle a_1, a_2\rangle, \langle a_1, b_2\rangle, \langle b_1, b_2\rangle\};+,\cdot,')$$

of $\mathfrak{A}_1 \times \mathfrak{A}_2$ *is a minimal generic of* $\mathcal{V}(1,2)$. *So* $g(\mathcal{V}(1,2)) = 3$.

Proof. (15.1) holds by Corollaries 3, 4 and Lemma 11; (15.2) holds by Corollaries 1′ and 2′; (15.3) is obvious. ∎

THEOREM 16. *We have*

(16.1) *If* $\mathfrak{A} \in \mathcal{V}(2,4)$, *then* $|A_b| = 1$ *and* $A_{(\cdot)} = A_{('')} = A_b$.
(16.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(2,4)$, *then* $A_{(+)} \setminus A_b \neq \emptyset \neq A \setminus A_{(+)}$.
(16.3) *The subdirect product*

$$\mathfrak{A}(2,4) = (\{\langle a_2, b_4\rangle, \langle b_2, a_4\rangle, \langle b_2, b_4\rangle\};+,\cdot,')$$

of $\mathfrak{A}_2 \times \mathfrak{A}_4$ *is a minimal generic of* $\mathcal{V}(2,4)$. *So* $g(\mathcal{V}(2,4)) = 3$.

Proof. (16.1) holds by Corollaries 1, 3 and 4; (16.2) holds by Corollary 2′ and Lemmas 11 and 9; (16.3) holds by (1.ii). ∎

THEOREM 17. *We have*

(17.1) *If* $\mathfrak{A} \in \mathcal{V}(1,2,4)$, *then* $A_{(\cdot)} = A_{('')} = A_b$.

(17.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,2,4)$, *then* $|A_b| \geq 2$ *and* $A_{(+)} \setminus A_b \neq \emptyset \neq A \setminus A_{(+)}$.

(17.3) *The subdirect product*

$$\mathfrak{A}(1,2,4) = (\{\langle a_1, a_2, b_4\rangle, \langle a_1, b_2, b_4\rangle, \langle a_1, b_2, a_4\rangle, \langle b_1, b_2, b_4\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_4$ *is a minimal generic of* $\mathcal{V}(1,2,4)$. *Consequently,* $g(\mathcal{V}(1,2,4)) = 4$.

Proof. (17.1) holds by Corollaries 3 and 4; (17.2) holds by Corollaries 1′, 2′, Lemmas 11 and 9; (17.3) is obvious. ∎

The proofs of Theorems 18–21 are analogous to those of Theorems 14–17, respectively. However, we must replace Lemma 11 by Lemma 12.

THEOREM 18. *We have*

(18.1) *If* $\mathfrak{A} \in \mathcal{V}(3)$, *then* $|A_b| = 1$ *and* $A_{(+)} = A_{('')} = A_b$ *and* $A = A_{(\cdot)}$.

(18.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(3)$, *then* $A_{(\cdot)} \setminus A_b \neq \emptyset$.

(18.3) *The algebra* $\mathfrak{A}_3$ *is a minimal generic of* $\mathcal{V}(3)$. *So* $g(\mathcal{V}(3)) = 2$.

THEOREM 19. *We have*

(19.1) *If* $\mathfrak{A} \in \mathcal{V}(1,3)$, *then* $A_{(+)} = A_{('')} = A_b$ *and* $A = A_{(\cdot)}$.

(19.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,3)$, *then* $|A_b| \geq 2$ *and* $(A_{(\cdot)} \setminus A_b) \neq \emptyset$.

(19.3) *The subdirect product*

$$\mathfrak{A}(1,3) = (\{\langle a_1, a_3\rangle, \langle a_1, b_3\rangle, \langle b_1, b_3\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_1 \times \mathfrak{A}_3$ *is a minimal generic of* $\mathcal{V}(1,3)$. *So* $g(\mathcal{V}(1,3)) = 3$.

THEOREM 20. *We have*

(20.1) *If* $\mathfrak{A} \in \mathcal{V}(3,4)$, *then* $|A_b| = 1$ *and* $A_{(+)} = A_{('')} = A_b$.

(20.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(3,4)$, *then* $A_{(\cdot)} \setminus A_b \neq \emptyset \neq A \setminus A_{(\cdot)}$.

(20.3) *The subdirect product*

$$\mathfrak{A}(3,4) = (\{\langle a_3, b_4\rangle, \langle b_3, a_4\rangle, \langle b_3, b_4\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_3 \times \mathfrak{A}_4$ *is a minimal generic of* $\mathcal{V}(3,4)$. *So* $g(\mathcal{V}(3,4)) = 3$.

THEOREM 21. *We have*

(21.1) *If* $\mathfrak{A} \in \mathcal{V}(1,3,4)$, *then* $A_{(+)} = A_{('')} = A_b$.

(21.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,3,4)$, *then* $|A_b| \geq 2$ *and* $A_{(\cdot)} \setminus A_b \neq \emptyset \neq A \setminus A_{(\cdot)}$.

(21.3) *The subdirect product*

$$\mathfrak{A}(1,3,4) = (\{\langle a_1, a_3, b_4\rangle, \langle a_1, b_3, b_4\rangle, \langle a_1, b_3, a_4\rangle, \langle b_1, b_3, b_4\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_1 \times \mathfrak{A}_3 \times \mathfrak{A}_4$ *is a minimal generic of* $\mathcal{V}(1,3,4)$. *Consequently,* $g(\mathcal{V}(1,3,4)) = 4$.

THEOREM 22. *We have*

(22.1) *If* $\mathfrak{A} \in \mathcal{V}(5,6)$, *then* $|A_b| = 1$ *and* $A_{(+)} = A_{(\cdot)} = A_b$ *and* $A = A_{('')}$.
(22.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(5,6)$, *then* $|A_{('')} \setminus A_b| \geq 2$.
(22.3) *The algebra* $\mathfrak{A}_6$ *is a minimal generic of* $\mathcal{V}(5,6)$. *So* $g(\mathcal{V}(5,6)) = 3$.

Proof. (22.1) holds by Corollaries 1–3 and Lemma 13; (22.2) holds by Lemma 8; (22.3) is obvious by (1.iv). ∎

THEOREM 23. *We have*

(23.1) *If* $\mathfrak{A} \in \mathcal{V}(1,5,6)$, *then* $A_{(+)} = A_{(\cdot)} = A_b$ *and* $A = A_{('')}$.
(23.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,5,6)$, *then* $|A_b| \geq 2$ *and* $|A_{('')} \setminus A_b| \geq 2$.
(23.3) *The algebra* $\mathfrak{A}_1 \times \mathfrak{A}_5$ *is a minimal generic of* $\mathcal{V}(1,5,6)$. *Consequently,* $g(\mathcal{V}(1,5,6)) = 4$.

Proof. (23.1) holds by Corollaries 2–3 and Lemma 13; (23.2) holds by Corollary $1'$ and Lemma 8; (23.3) is obvious, by (1.vi). ∎

THEOREM 24. *We have*

(24.1) *If* $\mathfrak{A} \in \mathcal{V}(4,5,6)$, *then* $|A_b| = 1$ *and* $A_{(+)} = A_{(\cdot)} = A_b$.
(24.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(4,5,6)$, *then* $|A_{('')} \setminus A_b| \geq 2$ *and* $A \setminus A_{('')} \neq \emptyset$.
(24.3) *The subdirect product*

$$\mathfrak{A}(4,6) = (\{\langle a_4, b_6\rangle, \langle b_4, b_6\rangle, \langle b_4, a_6\rangle, \langle b_4, c_6\rangle, \}; +, \cdot, ')$$

of $\mathfrak{A}_4 \times \mathfrak{A}_6$ *is a minimal generic of* $\mathcal{V}(4,5,6)$. *So* $g(\mathcal{V}(4,5,6)) = 4$.

Proof. (24.1) holds by Corollaries 1–3; (24.2) holds by Lemmas 8, 13 and 9; (24.3) is obvious, by (1.iv). ∎

THEOREM 25. *We have*

(25.1) *If* $\mathfrak{A} \in \mathcal{V}(1,4,5,6)$, *then* $A_{(+)} = A_{(\cdot)} = A_b$.
(25.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(1,4,5,6)$, *then* $|A_b| \geq 2$ *and* $|A_{('')} \setminus A_b| \geq 2$ *and* $A \setminus A_{('')} \neq \emptyset$.
(25.3) *The subdirect product*

$$\mathfrak{A}(1,4,5) = (\{\langle a_1, b_4, b_5\rangle, \langle b_1, b_4, b_5\rangle, \langle a_1, a_4, b_5\rangle, \langle a_1, b_4, a_5\rangle, \langle b_1, b_4, a_5\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_1 \times \mathfrak{A}_4 \times \mathfrak{A}_5$ *is a minimal generic of* $\mathcal{V}(1,4,5,6)$. *Consequently,* $g(\mathcal{V}(1,4,5,6)) = 5$.

Proof. (25.1) holds by Corollaries 2 and 3; (25.2) holds by Corollary $1'$, Lemmas 8, 13 and 9; (25.3) is obvious, by (1.vi). ∎

THEOREM 26. *We have*

(26.1) *If* $\mathfrak{A} \in \mathcal{V}(5)$, *then* $|A_b| = 1$ *and* $A_{(+)} = A_{(\cdot)} = A_b$ *and* $A = A_{('')} = \{x \in A : x' = x\}$.

(26.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(5)$, *then* $A_{('')} \setminus A_b \neq \emptyset$.

(26.3) *The algebra* $\mathfrak{A}_5$ *is a minimal generic of* $\mathcal{V}(5)$. *So* $g(\mathcal{V}(5)) = 2$.

Proof. (26.1) holds by Corollaries 1–3 and Lemma 14; (26.2) holds by Corollary 4′; (26.3) is obvious. ■

THEOREM 27. *We have*

(27.1) *If* $\mathfrak{A} \in \mathcal{V}(4,5)$, *then* $|A_b| = 1$ *and* $A_{(+)} = A_{(\cdot)} = A_b$.

(27.2) *If* $\mathfrak{A}$ *is a generic of* $\mathcal{V}(4,5)$, *then* $A_{('')} \setminus A_b \neq \emptyset \neq A \setminus A_{('')}$.

(27.3) *The subdirect product*

$$\mathfrak{A}(4,5) = (\{\langle a_4, b_5\rangle, \langle b_4, b_5\rangle, \langle b_4, a_5\rangle\}; +, \cdot, ')$$

of $\mathfrak{A}_4 \times \mathfrak{A}_5$ *is a minimal generic of* $\mathcal{V}(4,5)$. *So* $g(\mathcal{V}(4,5)) = 3$.

Proof. (27.1) holds by Corollaries 1–3; (27.2) holds by Corollary 4′, Lemmas 14 and 9; (27.3) is obvious. ■

Obviously we have:

(2.viii) A 1-element algebra of type τ_b is a minimal generic of the trivial variety $\mathcal{V}(\emptyset)$ (satisfying $x \approx y$).

It is known that

(2.ix) The algebra $\mathfrak{A}_1$ is a minimal generic of the variety $\mathcal{V}(1) = \mathcal{B}$.

In (2.vii) we noticed that $g(\mathcal{B}^c) = 6$, which was proved in [4]. Now having Corollaries 1′–3′ and Lemma 8 of the present paper the reader can easily see that $g(\mathcal{B}^c) \geq 6$, which together with the algebra $\mathfrak{A}(1,2,3,5)$ gives the statement of (2.vii).

We hope that the observations and methods of our paper will also be useful in other cases of finding minimal generics of varieties.

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REFERENCES

[1] G. Grätzer, *Universal Algebra*, 2nd ed., Springer, New York, 1979.

[2] J. Płonka, *Clone compatible identities and clone extensions of algebras*, Math. Slovaca 47 (1997), 231–249.

[3] —, *Subdirect decompositions of algebras from 2-clone extensions of varieties*, Colloq. Math. 77 (1998), 189–199.

[4] —, *On n-clone extensions of algebras*, Algebra Universalis 40 (1998), 1–17.

[5] —, *Lattices of subvarieties of the clone extensions of some varieties*, in: Contributions to General Algebra 11, Verlag Johannes Heyn, Klagenfurt, 1999, 161–171.

[6] J. Płonka, *Clone networks, clone extensions and biregularizations of varieties of algebras*, Algebra Colloq. 8 (2001), 327–344.
[7] —, *Clone extensions of varieties of algebras with nullary operations*, in: Contributions to General Algebra 15, Verlag Johannes Heyn, Klagenfurt, 2004, 109–118.
[8] —, *Subvarieties of the clone extension of the variety of distributive lattices*, Algebra Universalis 55 (2006), 175–186.
[9] —, *Subvarieties of the clone extension of the variety of Boolean algebras*, Southeast Asian Bull. Math. 31 (2007), 727–737.

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