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FREE POWERS OF THE FREE POISSON MEASURE

BY

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Abstract. We compute moments of the measures $(\varpi^{\boxtimes p})^{\boxplus t}$, where ϖ denotes the free Poisson law, and $\boxplus$ and $\boxtimes$ are the additive and multiplicative free convolutions. These moments are expressed in terms of the Fuss–Narayana numbers.

1. Introduction. *Free convolution* is a binary operation on the class $\mathcal{M}$ of probability measures on $\mathbb{R}$, which corresponds to the notion of free independence in noncommutative probability (see [3, 8, 12]). Namely, if X, Y are free noncommuting random variables, with distributions $\mu, \nu \in \mathcal{M}$ respectively, then the *additive free convolution* $\mu \boxplus \nu$ is the distribution of the sum $X + Y$. Similarly, if moreover $X \geq 0$ then the *multiplicative free convolution* $\mu \boxtimes \nu$ can be defined as the distribution of the product $\sqrt{X}Y\sqrt{X}$.

Here we can confine ourselves to the class $\mathcal{M}^{\mathbf{c}}$ of compactly supported measures in $\mathcal{M}$. Let $\mathcal{M}^{\mathbf{c}}_{+}$ denote the class of those $\mu \in \mathcal{M}^{\mathbf{c}} \setminus \{\delta_0\}$ with support in $[0, \infty)$. For $\mu \in \mathcal{M}^{\mathbf{c}}$ we define its *moment generating function*

$$M_\mu(z) := \sum_{m=0}^{\infty} s_m(\mu) z^m, \tag{1}$$

defined in some neighborhood of 0, where

$$s_m(\mu) := \int_{\mathbb{R}} x^m \, d\mu(x) \tag{2}$$

is the mth moment of μ. Then we define its *R-transform* $R_\mu(z)$ by the equation

$$M_\mu(z) = R_\mu(z M_\mu(z)) + 1. \tag{3}$$

If $R_\mu(z) = \sum_{m=1}^{\infty} r_m(\mu) z^m$ then the numbers $r_m(\mu)$ are called the *free cumulants* of μ. For $\mu, \nu \in \mathcal{M}^{\mathbf{c}}$ we define the *additive free convolution* $\mu \boxplus \nu$ and the *additive free power* $\mu^{\boxplus t}$ by

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z) \quad \text{and} \quad R_{\mu^{\boxplus t}}(z) = t R_\mu(z). \tag{4}$$

The latter is well defined at least for $t \geq 1$.

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The *free S-transform* (see [11]) of $\mu \in \mathcal{M}_+^{\mathbf{c}}$ is defined by the relation

$$R_\mu(zS_\mu(z)) = z \quad \text{or} \quad M_\mu(z(1+z)^{-1}S_\mu(z)) = 1+z. \tag{5}$$

Observe that

$$S_{\mu^{\boxplus t}}(z) = \frac{1}{t}S_\mu\left(\frac{z}{t}\right). \tag{6}$$

If $\mu,\nu \in \mathcal{M}^{\mathbf{c}}$ and μ has support contained in $[0,\infty)$ then the *multiplicative free convolution* $\mu \boxtimes \nu$ and the multiplicative free powers $\mu^{\boxtimes p}$ are defined by

$$S_{\mu\boxtimes\nu}(z) := S_\mu(z)S_\nu(z) \quad \text{and} \quad S_{\mu^{\boxtimes p}}(z) = S_\mu(z)^p. \tag{7}$$

The powers are well defined at least for $p \ge 1$.

For $c \in \mathbb{R}$, $c \neq 0$, and $\mu \in \mathcal{M}$ we define the *dilation* $\mathbf{D}_c\mu \in \mathcal{M}$ by $\mathbf{D}_c\mu(X) := \mu(c^{-1}X)$ for every Borel subset X of $\mathbb{R}$. Then we have

$$M_{\mathbf{D}_c\mu}(z) = M_\mu(cz), \quad R_{\mathbf{D}_c\mu}(z) = R_\mu(cz), \quad S_{\mathbf{D}_c\mu}(z) = \frac{1}{c}S_\mu(z). \tag{8}$$

The last formula, together with (6), leads to

PROPOSITION 1.1. *Assume that $\mu \in \mathcal{M}_+^{\mathbf{c}}$, $p,t>0$ and both the measures $(\mu^{\boxplus t})^{\boxtimes p}$ and $(\mu^{\boxtimes p})^{\boxplus t}$ exist. Then*

$$\mathbf{D}_{t^{p-1}}(\mu^{\boxtimes p})^{\boxplus t} = (\mu^{\boxplus t})^{\boxtimes p}.$$

Proof. If $S_\mu(z)$ is the free S-transform of μ then

$$\frac{1}{t}S_\mu\left(\frac{z}{t}\right)^p \quad \text{and} \quad \frac{1}{t^p}S_\mu\left(\frac{z}{t}\right)^p$$

are the free S-transforms of $(\mu^{\boxtimes p})^{\boxplus t}$ and $(\mu^{\boxplus t})^{\boxtimes p}$ respectively. ■

2. The free Poisson measure. Our aim is to study the additive and multiplicative free powers of the free Poisson measure

$$\varpi := \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,dx \qquad \text{on } [0,4].$$

It is known that $\varpi^{\boxplus t}$ is $\boxtimes$-infinitely divisible for $t \ge 1$ and $\varpi^{\boxtimes p}$ is $\boxplus$-infinitely divisible for $p \ge 1$ (see [12, 8, 2, 7]). Therefore the double powers $(\varpi^{\boxplus t})^{\boxtimes p}$ and $(\varpi^{\boxtimes p})^{\boxplus t}$ exist whenever $p,t>0$ and $\max\{p,t\} \ge 1$.

The additive free powers $\varpi^{\boxplus t}$, $t>0$, are well known:

$$\varpi^{\boxplus t} = \max\{1-t,0\}\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}dx \tag{9}$$

with the absolutely continuous part supported on $[(1-\sqrt{t})^2,(1+\sqrt{t})^2]$, as well as the corresponding functions:

$$M_{\varpi^{\boxplus t}}(z) = \frac{2}{1+(1-t)z+\sqrt{(1-(1+t)z)^2-4tz^2}} \tag{10}$$
$$= 1 + \sum_{m=1}^{\infty} z^m \sum_{k=1}^{m} \binom{m}{k}\binom{m}{k-1}\frac{t^k}{m},$$

$$R_{\varpi^{\boxplus t}}(z) = \frac{tz}{1-z}, \qquad S_{\varpi^{\boxplus t}}(z) = \frac{1}{t+z}. \tag{11}$$

For the multiplicative free powers $\varpi^{\boxtimes p}$, $p > 0$, it is known [2, 7] that

$$M_{\varpi^{\boxtimes p}}(z) = \sum_{m=0}^{\infty} \binom{m(p+1)+1}{m} \frac{z^m}{m(p+1)+1}, \tag{12}$$

$$R_{\varpi^{\boxtimes p}}(z) = \sum_{m=1}^{\infty} \binom{mp+1}{m} \frac{z^m}{mp+1}. \tag{13}$$

Explicit formulas for the measures $\varpi^{\boxtimes p}$ are known only for natural p [9, 10].

Our aim now is to study the measures

$$\varpi(p,t) := (\varpi^{\boxtimes p})^{\boxplus t},$$

where $p, t > 0$ and $\max\{p,t\} \geq 1$. First observe that

$$R_{\varpi(p,t)}(z) = t \sum_{m=1}^{\infty} \binom{mp+1}{m} \frac{z^m}{mp+1} \tag{14}$$

and

$$S_{\varpi(p,t)}(z) = t^{p-1}(t+z)^{-p}. \tag{15}$$

Our previous remarks lead to the following

Proposition 2.1. *Assume that* $p, t > 0$ *and* $\max\{p,t\} \geq 1$.

- *If* $p \geq 1$ *then* $\varpi(p,t)$ *is* $\boxplus$*-infinitely divisible.*
- *If* $t \geq 1$ *then* $\varpi(p,t)$ *is* $\boxtimes$*-infinitely divisible.* ■

In order to compute moments of $\varpi(p,t)$ we will use the *Lagrange inversion formula* which says that if a function $z = f(w)$ is analytic at the point $w = a$ and $f'(a) \neq 0$, $f(a) =: b$, then for the inverse function $w = g(z)$ we have

$$g(z) = a + \sum_{m=1}^{\infty} \frac{d^{m-1}}{dw^{m-1}} \left(\frac{w-a}{f(w)-b}\right)^m \bigg|_{w=a} \frac{(z-b)^m}{m!}. \tag{16}$$

Now we are ready to prove (see [6] for the special case $p = 2$)

THEOREM 2.2. *For $p,t>0$ with $\max\{p,t\}\geq 1$, define $\varpi(p,t) := (\varpi^{\boxtimes p})^{\boxplus t}$. Then*

$$M_{\varpi(p,t)}(z) = 1+\sum_{m=1}^{\infty} z^m \sum_{k=1}^{m} \binom{m}{k-1}\binom{mp}{m-k}\frac{t^k}{m}. \tag{17}$$

Proof. Putting in (16)

$$f(w) := t^{p-1}w(1+w)^{-1}(t+w)^{-p}$$

and $a=b=0$ we have, in view of (5) and (15), $M_{\varpi(p,t)}(z) = 1+g(z)$. Therefore

$$M_{\varpi(p,t)}(z) = 1+\sum_{m=1}^{\infty}\frac{d^{m-1}}{dw^{m-1}}(t^{m(1-p)}(1+w)^m(t+w)^{mp})\bigg|_{w=0}\frac{z^m}{m!}.$$

But now

$$\begin{aligned}(1+w)^m(t+w)^{mp} &= t^{mp}\left(\sum_{k=0}^{m}\binom{m}{k}w^k\right)\left(\sum_{k=0}^{\infty}\binom{mp}{k}\left(\frac{w}{t}\right)^k\right)\\ &= t^{mp}\sum_{k=0}^{\infty}w^k\sum_{i=0}^{k}\binom{m}{i}\binom{mp}{k-i}t^{i-k}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d^{m-1}}{dw^{m-1}}((1+w)^m(t+w)^{mp})\bigg|_{w=0} &= t^{mp}(m-1)!\sum_{i=0}^{m-1}\binom{m}{i}\binom{mp}{m-1-i}t^{i-m+1}\\ &= t^{mp}(m-1)!\sum_{k=1}^{m}\binom{m}{k-1}\binom{mp}{m-k}t^{k-m},\end{aligned}$$

which leads to our statement. ∎

Note that in view of Proposition 1.1 there is no point to study powers like

$$(((\varpi^{\boxtimes p_1})^{\boxplus t_1})^{\boxtimes p_2})^{\boxplus t_2 \cdots}$$

because all of them are dilations of some of $\varpi(p,t)$.

It would be interesting to verify the following

CONJECTURE. Assume that $p,t>0$. Then the sequence $\{s_m(p,t)\}_{m=0}^{\infty}$ defined by: $s_0(p,t) := 1$ and

$$s_m(p,t) := \sum_{k=1}^{m}\binom{m}{k-1}\binom{mp}{m-k}\frac{t^k}{m}$$

for $m\geq 1$, is positive definite if and only if $\max\{p,t\}\geq 1$.

The coefficients at (12) and (17) have a remarkable combinatorial meaning, which was found by Edelman [4]. Namely, fix $m,p\in\mathbb{N}$ and let $\mathrm{NC}^{(p)}(m)$

denote the set of all noncrossing partitions π of $\{1,\ldots,mp\}$ such that p divides the cardinality of every block of π. Then the cardinality of $\mathrm{NC}^{(p)}(m)$ is expressed as the *Fuss–Catalan number*:

$$|\mathrm{NC}^{(p)}(m)| = \binom{m(p+1)+1}{m}\frac{1}{m(p+1)+1}.$$

For other applications of these numbers we refer to [5].

For $\pi \in \mathrm{NC}^{(p)}(m)$ we define its *rank* $\mathrm{rk}(\pi) := m - |\pi|$. The elements of fixed rank are counted by the *Fuss–Narayana numbers*:

$$|\{\pi \in \mathrm{NC}^{(p)}(m) : \mathrm{rk}(\pi) = k-1\}| = \binom{m}{k-1}\binom{mp}{m-k}\frac{1}{m}.$$

There is a natural partial order on $\mathrm{NC}^{(p)}(m)$. Namely, we say that $\pi \in \mathrm{NC}^{(p)}(m)$ is *finer* than $\sigma \in \mathrm{NC}^{(p)}(m)$, and write $\pi \preceq \sigma$, if every block of π is contained in a block of σ. Then $\mathrm{NC}^{(p)}(m)$ equipped with $\preceq$ and rk becomes a *graded partially ordered set* (which means that for any $\pi, \sigma \in \mathrm{NC}^{(p)}(m)$ with $\pi \preceq \sigma$, every unrefinable chain $\pi = \pi_0 \prec \pi_1 \prec \cdots \prec \pi_r = \sigma$ from π to σ has the same length $r = \mathrm{rk}(\sigma) - \mathrm{rk}(\pi)$) and a *join-semilattice* (i.e. any two elements in $\mathrm{NC}^{(p)}(m)$ have a least upper bound).

More general structures, noncrossing partitions on Coxeter groups, were studied in [1].

3. Symmetrization of $(\varpi^{\boxtimes p})^{\boxplus t}$**.** For $\mu \in \mathcal{M}$ concentrated on $[0,\infty)$, we define its *symmetrization* $\mu^{\mathbf{s}}$ by $\int_{\mathbb{R}} f(x^2)\,d\mu^{\mathbf{s}}(x) = \int_{\mathbb{R}} f(x)\,d\mu(x)$ for every compactly supported continuous function $f : \mathbb{R} \to \mathbb{R}$. If $M_\mu(z)$ is the moment generating function of μ then $M_{\mu^{\mathbf{s}}}(z) = M_\mu(z^2)$, which means that $s_{2m}(\mu^{\mathbf{s}}) = s_m(\mu)$ and odd moments of $\mu^{\mathbf{s}}$ are zero.

Now we will compute the free cumulants for the symmetrization of $\varpi(p,t)$.

THEOREM 3.1. *Assume that $p, t > 0$ with $\max\{p,t\} \geq 1$. Then for the symmetrization $\varpi(p,t)^{\mathbf{s}}$ of $\varpi(p,t)$ we have*

$$R_{\varpi(p,t)^{\mathbf{s}}}(z) = \sum_{m=1}^{\infty} z^{2m} \sum_{k=1}^{m} \binom{-m}{k-1}\binom{mp}{m-k}\frac{t^k}{m} \tag{18}$$

$$= -\sum_{m=1}^{\infty} z^{2m} \sum_{k=1}^{m} \binom{m+k-2}{k-1}\binom{mp}{m-k}\frac{(-t)^k}{m}.$$

Proof. The cumulant generating function $R(z)$ for $\varpi(p,t)^{\mathbf{s}}$ satisfies

$$t^{p-1}R(z)(1+R(z)) = z^2(R(z)+t)^p. \tag{19}$$

To check this, it is sufficient to substitute $z \mapsto zM(z^2)$ and compare with (5) and (15). Therefore $R(z) = R_0(z^2)$, where R_0 satisfies

$$t^{p-1}R_0(z)(1+R_0(z)) = z(R_0(z)+t)^p. \tag{20}$$

To conclude, we use the Lagrange inversion formula as in the proof of Theorem 2.2 putting $f(w) := t^{p-1}w(1+w)(t+w)^{-p}$, $a = b = 0$ to get $R_0(z) = g(z)$. ∎

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