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SOME REMARKS ON UNIVERSALITY PROPERTIES OF ℓ_∞/c_0

BY

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Abstract. We prove that if $\mathfrak{c}$ is not a Kunen cardinal, then there is a uniform Eberlein compact space K such that the Banach space $C(K)$ does not embed isometrically into ℓ_∞/c_0. We prove a similar result for isomorphic embeddings. Our arguments are minor modifications of the proofs of analogous results for Corson compacta obtained by S. Todorčević. We also construct a consistent example of a uniform Eberlein compactum whose space of continuous functions embeds isomorphically into ℓ_∞/c_0, but fails to embed isometrically. As far as we know it is the first example of this kind.

1. Introduction. For a compact space K we denote by $C(K)$ the Banach space of continuous functions on K with the supremum norm. We say that a Banach space X is *universal* (resp. *isometrically universal*) for the class $\mathcal{K}$ of compact spaces if for any $K \in \mathcal{K}$ the space $C(K)$ embeds isomorphically (resp. isometrically) into X.

In this note we deal with the universality properties of the space ℓ_∞/c_0. Let us recall a classical result of Parovichenko that the space $C(\beta\omega \setminus \omega)$, which is isometric to ℓ_∞/c_0, is isometrically universal for the class of compact spaces of weight $\aleph_1$. It is natural to ask whether there are any other classes of spaces for which ℓ_∞/c_0 is universal. Clearly this question makes sense only if we restrict ourselves to spaces of weight $\leq \mathfrak{c}$.

It has been shown by C. Brech and P. Koszmider in [6] that consistently it is not the case for the class of uniform Eberlein compacta. In fact they proved more: consistently there is no universal space for the class of uniform Eberlein compacta.

Recently S. Todorčević found a beautiful connection between universality properties of ℓ_∞/c_0 and properties of the σ-field of subsets of $\mathbb{R}^n$ generated by sets of the form $A_1 \times \cdots \times A_n$ where $A_1, \ldots, A_n \subseteq \mathbb{R}$ (see [11]). More precisely, he proved that if $\mathfrak{c}$ is not a Kunen cardinal ([1]), then ℓ_∞/c_0 is not an isometrically universal space for Corson compacta. He also proved,

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([1]) See Section 2 for definitions and notation under another set-theoretical assumption involving σ-fields of subsets of $\mathbb{R}^n$, a similar result for isomorphic rather than isometric embeddings.

In this note we strengthen Todorčević's results by a simple modification of his argument. We prove that if $\mathfrak{c}$ is not a Kunen cardinal, then ℓ_∞/c_0 is not a universal space for the class of uniform Eberlein compacta. In particular, by a different approach, we get the above mentioned result of C. Brech and P. Koszmider. Our considerations lead to a consistent example of a space of continuous functions on a uniform Eberlein compactum, which distinguishes isometric from isomorphic embeddings into ℓ_∞/c_0. In particular the uniform Eberlein compactum we get is not a continuous image of $\beta\omega\setminus\omega$ yet its space of continuous functions embeds isomorphically into ℓ_∞/c_0. As far as we know it is the first example of this kind.

2. Preliminaries. We use the standard set-theoretical and topological notation. Given a set A and a positive integer n we denote by $[A]^n$ (resp. $[A]^{\le n}$) the family of all subsets of A of cardinality n (resp. $\le n$). By $\mathfrak{c}$ we denote the cardinal number $2^{\aleph_0}$ (the continuum).

Let us recall that a compact space is *uniform Eberlein* if it is homeomorphic to a subset of a Hilbert space in its weak topology (see [9]). Equivalently, a space is a uniform Eberlein compactum if it can be embedded into the space

$$B(\Gamma)=\Big\{x\in[-1,1]^\Gamma:\sum_{\gamma\in\Gamma}|x_\gamma|\le 1\Big\}$$

for some index set Γ. Indeed, the above space is homeomorphic to a ball in the space $(\ell_2(\Gamma),\text{ weak})$. A well known example of a uniform Eberlein compactum is the following. Take a natural number n and an infinite set Γ and put

$$\sigma_n(\Gamma)=\{x\in\{0,1\}^\Gamma:|\{\gamma\in\Gamma:x_\gamma\ne 0\}|\le n\}.$$

This space, homeomorphic to $B(\Gamma)\cap\{0,1/n\}^\Gamma$, is uniform Eberlein compact. The following, probably well known fact about uniform Eberlein compacta will be useful. It says that this class of spaces is closed under taking the one-point compactification of a discrete sum.

PROPOSITION 2.1. *Let T be an arbitrary set of indices and suppose K_t is a uniform Eberlein compactum for every $t\in T$. Then the space $K=\bigoplus_{t\in T}K_t\cup\{\infty\}$ (i.e. the one-point compactification of a discrete sum of K_t's) is uniform Eberlein compact.*

Proof. For every $t\in T$, there is an embedding $h_t:K_t\to B(\Gamma_t)$, for some Γ_t. Let Γ be the disjoint union of $\{\Gamma_t:t\in T\}$ and T. Then $h:K\to B(\Gamma)$

defined by

$$h(x)_\gamma = \begin{cases} \frac{1}{2}h_t(x)_\gamma & \text{if } \gamma \in \Gamma_t, \\ 0 & \text{if } \gamma \in \Gamma_s,\ s \neq t, \\ \frac{1}{2} & \text{if } \gamma = t, \\ 0 & \text{if } \gamma \in T,\ \gamma \neq t, \end{cases}$$

for $x \in K_t$ and $h_\gamma(\infty) = 0$ for every $\gamma \in \Gamma$ is the desired embedding. ∎

For an arbitrary set Γ and a natural number $k \geq 2$, we denote by $\mathcal{P}^k(\Gamma)$ the σ-field generated by sets of the form $A_1 \times \cdots \times A_k$ where $A_1, \ldots, A_k \subseteq \Gamma$. Following [1] we call a cardinal κ *Kunen* if $\mathcal{P}(\kappa \times \kappa) = \mathcal{P}^2(\kappa)$. It is clear that for an arbitrary set Γ, the equality $\mathcal{P}(\Gamma \times \Gamma) = \mathcal{P}^2(\Gamma)$ depends only on the cardinality of Γ, so it holds if and only if $|\Gamma|$ is Kunen. We refer the reader to [1] for more on Kunen cardinals. Let us only mention here that the statement '$\mathfrak{c}$ is a Kunen cardinal' is independent of ZFC.

Finally, let us recall that elements of ℓ_∞/c_0 are of the form $[x] = \{y \in \ell_\infty : x - y \in c_0\}$, where $x \in \ell_\infty$. For $[x] \in \ell_\infty/c_0$ we have $\|[x]\| = \limsup_n |x(n)|$.

3. Proofs. In this section we prove a strengthening of two theorems of Todorčević from [11]. Theorems 3.2 and 3.4 below are counterparts of Theorems 4.1 and 4.3 in [11], respectively. Although the main idea in our proofs is the same as in [11], for completeness we provide detailed reasonings.

For a binary relation $E \subseteq \mathbb{R}^2$ and a natural number n we set

$$K_n(E) = \{\chi_A \in \{0,1\}^{\mathbb{R}} : A \in [\mathbb{R}]^{\leq n},\ \forall a, b \in A\ [a < b \Rightarrow (a,b) \in E]\}.$$

It is a compact subspace of $\{0,1\}^{\mathbb{R}}$ and moreover it is a uniform Eberlein compact space since it embeds into $\sigma_n(\mathbb{R})$.

The following easy proposition plays a key role in the proof of the next theorem (see [11]). For a function $f : X \to Y$ and a natural number $n \geq 2$, f^n denotes the n-fold Cartesian product $f \times \cdots \times f : X^n \to Y^n$.

Proposition 3.1. *Let (X, τ) be a separable, metrizable topological space. Let $f : \mathbb{R} \to X$ be an injection and $S \subseteq \mathbb{R}^2$ be such that $f^2[S]$ is a Borel subset of $(f^2[\mathbb{R}^2], \tau \times \tau)$. Then $S \in \mathcal{P}^2(\mathbb{R})$.*

Theorem 3.2. *Suppose that, for every uniform Eberlein compact space K of weight at most $\mathfrak{c}$, the space $C(K)$ embeds isometrically into ℓ_∞/c_0. Then $\mathfrak{c}$ is a Kunen cardinal.*

Proof. Suppose the contrary and let E witness that $\mathfrak{c}$ is not Kunen. Since $E \cap \{(a,b) : a = b\}$ is in $\mathcal{P}^2(\mathbb{R})$, either $E_0 = E \cap \{(a,b) : a < b\}$ or $E_1 = E \cap \{(a,b) : a > b\}$ is not in $\mathcal{P}^2(\mathbb{R})$. By symmetry we can assume it is E_0. Consider the space $K_2(E_0)$ which is uniform Eberlein compact. Now the proof goes as in [11]. We define an injection $\phi : \mathbb{R} \to C(K_2(E_0))$ by

$\phi(r) = f_r$, where $f_r(x) = x(r)$ for $x \in K_2(E_0)$. Let $T : C(K_2(E_0)) \to \ell_\infty/c_0$ be an isometry (which exists by our assumption) and let $\psi : \ell_\infty/c_0 \to \ell_\infty$ be an arbitrary injection (a selector). For $a < b$ we have

$$\begin{aligned}(a,b) \in E_0 \quad &\text{iff} \quad \chi_{\{a,b\}} \in K_2(E_0) \quad \text{iff} \quad \|T(f_a) + T(f_b)\| = \|f_a + f_b\| > 1\\ &\text{iff} \quad \limsup_n |(\psi \circ T(f_a))(n) + (\psi \circ T(f_b))(n)| > 1,\end{aligned}$$

so putting $g = \psi \circ\ T \circ \phi$ we deduce that

$$g^2[\mathbb{R}^2] \cap \{(x,y) \in (\mathbb{R}^\omega)^2 : \limsup_n |x(n) + y(n)| > 1\}$$

is a Borel subset of $g^2[\mathbb{R}^2] \subseteq \ell_\infty \times \ell_\infty$ in the topology inherited from $(\mathbb{R}^\omega)^2$. Hence, by Proposition 3.1,

$$(g^2)^{-1}[\{(x,y) \in (\mathbb{R}^\omega)^2 : \limsup_n |x(n) + y(n)| > 1\}] \in \mathcal{P}^2(\mathbb{R}).$$

Since

$$E_0 = (g^2)^{-1}[\{(x,y) \in (\mathbb{R}^\omega)^2 : \limsup_n |x(n)+y(n)| > 1\}] \cap \{(x,y) \in \mathbb{R}^2 : x < y\}$$

and since $\{(x,y) \in \mathbb{R}^2 : x < y\} \in \mathcal{P}^2(\mathbb{R})$ we conclude that $E_0 \in \mathcal{P}^2(\mathbb{R})$, a contradiction. ∎

Notice that the result also follows under a weaker assumption: for every *scattered* uniform Eberlein compactum (of height 3) its space of continuous functions embeds isometrically into ℓ_∞/c_0.

To prove the result about isomorphic embeddings we need the following modification of Proposition 3.1.

PROPOSITION 3.3. *Let (X,τ) be a separable, metrizable topological space. Let $f : \mathbb{R} \to X$ be an injection and $A, B \subseteq \mathbb{R}^n$ for some integer $n \geq 2$. If $f^n[A]$ is separated from $f^n[B]$ by a Borel set in X^n, then A can be separated from B by a member of $\mathcal{P}^n(\mathbb{R})$.*

Following [11], for any binary relation $E \subseteq \mathbb{R}^2$ and integer $n \geq 2$, we set

$$E^{[n]} = \{(x_1, \dots x_n) \in \mathbb{R}^n : \forall i < j\ [x_i \neq x_j \text{ and } (x_i, x_j) \in E]\}.$$

In particular, $<^{[n]}$ is the set of all strictly increasing sequences of reals of length n. The complementary relation to E will be denoted by E^c.

THEOREM 3.4. *Suppose that, for every uniform Eberlein compact space K of weight at most $\mathfrak{c}$, the space $C(K)$ embeds isomorphically into ℓ_∞/c_0. Then for every binary relation $E \subseteq \mathbb{R}^2$ and for all but finitely many positive integers n, the set $E^{[n]}$ can be separated from $(E^c)^{[n]}$ by a member of $\mathcal{P}^n(\mathbb{R})$.*

Proof. Let $E \subseteq \mathbb{R}^2$ be arbitrary. First we will prove the theorem for the relation

$$E_0 = E \cap < = \{(x,y) \in \mathbb{R}^2 : x < y \text{ and } (x,y) \in E\}.$$

Since $E_0^{[n]} \subseteq <^{[n]}$ and $<^{[n]} \in \mathcal{P}^n(\mathbb{R})$, to get the desired separation between $E_0^{[n]}$ and $(E_0^c)^{[n]}$, it is enough to separate $E_0^{[n]}$ from $(E_0^c)^{[n]} \cap <^{[n]}$ by a member of $\mathcal{P}^n(\mathbb{R})$. To this end let us consider

$$K = \bigoplus_{n=1}^{\infty} K_n(E_0) \cup \{\infty\},$$

which is the one-point compactification of a discrete sum of the spaces $K_n(E_0)$. By Proposition 2.1, K is a uniform Eberlein compactum.

Let $T : C(K) \to l_\infty/c_0$ be an isomorphism (which exists by assumption). Then there exists a positive integer k such that $k > \|T\|\,\|T^{-1}\|$. Let $n \geq k$ be arbitrary.

As in the previous proof, we define an injection $\phi : \mathbb{R} \to C(K)$ by $\phi(r) = f_r$ where $f_r : K \to \mathbb{R}$ is defined by

$$f_r(x) = \begin{cases} x(r) & \text{if } x \in K_n(E_0), \\ 0 & \text{if } x \in K \setminus K_n(E_0). \end{cases}$$

Now, if $(x_1, \dots, x_n) \in E_0^{[n]}$ then $\chi_{\{x_1,\dots,x_n\}} \in K_n(E_0) \subseteq K$ so that $\|f_{x_1} + \cdots + f_{x_n}\| = n$. Hence

$$n = \|f_{x_1}+\cdots+f_{x_n}\| = \|T^{-1}T(f_{x_1}+\cdots+f_{x_n})\| \leq \|T^{-1}\|\,\|T(f_{x_1}+\cdots+f_{x_n})\|\ ,$$

so

$$\|T(f_{x_1} + \cdots + f_{x_n})\| \geq \frac{n}{\|T^{-1}\|} \geq \frac{k}{\|T^{-1}\|} > \|T\|.$$

If $(x_1, \dots, x_n) \in (E_0^c)^{[n]} \cap <^{[n]}$ then, for any $x \in K$, at most one function f_{x_i} has value 1 at x, so $\|f_{x_1} + \cdots + f_{x_n}\| = 1$. Hence

$$\|T(f_{x_1} + \cdots + f_{x_n})\| \leq \|T\|\,\|f_{x_1} + \cdots + f_{x_n}\| = \|T\|.$$

For an arbitrary injection $\psi : \ell_\infty/c_0 \to \ell_\infty$ we put $g = \psi \circ T \circ \phi$ and conclude that the set

$$\Big\{(x_1, \dots, x_n) \in (\ell_\infty)^n : \limsup_m |x_1(m) + \cdots + x_n(m)| > \|T\|\Big\}$$

separates $g^n[E_0^{[n]}]$ from $g^n[(E_0^c)^{[n]} \cap <^{[n]}]$ (recall that by g^n we mean the Cartesian product of n copies of g). Since it is Borel in the topology inherited from $(\mathbb{R}^\omega)^n$, the result follows from Proposition 3.3.

By symmetry the above argument also works for the relation

$$E_1 = E \cap > = \{(x, y) \in \mathbb{R}^2 : x > y \text{ and } (x, y) \in E\}.$$

We only need to change $<$ to $>$ in the definition of $K_n(E)$.

To get the result in full generality, observe that if n is sufficiently large then any sequence $(x_1, \dots, x_n) \in \mathbb{R}^n$ has a subsequence of length k which is strictly increasing or strictly decreasing. That is, if for any $A \in [n]^k$ (we identify here n with $\{1,\dots,n\}$) we put

$$F_0(A)=\{(x_1,\dots,x_n)\in\mathbb{R}^n:\forall i,j\in A\ [i<j\Rightarrow x_i<x_j]\},$$
$$F_1(A)=\{(x_1,\dots,x_n)\in\mathbb{R}^n:\forall i,j\in A\ [i<j\Rightarrow x_i>x_j]\},$$

then

$$\mathbb{R}^n=\bigcup_{A\in[n]^k}(F_0(A)\cup F_1(A)).$$

For $A\in[n]^k$ and $i\in\{0,1\}$ let $G_i(A)=E^{[n]}\cap F_i(A)$ and let $\pi_A:\mathbb{R}^n\to\mathbb{R}^A$ be the projection.

It is clear that we can identify $\pi_A(G_i(A))$ with $E_i^{[k]}$, so the first part of the proof yields the existence of a set $S_i(A)\in\mathcal{P}^k(\mathbb{R})$ which separates $\pi_A(G_i(A))$ from $\pi_A((E_i^c)^{[n]})$. It is not difficult to check that the set

$$\bigcup_{A\in[n]^k,\,i\in\{0,1\}}\pi_A^{-1}(S_i(A)),$$

which is clearly in $\mathcal{P}^n(\mathbb{R})$, separates $E^{[n]}$ from $(E^c)^{[n]}$. ∎

We should mention here that consistently the conclusion of 3.4 may not hold, i.e. consistently there is a set $E\subseteq\mathbb{R}^2$ such that, for every natural n, the set $E^{[n]}$ cannot be separated from $(E^c)^{[n]}$ by a member of $\mathcal{P}^n(\mathbb{R})$ (see [11, Remark 3.5]). Thus, from what we proved it follows that consistently there exists a uniform Eberlein compactum K such that $C(K)$ is not isomorphically embeddable into ℓ_∞/c_0.

We shall show now that consistently there is a uniform Eberlein compactum such that the space of continuous functions on it distinguishes isometric from isomorphic embeddings into ℓ_∞/c_0.

THEOREM 3.5. *If $\mathfrak{c}$ is not a Kunen cardinal then there exists a uniform Eberlein compact space K such that the space $C(K)$ embeds isomorphically but not isometrically into ℓ_∞/c_0.*

Proof. Since $\mathfrak{c}$ is not Kunen, there exists $E\subseteq\mathbb{R}^2$ which is not in the σ-field $\mathcal{P}^2(\mathbb{R})$. Without loss of generality, we can assume that $E\subseteq\{(a,b)\in\mathbb{R}^2:a<b\}$: see the proof of Theorem 3.2. From that proof it follows that $K=K_2(E)$ is a uniform Eberlein compact space such that $C(K)$ does not embed isometrically into ℓ_∞/c_0. On the other hand, $C(K)=C(K_2(E))$ always embeds isomorphically into ℓ_∞/c_0, since it is isomorphic to $c_0(\mathfrak{c})$ (see [8, Theorem 1.1]). ∎

4. Remarks. In this section we briefly discuss universality properties of ℓ_∞/c_0 for other two classes of compacta: continuous images of $\sigma_1(\mathfrak{c})^\omega$ and so called AD-compacta.

It was shown by Y. Benyamini, M. E. Rudin and M. Wage in [5] that a space K is uniform Eberlein compact of weight $\leq \kappa$ if and only if it is a continuous image of a closed subset of $\sigma_1(\kappa)^\omega$ (which is homeomorphic to $A(\kappa)^\omega$, where $A(\kappa)$ denotes the one-point compactification of a discrete space of size κ). In the same paper the authors asked whether we can replace a closed subset of $\sigma_1(\kappa)^\omega$ by $\sigma_1(\kappa)^\omega$ itself. This question was answered in the negative by M. Bell in [3]. He considered a space homeomorphic to the space

$$B = \{\chi_A \in \{0,1\}^{\omega_1} : A \in [\omega_1]^{\leq 2},\ \forall a, b \in A\ [a < b \text{ iff } a \prec b]\},$$

where $\prec$ is a well ordering on ω_1 ([3, Example 3.1]) and proved that it is not a continuous image of $\sigma_1(\omega_1)^\omega$ (see Example 4.2 below for a different space of this kind). It is not difficult to see that the above space is a continuous image of the space $K_2(\prec)$, where $\prec$ is a well ordering on $\mathbb{R}$. It was pointed out by S. Todorčević that consistently (in a model obtained by adding more than continuum many reals) the set

$$E = \{(a,b) \in [0,1]^2 : a < b \text{ iff } a \prec b\},$$

where $\prec$ is a well ordering on $[0,1]$, is not in the σ-field $\mathcal{P}^2(\mathbb{R})$ (see [11, Remark 3.5]). Thus from Theorem 3.2 it follows that (in this model) the space $C(K_2(\prec))$ does not embed isometrically into ℓ_∞/c_0. Clearly, the construction of the space $K_2(\prec)$ is very similar to the construction of the above mentioned example B of M. Bell (a well ordering on ω_1 is replaced by a well ordering on $\mathbb{R}$). However, let us point out that B is a continuous image of $\beta\omega \setminus \omega$ and therefore $C(B)$ embeds isometrically into ℓ_∞/c_0 (in ZFC).

It turns out that ℓ_∞/c_0 is isometrically universal for the class of continuous images of $\sigma_1(\mathfrak{c})^\omega$. So we can distinguish the class of uniform Eberlein compacta of weight at most $\mathfrak{c}$ from the class of continuous images of $\sigma_1(\mathfrak{c})^\omega$ in terms of universality properties of ℓ_∞/c_0.

Proposition 4.1. *If K is a continuous image of $\sigma_1(\mathfrak{c})^\omega$, then the space $C(K)$ embeds isometrically into ℓ_∞/c_0.*

Proof. $\sigma_1(\mathfrak{c})^\omega$ is a continuous image of $\beta\omega \setminus \omega$ (see [4, Theorem 2.5 and Example 5.3]) and since $C(\beta\omega\setminus\omega)$ is isometric to ℓ_∞/c_0, the result follows. ∎

Example 4.2. A simple counterexample to the question of Y. Benyamini, M. E. Rudin and M. Wage mentioned at the beginning of this section is the Alexandroff double of the Cantor set $C \subset [0,1]$, which we will denote by D. Since, by a result of J. Gerlits from [7], the character and the weight coincide for continuous images of $\sigma_1(\kappa)^\omega$, we conclude that D is not such an image. It is however a uniform Eberlein compactum. Indeed, for $x \in C$ and

$i = 0, 1$, let $\mathit{\Gamma} = C \cup \{2\}$ and define $f_{x,i} : \mathit{\Gamma} \to [0,1]$ by

$$f_{x,i}(\gamma) = \begin{cases} x & \text{if } \gamma = 2, \\ 0 & \text{if } \gamma \in C,\ \gamma \neq x, \\ 1 & \text{if } \gamma = x,\ i = 1, \\ 0 & \text{if } \gamma = x,\ i = 0. \end{cases}$$

One can easily verify that the space $\{f_{x,i} : x \in C,\ i = 0,1\}$ considered as a subspace of the product $[0,1]^{\mathit{\Gamma}}$ is homeomorphic to the space D (the functions $f_{x,0}$ correspond to nonisolated points of D and the functions $f_{x,1}$ correspond to the isolated ones). Thus D is a uniform Eberlein compactum.

The first countable space D has essentially different properties from the example B of M. Bell (and from $K_2(\prec)$). One can easily verify that the algebra of clopen subsets of D can be embedded into the algebra $P(\omega)/\mathrm{Fin}$, hence D is a continuous image of $\beta\omega\backslash\omega$ and $C(D)$ embeds isometrically into ℓ_∞/c_0.

The second author was informed about this example by M. Bell, who unfortunately never published it. As far as we know, in this context, it has never appeared in the literature before.

We should note that the space D was used, in a different context, by G. Plebanek in [10] to distinguish the class of Eberlein compacta from the class of AD-compacta (the definition is given below). He noted that D is not a continuous image of $\sigma_1(\kappa)^\omega$ and is an Eberlein compactum. He was not aware however of the question of Y. Benyamini, M. E. Rudin and M. Wage.

Another interesting class of compacta is the class of AD-compacta (see [10], [2]). Given a nonempty set X we say that a family $\mathcal{A}$ of its subsets is *adequate* if it satisfies the following two conditions:

(i) if $A \in \mathcal{A}$ and $B \subseteq A$ then $B \in \mathcal{A}$, and
(ii) if $A \subseteq X$ and every finite subset of A is in $\mathcal{A}$, then $A \in \mathcal{A}$.

Of course, we can associate in a natural way (identifying a set with its characteristic function) a family $\mathcal{A}$ with a space $K(\mathcal{A}) \subseteq \{0,1\}^X$. It is not difficult to check that $K(\mathcal{A})$ is a compact subspace of $\{0,1\}^X$ whenever $\mathcal{A}$ is adequate. We say that a compact space K is *adequate* if K is homeomorphic to $K(\mathcal{A})$ for some adequate family $\mathcal{A}$. We say that a space K is *AD-compact* if it is a continuous image of an adequate compactum. We refer to [10], [2] for the basic properties of AD-compacta.

M. Bell observed in [2] that consistently there exists an AD-compactum of weight $\leq \mathfrak{c}$ which is not a continuous image of $\beta\omega \setminus \omega$. From our previous considerations we can conclude more:

COROLLARY 4.3. *Suppose that, for every AD-compact space K of weight at most $\mathfrak{c}$, the space $C(K)$ embeds isometrically into ℓ_∞/c_0. Then $\mathfrak{c}$ is a Kunen cardinal.*

Proof. The space $K_2(E_0)$ considered in the proof of 3.2 is adequate compact. ∎

COROLLARY 4.4. *Suppose that, for every AD-compact space K of weight at most $\mathfrak{c}$, the space $C(K)$ embeds isomorphically into ℓ_∞/c_0. Then for every binary relation $E \subseteq \mathbb{R}^2$ and for all but finitely many positive integers n, the set $E^{[n]}$ can be separated from $(E^c)^{[n]}$ by a member of $\mathcal{P}^n(\mathbb{R}^2)$.*

Proof. Since for any set $E \subseteq \mathbb{R}$ the space $K_n(E)$ is adequate, it is AD-compact and by Theorem 2.1 in [10] the space $K = \bigoplus_{n=1}^\infty K_n(E_0) \cup \{\infty\}$ considered in the proof of 3.4 is also AD-compact. ∎

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