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NOTE ON BLOW-UP OF SOLUTIONS FOR A POROUS MEDIUM EQUATION WITH CONVECTION AND BOUNDARY FLUX

BY

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Abstract. De Pablo et al. [Proc. Roy. Soc. Edinburgh Sect. A 138 (2008), 513–530] considered a nonlinear boundary value problem for a porous medium equation with a convection term, and they classified exponents of nonlinearities which lead either to the global-in-time existence of solutions or to a blow-up of solutions. In their analysis they left open the case of a certain critical range of exponents. The purpose of this note is to fill this gap.

1. Introduction. De Pablo et al. [3] studied the initial-boundary value problem

$$\begin{cases} u_t = (u^m)_{xx} + \lambda (u^q)_x, & x \in (0,\infty),\ t \in (0,T), \\ -(u^m)_x(0,t) = u^p(0,t), & t \in (0,T), \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{1.1}$$

where u_0 is a continuous, non-negative, non-increasing and integrable function, and T is the maximal existence time which may be either finite or infinite. They established an almost complete characterization, in terms of the parameters $m \geq 1$, p, $q > 0$ and $\lambda > 0$, whether all solutions are global-in-time or there exist solutions which blow up in finite time. In particular, it is shown in [3] that for $p \neq q$ there exist both non-trivial global solutions and blowing up solutions in the range $p > \max\{q, \frac{1}{2}(m+1)\}$ while all solutions are global in the complementary range of p. If $p = q$, the parameter λ plays a role in this analysis and it has a critical value $\lambda_c = 1$: in the case $\lambda > 1$ all solutions are global while for $\lambda < 1$ all non-trivial solutions blow up if $\frac{1}{2}(m+1) < p \leq m+1$, and there are non-trivial global solutions and blowing up solutions if $p > m+1$. Finally, it is proved in [3] that in the case $\lambda = 1$, all solutions are global in the range $p = q \leq m+1$, and there are small global solutions when $p = q > m+1$.

As stated in [3, Remark 3.4], the question of blow-up of solutions in the range $p = q > m+1$ and $\lambda = 1$ was left open. Moreover, as pointed out in [3], there might not exist blowing up subsolutions of self-similar type.

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The aim of this note is to solve this problem, which together with the result from [3] provides a complete characterization of exponents $\lambda > 0$, $m \geq 1$ and $p, q > 0$ for which problem (1.1) has either global-in-time solutions or blowing up solutions. The following theorem contains the main result of this note.

Theorem. *For problem* (1.1) *in the case* $\lambda = 1$, $m \geq 1$ *and* $p = q > m+1$, *there exists a smooth, positive, non-increasing and integrable function* u_0 *such that the corresponding solution* u *of problem* (1.1) *blows up in finite time.*

To prove this theorem, we will employ ideas of Alikakos et al. [1] and of Fisher and Grant [4]. In our proof, the key step is to find a function whose mass is "large enough" in a neighborhood of $x = 0$ and such that the mass of the corresponding solution remains sufficiently large on some time interval in the same region. Moreover, we shall use the sub- and super-solution approach from [3], which is based on an important observation that the comparison principle for solutions to problem (1.1) is valid as long as the initial data are strictly ordered at $x = 0$ (see [3]).

2. Proof of Theorem. In the following we always assume that $\lambda = 1$, $m \geq 1$ and $p = q > m + 1$. First, we prove a lemma which provides a rough control of mass in the interval $[0, 1]$ of a certain large solution.

Lemma 2.1. *For fixed* $M > 0$ *and* $T > 0$, *there exists a positive smooth function* $\underline{u} \in C(0, T; L^1(\mathbb{R}^+))$ *such that* $\underline{u}(x, t)$ *is a subsolution of problem* (1.1) *satisfying*

$$\int_0^1 \underline{u}(x,t)\,dx \geq M \quad \textit{for } t \in (0,T). \tag{2.1}$$

Proof. Consider the function

$$\underline{u}(x,t) = \frac{c}{t+1} e^{-\varepsilon x}, \qquad (x,t) \in [0,\infty) \times [0,T), \tag{2.2}$$

where $c, \varepsilon > 0$ are parameters to be determined. Assuming that $\varepsilon \leq 1$ and choosing $c = (T + 1)Me$, we have

$$\int_0^1 \underline{u}(x,t)\,dx \geq \frac{c}{T+1} \int_0^1 e^{-1}\,dx = \frac{ce^{-1}}{T+1} \geq M.$$

Recall that a function $\underline{u}$ is a subsolution of problem (1.1) if it satisfies

$$\begin{cases} \underline{u}_t \leq (\underline{u}^m)_{xx} + (\underline{u}^p)_x, & (x,t) \in (0,\infty) \times (0,T), \\ (\underline{u}^m)_x(0,t) + \underline{u}^p(0,t) \geq 0, & t \in (0,T). \end{cases}$$

Inserting the function $\underline{u}$ from (2.2) into those inequalities, we obtain

$$-\frac{c}{(t+1)^2}e^{-\varepsilon x} \le \frac{c^m}{(t+1)^m}(m\varepsilon)^2 e^{-m\varepsilon x} - \frac{c^p}{(t+1)^p}p\varepsilon e^{-p\varepsilon x}, \tag{2.3}$$

$$-\frac{m\varepsilon c^m}{(t+1)^m} + \frac{c^p}{(t+1)^p} \ge 0. \tag{2.4}$$

Notice that $p > m+1 \ge 2$. By choosing $\varepsilon \le \frac{1}{pc^{p-1}}$, we have

$$-\frac{1}{(t+1)^2}ce^{-\varepsilon x} \le -\frac{1}{(t+1)^2}(c^p p\varepsilon)e^{-\varepsilon x} \le -\frac{c^p}{(t+1)^p}p\varepsilon e^{-p\varepsilon x}. \tag{2.5}$$

Since $\frac{c^m}{(t+1)^m}(m\varepsilon)^2 e^{-m\varepsilon x} > 0$, inequality (2.3) results immediately from (2.5).

On the other hand, we can also choose $\varepsilon \le \frac{c^{p-m}}{m(T+1)^{p-m}}$ such that

$$\frac{c^m m\varepsilon}{c^p} \le \frac{1}{(T+1)^{p-m}} \le \frac{1}{(t+1)^{p-m}},$$

which leads immediately to inequality (2.4).

To summarize the above arguments, if we take $c = (T+1)Me$ and

$$\varepsilon = \min\left\{1, \frac{1}{pc^{p-1}}, \frac{c^{p-m}}{m(T+1)^{p-m}}\right\},$$

then the function $\underline{u}$ of (2.2) is the desired subsolution of problem (1.1). ■

With this lemma, we are now in a position to prove the main theorem.

Proof of Theorem. Consider the solution of the equation

$$(u^m)_{xx} + (u^p)_x = 0 \quad \text{for } 0 < x < \infty$$

given by the explicit formula $\varphi(x) = \left(\frac{p-m}{m}x\right)^{\frac{1}{m-p}}$. Notice that $\varphi(x)$ blows up at $x = 0$.

Denote

$$M_c = \int_0^1 \varphi(x)\,dx = \left(\frac{p-m}{m}\right)^{\frac{1}{m-p}} \frac{p-m}{p-m-1}.$$

For $M = 2M_c$ and for fixed $\gamma \in (1/2, 1)$, we choose T so large that

$$\frac{1}{T} \le \left(\frac{1}{\gamma^{p-1}} - \frac{1}{\gamma^{m-1}}\right)\varphi^{p-1}(1). \tag{2.6}$$

For such M and T, according to Lemma 2.1, we have a positive smooth subsolution $\underline{u}$ of problem (1.1) on $(0,\infty)\times(0,T)$ which satisfies

$$\int_0^1 \underline{u}(x,t)\,dx \ge M \quad \text{for } t \in (0,T).$$

We claim that a solution u corresponding to a smooth and integrable initial datum $u_0(x) > \underline{u}(x,0)$ with $(u_0^m)_x(0) + (u_0^p)(0) = 0$ blows up in finite time. Assume on the contrary that $u(x,t)$ is global (for a local existence result, cf. [2]). By the comparison principle for problem (1.1), we have

$$u(x,t) \geq \underline{u}(x,t) = \frac{c}{t+1}e^{-\varepsilon x} \quad \text{for } (x,t) \in (0,\infty) \times (0,T),$$

where c and ε were determined in the proof of Lemma 2.1. Since the solution u has a positive lower bound, it is a classical solution of problem (1.1) by [5].

Now let

$$\Delta = \{(x,t) \in (0,1) \times (0,T) \mid x - (1 - t/T) > 0\}.$$

Define the functions $z, v : \overline{\Delta} \to \mathbb{R}$ by

$$z(x,t) = \int_0^{x-(1-t/T)} \frac{1}{\gamma}\varphi(s)\,ds,$$

$$v(x,t) = \int_0^x u(s,t)\,ds.$$

Furthermore, we define an auxiliary function $w : \overline{\Delta} \to \mathbb{R}$ by the formula

$$w(x,t) = e^{-t}(z(x,t) - v(x,t)).$$

Since u is a classical solution of problem (1.1), we see $w \in C^2(\Delta) \cap C(\overline{\Delta})$.

A direct computation yields

$$w(x,t) = e^{-t}(0 - v(x,t)) \leq 0 \quad \text{for } x - (1 - t/T) = 0.$$

Using inequality (2.1) and the comparison principle, we have

$$\begin{aligned} w(1,t) &= e^{-t}\left(\frac{1}{\gamma}\int_0^{t/T} \varphi(s)\,ds - \int_0^1 u(x,t)\,dx\right) \\ &\leq e^{-t}\left(\frac{1}{\gamma}M_c - \int_0^1 \underline{u}(x,t)\,dx\right) \\ &\leq e^{-t}\left(\frac{1}{\gamma}M_c - M\right) \leq 0, \quad 0 \leq t \leq T. \end{aligned}$$

Since $\lim_{x\to 0^+}\varphi(x) = +\infty$, using L'Hospital's rule we derive

$$w_x(0,T) = \lim_{x\to 0^+}\frac{w(x,T) - w(0,T)}{x} = \lim_{x\to 0^+} e^{-T}\left(\frac{1}{\gamma}\varphi(x) - u(x,T)\right) = +\infty.$$

So $w \leq 0$ in $\partial\Delta \setminus \{(x,T) \mid 0 < x < 1\}$. In addition, from the conditions $w(0,T) = 0$ and $w_x(0,T) = +\infty$, we see that $w(x,t)$ attains its positive maximum in the set $\Delta \cup \{(x,T) \mid 0 < x < 1\}$ at some point (x_0, t_0). This implies that at (x_0, t_0), the following relations hold true:

$$w(x_0, t_0) > 0, \tag{2.7}$$

$$w_t(x_0, t_0) \geq 0, \tag{2.8}$$

$$w_x(x_0, t_0) = 0, \tag{2.9}$$

$$w_{xx}(x_0, t_0) \leq 0. \tag{2.10}$$

Next, we show that for our choice of γ and T we have $w_t(x_0, t_0) < 0$, which contradicts (2.8). Indeed, by (2.9), we have $z_x(x_0, t_0) = v_x(x_0, t_0)$. Thus $\frac{1}{\gamma}\varphi(\xi) = u(x_0, t_0)$, where $\xi = x_0 - (1 - t_0/T)$. Therefore

$$\begin{aligned} v_t(x_0, t_0) &= \int_0^{x_0} u_t(x, t_0)\, dx = \int_0^{x_0} [(u^m)_{xx}(x, t_0) + (u^p)_x(x, t_0)]\, dx \\ &= (u^m)_x(x_0, t_0) + u^p(x_0, t_0) \\ &= m\left(\frac{1}{\gamma}\varphi(\xi)\right)^{m-1} u_x(x_0, t_0) + \left(\frac{1}{\gamma}\varphi(\xi)\right)^p. \end{aligned}$$

On the other hand, it follows from (2.10) that $\frac{1}{\gamma}\varphi'(\xi) - u_x(x_0, t_0) \leq 0$, which implies that

$$v_t(x_0, t_0) \geq m\left(\frac{1}{\gamma}\varphi\right)^{m-1} \frac{1}{\gamma}\varphi'(\xi) + \left(\frac{1}{\gamma}\varphi\right)^p(\xi).$$

Recalling that φ satisfies $(\varphi^m)' + \varphi^p = 0$, we deduce

$$v_t(x_0, t_0) \geq \left(\frac{1}{\gamma^p} - \frac{1}{\gamma^m}\right)\varphi^p(\xi),$$

which gives

$$\begin{aligned} z_t(x_0, t_0) - v_t(x_0, t_0) &\leq \frac{1}{\gamma T}\varphi(\xi) - \left(\frac{1}{\gamma^p} - \frac{1}{\gamma^m}\right)\varphi^p(\xi) \\ &= \frac{\varphi(\xi)}{\gamma}\left(\frac{1}{T} - \left(\frac{1}{\gamma^{p-1}} - \frac{1}{\gamma^{m-1}}\right)\varphi^{p-1}(\xi)\right). \end{aligned}$$

Since φ is decreasing, and since $1/\gamma > 1$ and $p > m + 1 \geq 2$, we have

$$z_t(x_0, t_0) - v_t(x_0, t_0) \leq \frac{\varphi(\xi)}{\gamma}\left(\frac{1}{T} - \left(\frac{1}{\gamma^{p-1}} - \frac{1}{\gamma^{m-1}}\right)\varphi^{p-1}(1)\right).$$

Consequently, by the choice of T in (2.6), we obtain

$$z_t(x_0, t_0) - v_t(x_0, t_0) \leq 0. \tag{2.11}$$

Moreover, by inequality (2.7), we have

$$\begin{aligned} w_t(x_0, t_0) &= -w(x_0, t_0) + e^{-t_0}(z_t(x_0, t_0) - v_t(x_0, t_0)) \\ &< e^{-t_0}(z_t(x_0, t_0) - v_t(x_0, t_0)). \end{aligned}$$

This estimate together with (2.11) leads to the inequality $w_t(x_0, t_0) < 0$, which contradicts (2.8).

This contradiction implies that the solution $u(x,t)$, which was assumed to be global, has to blow up in finite time, no later than at time

$$\varphi^{1-p}(1)/\left(\frac{1}{\gamma^{p-1}} - \frac{1}{\gamma^{m-1}}\right) < T$$

(cf. (2.6)). ∎

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