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A SHARP BOUND FOR THE SCHWARZIAN DERIVATIVE OF CONCAVE FUNCTIONS

BY

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Abstract. We derive a sharp bound for the modulus of the Schwarzian derivative of concave univalent functions with opening angle at infinity less than or equal to $\pi\alpha$, $\alpha \in [1, 2]$.

1. Introduction. Let $\mathbb{C}$ be the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Let f be an analytic and locally univalent function in $\mathbb{D}$. For such functions f, the Schwarzian derivative and its norm are defined by

$$S_f(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

and

$$\|S_f\| := \sup_{z\in\mathbb{D}} (1 - |z|^2)^2 |S_f(z)|.$$

These quantities are of importance in the theory of Teichmüller spaces. The fundamental results on the Schwarzian derivative can be found in the works of Z. Nehari (see [N1] and [N2]), R. Kühnau (see [K]), and L. V. Ahlfors and G. Weill (see [AW-1]). We also refer to the articles [CDO, CDMMBO, KS] for recent developments in this area of research. We summarize the work of Nehari, Kühnau and Ahlfors–Weill below:

Theorem A. *Let f be analytic and locally univalent in $\mathbb{D}$. If f is univalent in $\mathbb{D}$ then $\|S_f\| \le 6$; conversely, if $\|S_f\| \le 2$, then f is univalent. Let $0 \le k < 1$. If f extends to a k-quasiconformal mapping of the Riemann sphere $\overline{\mathbb{C}}$, then $\|S_f\| \le 6k$. Conversely, if $\|S_f\| \le 2k$, then f extends to a k-quasiconformal mapping of $\overline{\mathbb{C}}$.*

We clarify here that a mapping $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called *k-quasiconformal* if f is a sense preserving homeomorphism of $\overline{\mathbb{C}}$ and has locally integrable partial derivatives on $\mathbb{C} \setminus \{f^{-1}(\infty)\}$, with $|f_{\overline{z}}| \le k|f_z|$ a.e.

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Now, we define the class of functions which is our main concern in this article. A function f is said to be *concave with opening angle at infinity less than or equal to* $\pi\alpha$, $\alpha \in [1,2]$, if it satisfies the following conditions:

(i) f is analytic and univalent in $\mathbb{D}$.
(ii) f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex and satisfies $f(0) = 0 = f'(0) - 1$ and $f(1) = \infty$.
(iii) The opening angle of $f(\mathbb{D})$ at infinity is less than or equal to $\pi\alpha$, $\alpha \in [1,2]$.

We denote this class by $\mathrm{Co}(\alpha)$. Various results on $\mathrm{Co}(\alpha)$ can be found in [AW-2], [B], [BPW] and [W]. In [AW-2] and [W], the following characterization for functions in $\mathrm{Co}(\alpha)$ was proved:

THEOREM B. *A function f belongs to* $\mathrm{Co}(\alpha)$ *if and only if* $f(0) = f'(0) - 1 = 0$ *and there exists a holomorphic function* $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$ *such that*

$$\frac{f''(z)}{f'(z)} = \frac{\alpha+1}{1-z} + \frac{(\alpha-1)\varphi(z)}{1+z\varphi(z)}, \qquad z \in \mathbb{D}. \tag{1.1}$$

In this note, our main aim is to find a sharp bound for the modulus of the Schwarzian derivative for functions in $\mathrm{Co}(\alpha)$. This result will yield a sharp norm estimate for the Schwarzian derivative of concave mappings, which will help us to comment on quasiconformal extension and get a pair of two-point distortion conditions of such mappings. These are the contents of Section 2.

2. Results.

The main result of this article is the following theorem:

MAIN THEOREM 2.1. *Let* $\alpha \in [1,2]$, $f \in \mathrm{Co}(\alpha)$, *and* $z \in \mathbb{D}$. *Then*

$$\left| \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \right| \le \frac{2(\alpha^2-1)}{(1-|z|^2)^2}. \tag{2.1}$$

Equality is attained in (2.1) *if and only if*

$$f(\zeta) = \int_0^\zeta \frac{(1+te^{i\theta_0})^{\alpha-1}}{(1-t)^{\alpha+1}}\,dt, \qquad \zeta \in \mathbb{D},$$

where

$$e^{i\theta_0} = \frac{1-2\overline{z}+|z|^2}{1-2z+|z|^2}.$$

Proof. A little computation using the representation formula (1.1) in Theorem B yields

$$\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$
$$= (\alpha-1)\left(\frac{\varphi'(z)}{(1+z\varphi(z))^2} - \frac{1}{2}(\alpha+1)\,\frac{(1+\varphi(z))^2}{(1-z)^2(1+z\varphi(z))^2}\right)$$
$$=: (\alpha-1)M_1(z,\varphi).$$

In the further considerations, we exclude the trivial case $\alpha = 1$. Since it is known (see [C]) that for a holomorphic function $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$ we have

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D},$$

we get an upper bound for $|M_1(z,\varphi)|$ if we assume that the first term in the sum defining $M_1(z,\varphi)$ has the same argument as the second one:

$$|M_1(z,\varphi)| \le \left|\frac{(1+\varphi(z))^2}{(1-z)^2}\left(\frac{1-|\varphi(z)|^2}{1-|z|^2}\,\frac{|1-z|^2}{|1+\varphi(z)|^2} + \frac{1}{2}(\alpha+1)\right)\right|\frac{1}{|1+z\varphi(z)|^2}$$
$$= \left(\frac{1-|\varphi(z)|^2}{1-|z|^2} + \frac{1}{2}(\alpha+1)\frac{|1+\varphi(z)|^2}{|1-z|^2}\right)\frac{1}{|1+z\varphi(z)|^2} =: M_2(z,\varphi).$$

Our aim is to prove that, for fixed $z \in \mathbb{D}$ and $|\varphi(z)| \le 1$,

$$M_2(z,\varphi) \le \frac{2(\alpha+1)}{(1-|z|^2)^2}. \tag{2.2}$$

Further we will show that equality occurs in (2.2) if and only if

$$\varphi(z) = \frac{1-2\overline{z}+|z|^2}{1-2z+|z|^2} = e^{i\theta_0}, \tag{2.3}$$

where $\theta_0 \in [0, 2\pi)$. This will imply $\varphi(z) = e^{i\theta_0}$ for all $z \in \mathbb{D}$, due to the maximum principle for analytic functions. We divide the proof of these claims into two parts.

Part A. First, for fixed $z \in \mathbb{D}$, we consider the image under $M_2(z,\varphi)$ of the circle $\{\varphi : |\varphi| = 1\}$. To this end, let

$$w = \frac{1+\varphi}{1+z\varphi}.$$

It is easily seen that $|\varphi| = 1$ is equivalent to

$$\left|w - \frac{1-\overline{z}}{1-|z|^2}\right| = \frac{|1-z|}{1-|z|^2}.$$

This implies that

$$\left|\frac{w}{1-z}\right| \le \frac{2}{1-|z|^2}, \tag{2.4}$$

where equality occurs if and only if

$$w = \frac{2(1-\overline{z})}{1-|z|^2},$$

or

$$\varphi = \frac{1-2\overline{z}+|z|^2}{1-2z+|z|^2}.$$

Hence, for fixed $z \in \mathbb{D}$ and $|\varphi| = 1$, the inequality (2.4) will imply the validity of (2.2), and equality is attained in (2.2) if and only if $\varphi = e^{i\theta_0}$, where θ_0 is given by the equation (2.3).

PART B. Now, for fixed $z \in \mathbb{D}$, we consider the curve in the φ-plane which is defined by

$$M_2(z,\varphi) = \frac{2(\alpha+1)}{(1-|z|^2)^2}. \tag{2.5}$$

Hereafter, we use the abbreviation $a = \frac{1}{2}(\alpha+1) \in \left(1, \frac{3}{2}\right]$. A little computation reveals that (2.5) is equivalent to

$$B\varphi\overline{\varphi} + C\varphi + \overline{C}\overline{\varphi} + D = 0,$$

where

$$\begin{aligned} B &= a(1-|z|^2)^2 - 4a|z|^2|1-z|^2 - |1-z|^2(1-|z|^2), \\ C &= a(1-|z|^2)^2 - 4az|1-z|^2, \\ D &= |1-z|^2(1-|z|^2) + a(1-|z|^2)^2 - 4a|1-z|^2. \end{aligned}$$

We wish to analyze the set in the φ-plane described by (2.5). To this end, first we claim that $C \neq 0$; indeed, if $C = 0$, then either

$$z = r \quad \text{and} \quad a(1-r^2)^2 - 4ar(1-r)^2 = a(1-r)^4 = 0,$$

or

$$z = -r \quad \text{and} \quad a(1-r^2)^2 + 4ar(1+r)^2 = a(1+r)^4 = 0.$$

We see that both are impossible. This proves $C \neq 0$. Next, we consider the following two cases:

CASE (i): $B = 0$, which is equivalent to

$$|1-z|^2 = \frac{a(1-|z|^2)^2}{4a|z|^2 + 1 - |z|^2} =: R^2. \tag{2.6}$$

Since this equation describes the circle with center 1 and radius R, we have to decide whether it is possible that for fixed $|z| = r \in [0,1)$, the inequalities $-r \leq 1 - R \leq r$ are satisfied. They imply $(1-r)^2 \leq R^2 \leq (1+r)^2$ and we see that the left inequality is always true for $r \in [0,1)$, whereas the right one is satisfied for $r \in \left[\frac{a-1}{3a-1}, 1\right)$. Hence, the equation $B = 0$ is valid for the intersection points of the circle $\{z : |z| = r\}$ with the circle given by (2.6). So for $B = 0$, the equation (2.5) represents a straight line that divides the plane into two open half-planes. According to Part A, the closed disc $\{\varphi : |\varphi| \le 1\}$ lies in the closed half-plane

$$M_2(z, \varphi) \le \frac{2(\alpha+1)}{(1-|z|^2)^2}$$

and the straight line defined by formula (2.5) has only the point $\varphi = e^{i\theta_0}$ in common with the unit circle $\{\varphi : |\varphi| = 1\}$. This proves the assertion of the theorem for $B = 0$.

Case (ii): $B \neq 0$. Here the equation (2.5) represents a circle if and only if $C\overline{C} - BD > 0$. A straightforward computation yields

$$C\overline{C} - BD = (2a-1)^2(1-|z|^2)^2|1-z|^4,$$

which is always > 0. Hence, whenever $B \neq 0$, (2.5) is the equation of a circle, which divides the φ-plane into the corresponding inner and outer domains. Again, according to Part A, the closed disc $\{\varphi : |\varphi| \le 1\}$ lies in the region defined by

$$M_2(z, \varphi) \le \frac{2(\alpha+1)}{(1-|z|^2)^2},$$

and the only intersection point of the circle (2.5) with the unit circle $\{\varphi : |\varphi| = 1\}$ is the point $\varphi = e^{i\theta_0}$. This proves the assertion of the theorem for $B \neq 0$.

To get the extremal function as given in the theorem, we only have to integrate the differential equation (1.1). ■

Remark. We note that $\|S_f\| \le 2(\alpha^2 - 1)$ for $f \in \mathrm{Co}(\alpha)$. In the case of $\alpha = 2$, $\|S_f\| \le 6$, which is the bound obtained by Nehari for the norm of the Schwarzian derivative for univalent functions. This is a natural consequence of the fact that the Koebe function, which is extremal in that problem, belongs to the class Co(2).

Corollary 2.2. *Let $\alpha \in [1, \sqrt{2})$, and $f \in \mathrm{Co}(\alpha)$. Then f extends to an (α^2-1)-quasiconformal mapping.*

Proof. As $\alpha \in [1, \sqrt{2})$, for $f \in \mathrm{Co}(\alpha)$ we have $\|S_f\| \le 2(\alpha^2 - 1) =: 2k$, $k \in [0, 1)$. Now an application of Theorem A proves the corollary. ■

For, $z_1, z_2 \in \mathbb{D}$, let the hyperbolic metric $d(z_1, z_2)$ be defined by

$$d(z_1, z_2) = \frac{1}{2}\log\frac{1+\rho(z_1,z_2)}{1-\rho(z_1,z_2)}, \quad \text{where } \rho(z_1, z_2) = \left|\frac{z_1 - z_2}{1-\overline{z_1}z_2}\right|.$$

We also define the following quantity for an analytic and locally univalent function f in $\mathbb{D}$:

$$\Delta_f(z_1,z_2) := \frac{|f(z_1)-f(z_2)|}{\{(1-|z_1|^2)|f'(z_1)|\}^{1/2}\{(1-|z_2|^2)|f'(z_2)|\}^{1/2}}, \qquad z_1,z_2\in\mathbb{D}.$$

Now, in view of the above theorem and [CDMMBO, Theorem 1], we obtain a pair of two-point distortion conditions for functions in $\mathrm{Co}(\alpha)$ for a certain range of α:

COROLLARY 2.3. *Let* $\alpha\in(\sqrt{2},2]$ *and* $f\in\mathrm{Co}(\alpha)$. *Then*

$$\Delta_f(z_1,z_2) \ge \frac{1}{\sqrt{\alpha^2-2}}\sin(\sqrt{\alpha^2-2}\,d(z_1,z_2)) \tag{2.7}$$

for all $z_1,z_2\in\mathbb{D}$ *with* $d(z_1,z_2)\le \pi/\sqrt{\alpha^2-2}$, *and*

$$\Delta_f(z_1,z_2) \le \frac{1}{\alpha}\sinh(\alpha\, d(z_1,z_2)) \tag{2.8}$$

for all $z_1,z_2\in\mathbb{D}$. *Both inequalities are sharp.*

Proof. Since $f\in\mathrm{Co}(\alpha)$, $\alpha\in(\sqrt{2},2]$, by Theorem 2.1 we have

$$\|S_f\|\le 2(1+\delta^2), \quad \text{where} \quad \delta^2=\alpha^2-2>0.$$

Now, the corollary follows as an application of [CDMMBO, Theorem 1], with $\delta^2=\alpha^2-2$. ∎

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