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THE ESSENTIAL COVER AND THE ABSOLUTE COVER OF A SCHEMATIC SPACE

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Dedicated to B. V. M.

Abstract. A theorem of Gleason states that every compact space admits a projective cover. More generally, in the category of topological spaces with continuous maps, covers exist with respect to the full subcategory of extremally disconnected spaces. Such a cover of a space is called its *absolute*. We prove that the absolute exists within the category of *schematic* spaces, i.e. the spaces underlying a scheme. For a schematic space, we use the absolute to generalize Bourbaki's concept of irreducible component, so that embedded and multiple components may arise. We introduce the *essential cover* of a schematic space, and show that it parametrizes the generalized components.

Introduction. Spectral spaces arise as spectra of commutative rings or abelian l-groups [17, 10]. By Stone's duality theorem [26], the spectrum of a bounded distributive lattice is also a spectral space. Hochster [17] has shown that every spectral space occurs as the spectrum of a commutative ring. By definition, a T_0-space X is said to be *spectral* if every closed irreducible set is generic, and the quasi-compact open sets form a basis $\mathcal{D}(X)$ of X which is closed under finite intersections. In particular, the empty intersection, i.e. X itself, is quasi-compact. If we drop the assumption that $X \in \mathcal{D}(X)$, we get precisely the class of spaces that arise as underlying spaces of schemes [17]. Therefore, we call such spaces *schematic.*

The morphisms in the category **GSp** of schematic spaces are *spectral* maps $f\colon X \to Y$, i.e. those for which $\mathcal{D}(f) := f^{-1}$ maps $\mathcal{D}(Y)$ into $\mathcal{D}(X)$. By Stone's duality theorem [26], this gives a duality $\mathcal{D}$ between **GSp** and the category $\mathbf{DL}_0$ of distributive lattices with 0. We call $f\colon X \to Y$ *dense* if $f(X)$ is dense in Y, and *essentially dense* if, in addition, $g\colon Z \to X$ is dense whenever fg is dense.

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In this paper, we prove that any schematic space X admits an essentially dense spectral map $e\colon \widetilde{X} \to X$ which factors through every essentially dense spectral map $Y \to X$ such that every essentially dense spectral map $Z \to \widetilde{X}$ is a retraction. We show that such an *essential cover* e is unique, up to isomorphism (Theorem 3).

Our motivation comes from the theory of abelian *l-groups*, i.e. abelian groups with a lattice structure such that the translations $x \mapsto a + x$ are lattice automorphisms. To allow precise statements, let us recall first some pertinent facts on abelian l-groups. For details, the reader is referred to books like [1, 3, 13, 15].

The abelian l-groups form a category $\mathbf{Ab}_l$, morphisms being the group homomorphisms which are also lattice homomorphisms. A subgroup A of an abelian l-group G is said to be an *l-ideal* if $|x| \le |a|$ with $a \in A$ implies that $x \in A$, where $|a| := a \vee (-a)$. The set of l-ideals of G is denoted by $\mathcal{C}(G)$. For any $A \in \mathcal{C}(G)$, there is a largest $B \in \mathcal{C}(G)$ with $A \cap B = 0$, the *polar* $A^{\perp}$ of A. An l-ideal P is called *prime* if the abelian l-group G/P is totally ordered, or equivalently, if $P = A \cap B$ with $A, B \in \mathcal{C}(G)$ implies that $P = A$ or $P = B$. The set $\operatorname{Spec} G$ of proper prime l-ideals is a schematic space with $\mathcal{D}(\operatorname{Spec} G) = \{S(a) \mid a \in G\}$, where $S(a) := S(\{a\})$, and

$$S(A) := \{P \in \operatorname{Spec} G \mid A \not\subset P\}$$

for any subset $A \subset G$. By [20, Proposition 1.19], the map $A \mapsto S(A)$ defines a lattice isomorphism

$$\mathcal{C}(G) \xrightarrow{\sim} \mathcal{O}(\operatorname{Spec} G)$$

onto the lattice of open sets in $\operatorname{Spec} G$.

The space $\operatorname{Spec} G$ was introduced by Keimel [19, 3] and is very useful for the study of abelian l-groups. Unfortunately, it is not functorial. By adding the "infinite" prime G to $\operatorname{Spec} G$, we get a spectral space $\operatorname{Spec}^* G$ with $\mathcal{D}(\operatorname{Spec}^* G) = \mathcal{D}(\operatorname{Spec} G) \cup \{\operatorname{Spec}^* G\}$ such that $\operatorname{Spec} G$ is a schematic subspace. We call $\operatorname{Spec}^* G$ the *spectrum* of G. For a morphism $f\colon G \to H$ in $\mathbf{Ab}_l$, the map

$$\operatorname{Spec}^* f\colon \operatorname{Spec}^* H \to \operatorname{Spec}^* G$$

with $(\operatorname{Spec}^* f)(P) := f^{-1}(P)$ is spectral since

$$(\operatorname{Spec}^* f)^{-1}(S(a)) = S(f(a))$$

for all $a \in G$. Thus we have a functor

$$\operatorname{Spec}^*\colon \mathbf{Ab}_l \to \mathbf{Sp}$$

into the category $\mathbf{Sp}$ of spectral spaces.

Not every spectral space is homeomorphic to the spectrum of an abelian l-group. Some necessary conditions are known [10], but it is still open which spectral spaces actually occur.

The lattice isomorphism $\mathcal{C}(G) \cong \mathcal{O}(\operatorname{Spec} G)$ implies that for $A \subset G$, the equality

$$\overline{S(A)} = (\operatorname{Spec} G) \smallsetminus S(A^{\perp})$$

holds in $\operatorname{Spec} G$ (see [3, Proposition 3.4.1]). Now an abelian l-group G is said to be *strongly projectable* [1] if every polar $A^{\perp}$ of G is a direct summand. Thus by the above equality, G is strongly projectable if and only if $\operatorname{Spec} G$ is extremally disconnected.

An l-subgroup G of an abelian l-group H is said to be *large* if each non-zero $A \in \mathcal{C}(H)$ intersects G non-trivially. Then $H \supset G$ is also called an *essential extension* of G. If H is strongly projectable with a large l-subgroup G such that there is no other strongly projectable l-group between G and H, then H is called a *strongly projectable hull* of G. The existence and uniqueness of the strongly projectable hull was proved by Conrad [12]. An explicit construction was given by Chambless [9].

In [24], the first author proves that every topological space X admits a cover $p\colon P \to X$ with respect to the full subcategory of extremally disconnected spaces, and that p coincides with the *absolute* [23, 25, 27] of X (see Section 5). Furthermore, he characterizes the strongly projectable hull in terms of spectra. Specifically, [24, Theorem 4] states that a morphism $f\colon G \to H$ in $\mathbf{Ab}_l$ describes the strongly projectable hull if and only if $\operatorname{Spec}^* f$ induces a map $\operatorname{Spec} H \to \operatorname{Spec} G$ which is the absolute of $\operatorname{Spec} G$. In other words, the absolute of $\operatorname{Spec} G$ lifts uniquely to $\mathbf{Ab}_l$.

More generally, we will show that the absolute of a schematic space X is again schematic (Proposition 12). Like the essential cover $e\colon \widetilde{X} \to X$, the absolute $p\colon P \to X$ is essentially dense, and both P and $\widetilde{X}$ are extremally disconnected. While p is always surjective, the image of e exhibits an interesting invariant of X, namely, $X_{\min} := e(\widetilde{X})$ is the smallest dense schematic subspace of X (Theorem 1). If $X_{\min} = X$, we call X *minimal.* For example, if X is extremally disconnected, $X_{\min}$ consists of the generic points of the irreducible components [8] of X. Although spectral maps of the form $\operatorname{Spec}^* f$ with $f \in \mathbf{Ab}_l$ are closed [24], the spectrum of an abelian l-group need not be minimal.

For an arbitrary topological space X with absolute $p\colon P \to X$, we show that the quasicomponents C of P map bijectively onto closed irreducible subspaces $p(C)$ of X including the irreducible components of X. Therefore, the absolute leads us to redefine the collection of irreducible components of X to be the family of those $p(C)$, parametrized by the quasicomponents of P. In case that X is schematic, this yields a new interpretation of the essential cover $e\colon \widetilde{X} \to X$. Namely, $\widetilde{X}$ can be identified with the space $Q(P)$ of quasicomponents of P, and under this identification, e maps each quasicomponent to the generic point of the corresponding irreducible component of X (Theorem 5). Thus in a word, the essential cover of a schematic space X can be viewed as the space of irreducible components of X in the new sense, i.e. with embedded components and multiplicities given by the absolute.

As every abelian l-group G has a largest prime, the spectrum $\operatorname{Spec}^* G$ is Hausdorff if and only if $G = 0$. By [3, Theorem 14.1.2], the Keimel spectrum $\operatorname{Spec} G$ is Hausdorff if and only if G is hyperarchimedean. Thus in general, the essential cover of $X := \operatorname{Spec}^* G$ (or $\operatorname{Spec} G$) cannot be lifted to $\mathbf{Ab}_l$. Nevertheless, the Stone space $\widetilde{X}$ is an important invariant of G. In fact, the above formula for $\overline{S(A)}$ gives

$$S(A^{\perp\perp}) = \operatorname{int} \overline{S(A)}$$

for all $A \in \mathcal{C}(G)$. Therefore, the Boolean algebra $\mathcal{P}(G)$ of polars of G is isomorphic to the complete Boolean algebra of regular open sets in $\operatorname{Spec}^* G$. So the space $\widetilde{X}$ in the essential cover $\widetilde{X} \to X := \operatorname{Spec}^* G$ just the *Stone space* of G, i.e. the Stone dual of $\mathcal{P}(G)$.

In general, the essential cover $e\colon \widetilde{X} \to X$ is not functorial, and not even a precover with respect to a suitable full subcategory of **GSp**. However, the map e always belongs to the subcategory **SSp** of *skeletal* maps, i.e. spectral maps for which the inverse image of a dense open set is dense. In the subcategory **SSp**, e is not only a precover, but even functorial. Here the full subcategory of extremally disconnected locally Stone spaces, i.e. the spaces of the form $\widetilde{X}$, is coreflective (Theorem 4).

By [24, Proposition 19], a morphism $f\colon G \to H$ in $\mathbf{Ab}_l$ represents a large embedding if and only if $\operatorname{Spec}^* f$ is essentially dense. This shows that the Stone space of G is invariant under essential extensions. For archimedean l-groups, essential extensions are characterized by this property (see [3, Theorem 11.1.5]). Now Bernau's theorem [2, 11] implies that every archimedean l-group G admits a unique largest essential extension $G \hookrightarrow D(\widetilde{X})$, where $\widetilde{X}$ is the Stone space of G, and $D(\widetilde{X})$ denotes the l-group of almost finite continuous functions $\widetilde{X} \to [-\infty, \infty]$ (see [3, 13.2]). So it is natural to ask to what extent the embedding $G \hookrightarrow D(\widetilde{X})$ can be characterized in terms of the spectra.

Conrad [11] has shown that $D(\widetilde{X})$ is the lateral completion $((G^d)^\wedge)^L$ of $(G^d)^\wedge$, the Dedekind–MacNeille completion of the divisible hull G^d of G. Moreover, the proof of [24, Theorem 4] shows that the reason why the absolute $p\colon P \to X := \operatorname{Spec} G$ corresponds to the strongly projectable hull of G, and not to a bigger strongly projectable l-subgroup of $D(\widetilde{X})$, is just that p is *separated*, i.e. that the diagonal $P \to P \times_X P$ is closed. Now the passage from G to G^d does not affect the spectrum. Hence the embedding $G \hookrightarrow (G^d)^\wedge$ induces a spectral map $\operatorname{Spec}(G^d)^\wedge \to \operatorname{Spec} G$ by virtue of [3, Theorem 11.3.7]. Furthermore, complete l-groups are archimedean and strongly projectable

([1, Theorem 8.2.3]). So there might be a non-separated version of the absolute that generalizes $\operatorname{Spec}(G^d)^\wedge \to \operatorname{Spec} G$.

On the other hand, it follows by [3, Theorem 11.3.7], that $(G^d)^\wedge$ is an l-ideal of $D(\widetilde{X})$. Hence $(G^d)^\wedge \hookrightarrow D(\widetilde{X})$ does not induce a map $\operatorname{Spec} D(\widetilde{X}) \to \operatorname{Spec}(G^d)^\wedge$. Therefore, a spectral analogue of the embedding $G \hookrightarrow D(\widetilde{X})$ cannot be expected.

1. Preliminaries. For a topological space X and a subset D, we write $\overline{D}$ for the closure, and $\operatorname{int} D$ for the interior of D. We denote the bounded lattice of open sets by $\mathcal{O}(X)$. Thus every continuous map $f\colon X \to Y$ between topological spaces X, Y induces a morphism of bounded lattices

$$f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X). \tag{1}$$

We call f *dense* if the image $f(X)$ of f is dense in Y. This means that the implication $U \neq \emptyset \Rightarrow f^{-1}(U) \neq \emptyset$ holds for all $U \in \mathcal{O}(Y)$. Therefore, any pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ of continuous maps satisfies

$$f, g \text{ dense} \;\Rightarrow\; gf \text{ dense} \;\Rightarrow\; g \text{ dense}. \tag{2}$$

Definition 1. We call a continuous map $g\colon Y \to Z$ *essentially dense* if g is dense, and the implication

$$gf \text{ dense} \;\Rightarrow\; f \text{ dense} \tag{3}$$

holds for every continuous map $f\colon X \to Y$.

The following is easy to verify (see [24]):

Proposition 1. *Let $f\colon X \to Y$ be a continuous map between topological spaces. The following are equivalent.*

(a) *f is essentially dense.*
(b) *f is dense, and for every non-empty $U \in \mathcal{O}(X)$, there exists a non-empty $V \in \mathcal{O}(Y)$ with $f^{-1}(V) \subset U$.*
(c) *The equivalence $\overline{f(D)} = Y \Leftrightarrow \overline{D} = X$ holds for all $D \subset X$.*

Remark. If $f\colon X \to Y$ is closed, the equivalent conditions of Proposition 1 state that f is surjective and *irreducible*, i.e. $f(A) \neq Y$ for every proper closed subspace $A \subset X$ (cf. [6]).

There is a close relationship between essentially dense maps and regular open sets. Recall that an open set U of a topological space X is said to be *regular* if $\operatorname{int} \overline{U} = U$. It is well-known (cf. [16, 3.1], [21, Theorem 1.37]) that the regular open sets form a complete Boolean algebra $\mathfrak{B}(X)$ with lattice operations

$$U \wedge V = U \cap V, \qquad U \vee V = \operatorname{int} \overline{U \cup V}. \tag{4}$$

For a continuous map $f\colon X \to Y$ and open sets $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$, we define

$$f_*U := \bigcup\{W \in \mathcal{O}(Y) \mid f^{-1}(W) \subset U\}, \tag{5}$$

$$f^{-1}[V] := \operatorname{int} \overline{f^{-1}(V)}. \tag{6}$$

Proposition 2. *A continuous map $f\colon X \to Y$ is dense if and only if*

$$V \subset f_*f^{-1}[V] \subset \overline{V} \tag{7}$$

for all $V \in \mathcal{O}(Y)$. A dense $f\colon X \to Y$ is essentially dense if and only if

$$U \subset f^{-1}[f_*U] \subset \overline{U} \tag{8}$$

for all $U \in \mathcal{O}(X)$.

Proof. Let us show first that

$$V \subset f_*f^{-1}[V], \qquad f^{-1}[f_*U] \subset \overline{U} \tag{9}$$

for any f. The first inclusion says that $f^{-1}(V) \subset f^{-1}[V]$ for all $V \in \mathcal{O}(Y)$, which is trivial. Secondly, if $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$, then $f^{-1}(V) \subset U$ implies that $f^{-1}(V) \subset \overline{U}$. Hence $f^{-1}(f_*U) \subset \overline{U}$, and thus $f^{-1}[f_*U] \subset \overline{U}$.

Assume that (7) holds for $V = \emptyset$. Then $Y \smallsetminus \overline{f(X)} = f_*f^{-1}[\emptyset] \subset \emptyset$, which implies that f is dense. Conversely, let f be dense. Suppose that (7) does not hold for some $V \in \mathcal{O}(Y)$. Then there is some $W \in \mathcal{O}(Y)$ with $f^{-1}(W) \subset f^{-1}[V]$ and $W \not\subset \overline{V}$. Thus $W \smallsetminus \overline{V} \in \mathcal{O}(Y)$ is non-empty. Hence $\emptyset \neq f^{-1}(W \smallsetminus \overline{V}) \subset f^{-1}[V] \subset \overline{f^{-1}(V)}$. So we get $f^{-1}(W \smallsetminus \overline{V}) \cap f^{-1}(V) \neq \emptyset$, a contradiction.

Now let f be essentially dense and $U \in \mathcal{O}(X)$. Then every non-empty open $U' \subset U$ contains an inverse image $f^{-1}(V)$ with $\emptyset \neq V \in \mathcal{O}(Y)$. Hence $\emptyset \neq f^{-1}(V) \subset U' \cap f^{-1}(f_*U)$. This proves that $U \subset \overline{f^{-1}(f_*U)}$, which yields (8). Conversely, let f be dense such that (8) holds for every $U \in \mathcal{O}(X)$. If $U \neq \emptyset$, this gives $U \subset \overline{f^{-1}(f_*U)}$, whence $U \cap f^{-1}(f_*U) \neq \emptyset$. So there exists some $V \in \mathcal{O}(Y)$ with $\emptyset \neq f^{-1}(V) \subset U$. By Proposition 1, it follows that f is essentially dense. ∎

Corollary. *Let $f\colon X \to Y$ be continuous. If f is essentially dense, the maps* (5) *and* (6) *induce a lattice isomorphism $\mathfrak{B}(X) \cong \mathfrak{B}(Y)$.*

Proof. Assume that f is essentially dense. For a given $U \in \mathfrak{B}(X)$, we show first that $f_*U \in \mathfrak{B}(Y)$. Thus we have to verify $f^{-1}(W) \subset U$ for every open $W \subset \overline{f_*U}$. Since U is regular, it suffices to prove $f^{-1}(W) \subset \overline{U}$ for such W. Suppose that $f^{-1}(W) \not\subset \overline{U}$ for some open $W \subset \overline{f_*U}$. By Proposition 1, there exists a non-empty $V \in \mathcal{O}(Y)$ with $f^{-1}(V) \subset f^{-1}(W) \smallsetminus \overline{U}$. Hence $f^{-1}(V \cap f_*U) = \emptyset$. Since f is dense, this gives $V \cap f_*U = \emptyset$. Therefore, we get $V \cap \overline{f_*U} = \emptyset$, and thus $V \cap W = \emptyset$, which contradicts $f^{-1}(V) \subset f^{-1}(W)$. Now the corollary follows immediately by (7) and (8). ∎

Example 1. If a continuous map $f\colon X \to Y$ induces an isomorphism $\mathfrak{B}(X) \cong \mathfrak{B}(Y)$, then (7) holds for $V = \emptyset$. Therefore, f is dense. However, f need not be essentially dense. For example, let $\mathbb{S} = \{0, 1\}$ be the *Sierpiński space* with $\mathcal{O}(\mathbb{S}) = \{\emptyset, \{1\}, \mathbb{S}\}$. Then $\mathfrak{B}(\mathbb{S}) = \{\emptyset, \mathbb{S}\}$, and the map $f : \mathbb{S} \to \mathbb{S}$ with $f(\mathbb{S}) = \{1\}$ induces an isomorphism $\mathfrak{B}(\mathbb{S}) \cong \mathfrak{B}(\mathbb{S})$. However, the inclusion (8) does not hold for $U = \{1\}$.

2. Schematic spaces. In view of the preceding section, it is natural to ask which topological spaces X admit an essentially dense map $X' \to X$ which is maximal in a suitable sense. For arbitrary spaces X, it is not likely to find such an X' since every dense subspace Y gives rise to an essentially dense map $Y \hookrightarrow X$, while the intersection of dense subsets need not be dense. We will see that this problem does not arise if X is *schematic*, i.e. a T_0-space for which the set $\mathcal{D}(X)$ of quasi-compact open subsets is a basis with

$$U, V \in \mathcal{D}(X) \;\Rightarrow\; U \cap V \in \mathcal{D}(X), \tag{10}$$

and every non-empty closed irreducible set $A \subset X$ has a *generic point* x, i.e. $A = \overline{\{x\}}$. In [10], such spaces X are called "generalized spectral spaces". Since $\mathcal{D}(X)$ is closed under finite unions, it follows that $\mathcal{D}(X)$ is a lattice with respect to union and intersection. A continuous map $f\colon X \to Y$ between schematic spaces is said to be *spectral* if $f^{-1}(V)$ belongs to $\mathcal{D}(X)$ whenever $V \in \mathcal{D}(Y)$. The category of schematic spaces with spectral maps as morphisms will be denoted by **GSp**. The *spectral spaces*, i.e. the quasi-compact spaces in **GSp**, form a full subcategory **Sp**. A topological space X is a Hausdorff schematic space if and only if X is a *locally Stone space*, i.e. locally compact and totally disconnected. Similarly, a Hausdorff spectral space is the same as a *Stone space*, i.e. a compact totally disconnected space.

For $X \in \mathbf{GSp}$, let $\mathcal{R}(X)$ denote the ring of subsets generated by $\mathcal{D}(X)$, i.e. $\mathcal{R}(X)$ consists of the finite unions of differences $U \smallsetminus V$ with $U, V \in \mathcal{D}(X)$. Then $\mathcal{R}(X)$ is a basis of open sets for the *patch topology* on X (cf. [17, 10]). With this topology, X becomes a locally Stone space πX (use [17, Theorem 1]), and

$$\mathcal{R}(X) = \mathcal{D}(\pi X). \tag{11}$$

Instead of $\mathfrak{B}(X)$, we consider the sublattice

$$\mathfrak{C}(X) := \{U \in \mathfrak{B}(X) \mid \exists V \in \mathcal{D}(X)\colon U \subset \overline{V}\}. \tag{12}$$

Recall that a topological space is said to be *extremally disconnected* if the closure of every open set is open.

Proposition 3. *A topological space X is locally compact and extremally disconnected if and only if X is a schematic space with $\mathcal{D}(X) = \mathfrak{C}(X)$.*

Proof. Let $X \in \mathbf{GSp}$ satisfy $\mathcal{D}(X) = \mathfrak{C}(X)$. We show first that every $U \in \mathcal{D}(X)$ is closed. To this end, it suffices to verify that $\overline{U} \cap V = U \cap V$ for each $V \in \mathcal{D}(X)$. Since $V \smallsetminus \overline{U} \in \mathfrak{C}(X) = \mathcal{D}(X)$, we get $V' := (V \cap U) \cup (V \smallsetminus \overline{U}) \in \mathcal{D}(X) = \mathfrak{C}(X)$. Hence $V' = \operatorname{int} \overline{V'} = \operatorname{int} \overline{V} = V$, and thus $V \cap \overline{U} = V \cap U$. This shows that U is closed. Since $\mathcal{D}(X)$ is a basis of X, we infer that X is locally Stone. For any $W \in \mathcal{O}(X)$ and $U \in \mathcal{D}(X)$, it follows that $\operatorname{int}(\overline{W} \cap U) \in \mathfrak{C}(X) = \mathcal{D}(X)$. Therefore, $W \cap U \subset \operatorname{int}(\overline{W} \cap U) \subset \overline{W} \cap U \subset \overline{W \cap U}$ yields $\overline{W} \cap U = \operatorname{int}(\overline{W} \cap U) \in \mathcal{O}(X)$. Hence $\overline{W}$ is open. The converse is trivial. ∎

Similar to [17, Section 2], we define a *patch* of $X \in \mathbf{GSp}$ to be a subspace of X which is closed in πX.

PROPOSITION 4. *The image $f(X)$ of a morphism $f\colon X \to Y$ in* $\mathbf{GSp}$ *is a patch of Y. For every patch Z of Y, the inclusion $Z \hookrightarrow Y$ is a morphism in* $\mathbf{GSp}$.

Proof. Let $y \in Y$ belong to the closure of $f(X)$ in πY. Choose $U \in \mathcal{D}(Y)$ with $y \in U$. Then $f^{-1}(U)$ is a spectral space. Therefore, [17, Theorem 1] implies that $U \cap f(X) = f(f^{-1}(U))$ is compact in πY. Since y belongs to the closure of $U \cap f(X)$ in πY, it follows that $y \in U \cap f(X) \subset f(X)$. This proves that $f(X)$ is a patch of Y.

Now let Z be a patch of Y. Then πZ is a locally Stone subspace of πY, and the inclusion $\pi Z \hookrightarrow \pi Y$ is spectral. Therefore, $\mathcal{D}(Z)$ consists of the intersections $U \cap Z$ with $U \in \mathcal{D}(Y)$. Consequently, the T_0-space Z has $\mathcal{D}(Z)$ as a basis which satisfies (10). Let $A \neq \emptyset$ be a closed irreducible subset of Z. Then $\overline{A}$ is irreducible in Y. Hence $\overline{A} = \overline{\{y\}}$ for some $y \in Y$. If $y \notin A$, there exist $U, V \in \mathcal{D}(Y)$ with $y \in U \smallsetminus V$ and $(U \smallsetminus V) \cap A = \emptyset$. This gives $U \cap A \subset V$. On the other hand, $y \notin V$ implies that $\overline{A} \cap V = \emptyset$. Hence $U \cap A = \emptyset$, a contradiction. Thus $y \in A$, which proves that $A \subset Z$ is generic. Therefore, Z is a schematic space, and the inclusion $Z \hookrightarrow Y$ is spectral. ∎

DEFINITION 2. We call a schematic space *minimal* if it does not contain a proper dense patch.

For a schematic space X, we define the patch

$$X_{\min} := X \smallsetminus \bigcup \{U \smallsetminus V \mid U, V \in \mathcal{D}(X),\ U \subset \overline{V}\}. \tag{13}$$

THEOREM 1. *Let X be a schematic space. Then $X_{\min}$ is dense in X and is contained in every dense patch of X. Moreover, X is minimal if and only if $\mathcal{D}(X) \subset \mathfrak{C}(X)$.*

Proof. We show first that $X_{\min}$ is contained in every dense patch Y of X. In fact, $X \smallsetminus Y \in \mathcal{O}(\pi X)$ implies that $X \smallsetminus Y$ is a union of differences $U \smallsetminus V$ with $U, V \in \mathcal{D}(X)$. For any such difference $U \smallsetminus V$, we have $U \smallsetminus \overline{V} \subset X \smallsetminus Y$, and thus $(U \smallsetminus \overline{V}) \cap Y = \emptyset$. Since $U \smallsetminus \overline{V}$ is open, this gives $U \smallsetminus \overline{V} = \emptyset$, i.e. $U \subset \overline{V}$. Hence $X_{\min} \subset Y$.

To show that $X_{\min}$ is dense in X, let $W \in \mathcal{D}(X)$ be such that $W \cap X_{\min} = \emptyset$. Since W is compact in πX, there are $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ in $\mathcal{D}(X)$ with $U_i \subset \overline{V_i}$ such that $W \subset \bigcup_{i=1}^{n} (U_i \smallsetminus V_i)$. Then $U_i \cap W \subset \overline{V_i \cap W}$ for all i. Thus if we replace U_i by $U_i \cap W$ and V_i by $V_i \cap W$, we have $U_i, V_i \in \mathcal{D}(W)$ with $U_i \subset \overline{V_i}$ and

$$W = (U_1 \smallsetminus V_1) \cup \cdots \cup (U_n \smallsetminus V_n). \tag{14}$$

We will show by induction that $W = \emptyset$. First, (14) implies $V_1 \cap \cdots \cap V_n = \emptyset$. We set $V_i' := V_1 \cap \cdots \cap V_i$. Assume that $V_i' \cap V_j = \emptyset$ for all $j > i$. Since $V_i' \cap (U_j \smallsetminus V_j) = \emptyset$ for $j \le i$, (14) gives $V_i' \subset (U_{i+1} \smallsetminus V_{i+1}) \cup \cdots \cup (U_n \smallsetminus V_n)$. Thus if $j > i$, then $V_i' \cap (U_j \smallsetminus V_j) = V_i' \cap U_j \in \mathcal{D}(W)$. Hence $V_i' \cap (U_j \smallsetminus V_j) = \emptyset$. Therefore, we get $V_i' = \emptyset$. By induction and symmetry, this yields $V_i = \emptyset$ for all i. So we get $U_i \subset \overline{V_i} = \emptyset$, and consequently, $W = \emptyset$. Thus we have proved that $X_{\min}$ is the smallest dense patch of X.

Next let X be minimal. If $U \in \mathcal{D}(X) \smallsetminus \mathfrak{C}(X)$, then $U \subsetneq V \subset \overline{U}$ for some $V \in \mathcal{D}(X)$. Hence $U \cup (X \smallsetminus V)$ is a dense patch of X, and so $U \cup (X \smallsetminus V) = X$, i.e. $U = V$, a contradiction. Thus $\mathcal{D}(X) \subset \mathfrak{C}(X)$. Conversely, the inclusion $\mathcal{D}(X) \subset \mathfrak{C}(X)$ implies that $X_{\min} = X$, whence X is minimal. ∎

For an important class of schematic spaces X, the subspace $X_{\min}$ admits a simpler description. Note first that every T_0-space X is partially ordered by the *specialization order*

$$x \le y \;\Leftrightarrow\; \overline{\{x\}} \subset \overline{\{y\}}. \tag{15}$$

For example, the Sierpiński space $\mathbb{S} = \{0, 1\}$ (see Example 1) satisfies $0 < 1$. The following Proposition 5 can be derived from Hochster's theorem [17] that every spectral space occurs as the spectrum of a commutative ring R, together with Kaplansky's observation [18] that $\operatorname{Spec} R$ satisfies (K1) and (K2) of Proposition 5 with respect to inclusion. We will give a direct proof.

Proposition 5. *With respect to the specialization order, every spectral space X satisfies*

(K1) *A non-empty chain $C \subset X$ has a supremum and an infimum. Moreover, $\overline{\{\sup C\}} = \overline{C}$ and $\overline{\{\inf C\}} = \bigcap_{x \in C} \overline{\{x\}}$.*

(K2) *For any pair $x < y$ in X, there exist $x', y' \in X$ with $x \le x' < y' \le y$ such that there is no point properly between x' and y'.*

Proof. Let $C \neq \emptyset$ be a chain in X. Then $\overline{C} \subset X$ is closed and irreducible. In fact, suppose that $\overline{C} = A \cup B$ with closed sets $A, B \subset X$. If $A \neq \overline{C}$, there is a point $x \in C \smallsetminus A$. Hence $C \subset \bigcup_{x \le y \in C} \overline{\{y\}} \subset B$, and thus $B = \overline{C}$. Therefore, $\overline{C}$ contains a generic point z, and thus $z = \sup C$. As X is quasi-compact, the intersection $C_0 := \bigcap_{x \in C} \overline{\{x\}}$ is non-empty. Every $U \in \mathcal{O}(X)$ which intersects

C_0 contains C. Therefore, if $U, V \in \mathcal{D}(X)$ both intersect C_0, then $C \subset U \cap V$. Since $U \cap V$ is quasi-compact, we get $C_0 \cap U \cap V \neq \emptyset$. Hence C_0 is irreducible, and its generic point is the infimum of C. This proves (K1).

For $x < y$ in X, there exists $U \in \mathcal{D}(X)$ with $y \in U$ and $x \notin U$. Since U is a spectral subspace of X, the Kuratowski–Zorn lemma yields a minimal $y' \in U$ with $x < y' \leq y$. By (K1), every chain $C \subset X \smallsetminus U$ with $x \in C$ has a supremum in $X \smallsetminus U$. Therefore, again by the Kuratowski–Zorn lemma, we find a maximal $x' < y'$ with $x' \geq x$. ∎

The next result shows that for extremally disconnected schematic spaces X, the subspace $X_{\min}$ is a locally Stone space and consists of the "most general" points of X. Recall that a topological space is 0-*dimensional* if it has a basis of closed open sets.

Proposition 6. *Let X be a schematic space. The maximal points of X form a 0-dimensional dense Hausdorff subspace μX of $X_{\min}$. If X is extremally disconnected, then $\mu X = X_{\min}$.*

Proof. Let $\mu X \subset X$ be the subspace of maximal points with respect to the specialization order (15). Since every $x \in X$ belongs to a spectral subspace $U \in \mathcal{D}(X)$ of X, Proposition 5 implies that $X = \bigcup_{x \in \mu X} \overline{\{x\}}$. This shows that μX is dense in X. For different x, y in μX, suppose that $U \cap V \neq \emptyset$ for all $U, V \in \mathcal{D}(X)$ with $x \in U$ and $y \in V$. Since πX is locally compact, this implies that there is a common point z in all these intersections $U \cap V$. Hence $z \geq x, y$, a contradiction. This proves that μX is Hausdorff. If $x \in U \cap \mu X$ and $y \in \overline{U} \cap \mu X$ with $U \in \mathcal{D}(X)$, then $y \leq x'$ for some $x' \in U$. Since y is maximal, this yields $y \in U$. Thus $\overline{U} \cap \mu X = U \cap \mu X$, which shows that μX is 0-dimensional.

Now let $x \in \mu X$ be given. Then $\bigcap \{W \in \mathcal{D}(X) \mid x \in W\} = \{x\}$. Suppose that $x \in U \smallsetminus V$ with $U, V \in \mathcal{D}(X)$. By the compactness of πV, this implies that $W \cap V = \emptyset$ for some neighbourhood $W \in \mathcal{D}(X)$ of x. Hence $x \in U \cap W \subset U \smallsetminus V$. So we get $\mu X \subset X_{\min}$.

Finally, let X be extremally disconnected. Suppose that $y > x$. Then $y \notin \overline{\{x\}}$, and so there is some $V \in \mathcal{D}(X)$ with $y \in V$ and $x \notin V$. Since $\overline{V}$ is open and $x \in \overline{\{y\}} \subset \overline{V}$, we find some $U \in \mathcal{D}(X)$ with $x \in U \subset \overline{V}$. Hence $x \in U \smallsetminus V$, and thus $x \notin X_{\min}$. This proves that $\mu X = X_{\min}$. ∎

The following example shows that $X_{\min}$ need not be a Hausdorff space.

Example 2. For an infinite set X', consider the spectral space $X := X' \sqcup \{0\}$ (disjoint union) such that $\mathcal{D}(X)$ consists of the finite subsets of X' together with X. Then the elements of X' are pairwise incomparable, and $0 < x$ for all $x \in X'$. By Theorem 1, the whole space X is minimal, and thus $\mu X = X' \neq X_{\min}$.

3. The essential cover. Theorem 1 provides a step toward a maximal essentially dense map $\widetilde{X} \to X$ for a schematic space X. To construct $\widetilde{X} \to X$, we use Stone duality. Let $\mathbf{DL}_0$ denote the category of distributive 0-lattices (i.e. with a smallest element 0). Morphisms in $\mathbf{DL}_0$ are 0-preserving lattice homomorphisms $f\colon D \to D'$ such that for every $b \in D'$, there is an element $a \in D$ with $f(a) \geq b$. If D, D' belong to the full subcategory $\mathbf{DL}_b$ of bounded distributive lattices, the latter condition reduces to $f(1) = 1$. Stone's duality theorem [26] implies that the functor $\mathcal{D}$ gives a duality

$$\mathcal{D}\colon \mathbf{GSp}^{\mathrm{op}} \xrightarrow{\sim} \mathbf{DL}_0. \tag{16}$$

To get the inverse of $\mathcal{D}$, note that the Sierpiński space $\mathbb{S}$ can be regarded as an object in $\mathbf{DL}_b$. Therefore, any $D \in \mathbf{DL}_0$ gives rise to a morphism set

$$\operatorname{Spec} D := \operatorname{Hom}_{\mathbf{DL}_0}(D, \mathbb{S}), \tag{17}$$

which can be viewed as a subspace of the product space $\mathbb{S}^D \in \mathbf{Sp}$. If D is bounded, $\operatorname{Spec} D$ is a patch in $\mathbb{S}^D$. Otherwise, the closure of $\operatorname{Spec} D$ in $\pi(\mathbb{S}^D)$ is $(\operatorname{Spec} D) \cup \{0\}$. Hence $\operatorname{Spec} D$ is schematic. Thus (17) defines a functor $\operatorname{Spec}\colon \mathbf{DL}_0 \to \mathbf{GSp}^{\mathrm{op}}$ which is inverse to (16).

Definition 3. We call a morphism $f\colon D \to D'$ in $\mathbf{DL}_0$ *codense* if $f(a) = 0$ implies that $a = 0$. We say that f is *essentially codense* if f is codense and for every $b > 0$ in D', there is a non-zero $a \in D$ with $f(a) \leq b$.

As an immediate consequence of Proposition 1, we have

Proposition 7. *Let $f\colon X \to Y$ be a morphism of schematic spaces.*

(a) *f is dense if and only if $\mathcal{D}(f)\colon \mathcal{D}(Y) \to \mathcal{D}(X)$ is codense.*
(b) *f is essentially dense if and only if $\mathcal{D}(f)$ is essentially codense.*

Although continuous maps between schematic spaces need not be spectral, the definition of essentially dense maps carries over to $\mathbf{GSp}$.

Corollary. *A dense morphism $f\colon X \to Y$ in* $\mathbf{GSp}$ *is essentially dense if and only if the implication* (3) *holds for every $g\colon Y \to Z$ in* $\mathbf{GSp}$.

Proof. By Proposition 7, it suffices to prove the dual assertion. Thus let $g\colon D \to D'$ be a codense morphism in $\mathbf{DL}_0$. Assume first that g is essentially codense, and let $f\colon D' \to D''$ be a morphism in $\mathbf{DL}_0$ such that fg is codense. If $0 < b \in D'$, we find a non-zero $a \in D$ with $g(a) \leq b$. Hence $f(b) \geq f(g(a)) > 0$, which shows that f is codense. Conversely, assume that the dual of (3) is satisfied. Let $b \in D'$ be non-zero. Then $x \mapsto x \vee b$ defines a morphism $f\colon D' \to [b, -)$ in $\mathbf{DL}_0$. If $g(a) \not\leq b$ for all $a > 0$ in D, then fg is codense, while f is not since $b > 0$. ∎

We say that a morphism $f\colon X \to Y$ in any category is a *retraction* (resp. *section*) if there is a morphism $g\colon Y \to X$ with $fg = 1$ (resp. $gf = 1$).

THEOREM 2. *Let X be a schematic space. The following are equivalent.*

(a) *X is locally compact and extremally disconnected.*
(b) *Every dense morphism $Y \to X$ in* **GSp** *is a retraction.*
(c) *Every essentially dense morphism $Y \to X$ in* **GSp** *is a retraction.*

Proof. (a)⇒(b): Let $f\colon Y \to X$ be a dense morphism in **GSp**. By Proposition 4, $f(Y)$ is closed in $\pi X = X$. Hence f is surjective. To prove that f is a retraction, we can replace f by $\pi Y \to Y \xrightarrow{f} X$, where $\pi Y \to Y$ is the natural bijection. Therefore, we can assume, without loss of generality, that Y is Hausdorff. Let $\mathcal{C}$ be a chain of closed subspaces $A \subset Y$ with $f(A) = X$. We show that the intersection $C := \bigcap \mathcal{C}$ also satisfies $f(C) = X$. If $f(C) \neq X$, then Proposition 4 implies that $f(C)$ is a proper closed subset of X. So we find a non-empty $U \in \mathcal{D}(X)$ with $U \cap f(C) = \emptyset$. Hence $f^{-1}(U) \cap A \neq \emptyset$ for all $A \in \mathcal{C}$, but $f^{-1}(U) \cap C = \emptyset$. Since $f^{-1}(U)$ is compact, this is a contradiction. By the Kuratowski–Zorn lemma, we thus obtain a minimal closed subset $Z \subset Y$ with $f(Z) = X$. If $\overline{f(D)} = X$ for a subset $D \subset Z$, then $f(\overline{D}) = \overline{f(\overline{D})} = X$, and thus $\overline{D} = Z$. Therefore, Proposition 1 implies that $Z \hookrightarrow Y \xrightarrow{f} X$ is essentially dense. So there is no loss of generality if we assume that Y is a locally Stone space and $f\colon Y \twoheadrightarrow X$ is essentially dense.

Now we proceed as in [16], using the fact that f is a closed map. Suppose that there are different $x, y \in Y$ with $f(x) = f(y)$. Then there are disjoint $U, V \in \mathcal{D}(Y)$ with $x \in U$ and $y \in V$. By Proposition 2, the open sets f_*U and f_*V are disjoint, and their inverse images are dense in U and V, respectively. Hence $f(x) \in \overline{f_*U}$ and $f(y) \in \overline{f_*V}$, and thus $\overline{f_*U} \cap \overline{f_*V} \neq \emptyset$. Since X is extremally disconnected, this is a contradiction. So we have shown that f is bijective, which completes the proof of (b). The implication (b)⇒(c) is trivial.

(c)⇒(a): By Stone duality, condition (c) implies that every essentially codense morphism $\mathcal{D}(X) \to D$ in $\mathbf{DL}_0$ is a section. For a dense open set $U \subset X$, we set

$$\mathcal{O}'(U) := \{W \in \mathcal{O}(U) \mid \exists V \in \mathcal{D}(X)\colon W \subset V\}$$

and consider the morphism

$$p\colon \mathcal{D}(X) \to \mathcal{O}'(U)$$

given by $p(V) := V \cap U$. If $p(V) = \emptyset$, then $V \subset X \setminus U$, which yields $V = \emptyset$. Thus p is essentially codense. So (c) implies that p is injective. Therefore, if $V, W \in \mathcal{D}(X)$ satisfy $W \subset V \subset \overline{W}$, then $U := W \cup (X \setminus \overline{W})$ is dense in X, and $V \cap U = W \cap U$, whence $V = W$. So we get $\mathcal{D}(X) \subset \mathfrak{C}(X)$. By Theorem 1, this means that X is minimal.

Next we consider the inclusion

$$i\colon \mathcal{D}(X) \hookrightarrow \mathcal{R}(X)$$

in $\mathbf{DL}_0$, where $\mathcal{R}(X)$ is given by (11). Since X is minimal, i is essentially codense. Hence there exists a morphism $f\colon \mathcal{R}(X) \to \mathcal{D}(X)$ in $\mathbf{DL}_0$ with $fi = 1$. For $U, V \in \mathcal{D}(X)$, we have $U \smallsetminus V \subset U$ and $(U \smallsetminus V) \cap V = \emptyset$. Hence $f(U \smallsetminus V) \subset U$ and $f(U \smallsetminus V) \cap V = \emptyset$. Thus $f(U \smallsetminus V) \subset U \smallsetminus V$, and $f(U \smallsetminus V)$ is the largest $W \in \mathcal{D}(X)$ with $W \subset U \smallsetminus V$. Hence $U \smallsetminus \overline{V} \in \mathcal{D}(X)$. Since $U \subset \overline{(U \smallsetminus \overline{V}) \cup V}$ and $(U \smallsetminus \overline{V}) \cup V \in \mathcal{D}(X)$, the minimality of X implies that $U \subset (U \smallsetminus \overline{V}) \cup V$. Therefore, $U \smallsetminus V \subset U \smallsetminus \overline{V}$, which gives $U \cap \overline{V} \subset U \cap V$. As this holds for all $U \in \mathcal{D}(X)$, we get $\overline{V} = V$. Hence X is locally Stone.

Finally, let us consider the inclusion

$$j\colon \mathcal{D}(X) \hookrightarrow \mathcal{O}'(X).$$

Since $j \in \mathbf{DL}_0$ is essentially codense, there exists a morphism $g\colon \mathcal{O}'(X) \to \mathcal{D}(X)$ in $\mathbf{DL}_0$ such that $gj = 1$. Let $U \in \mathcal{O}'(X)$ be given. Then every $V \in \mathcal{D}(X)$ with $V \subset U$ satisfies $V \subset g(U)$. Hence $U \subset g(U)$. Similarly, every $V \in \mathcal{D}(X)$ with $V \supset U$ satisfies $V \supset g(U)$. This shows that $g(U)$ is the smallest $V \in \mathcal{D}(X)$ with $V \supset U$. For any $U \in \mathcal{O}(X)$ and $W \in \mathcal{D}(X)$, suppose that $g(U \cap W) \not\subset \overline{U \cap W}$. Then there is a non-empty $V \in \mathcal{D}(X)$ with $V \subset g(U \cap W) \smallsetminus \overline{U \cap W}$. Since V is closed, this contradicts the minimality of $g(U \cap W)$. Hence $\overline{U \cap W} = g(U \cap W) \in \mathcal{O}(X)$. As this holds for every $W \in \mathcal{D}(X)$, it follows that $\overline{U}$ is open. ∎

Now we are ready to prove the main result of this section.

THEOREM 3. *Let X be a schematic space. Up to isomorphism, there is a unique essentially dense morphism $e\colon \widetilde{X} \to X$ in* $\mathbf{GSp}$ *such that e factors through every essentially dense morphism $Y \to X$, and every essentially dense morphism $Y \to \widetilde{X}$ in* $\mathbf{GSp}$ *is a retraction.*

Proof. Consider the map

$$r\colon \mathcal{O}(X) \to \mathfrak{B}(X) \tag{18}$$

with $r(U) := \operatorname{int} \overline{U}$. We show first that r is a morphism in $\mathbf{DL}_b$. For $U, V \in \mathcal{O}(X)$, we have

$$\operatorname{int} \overline{U \cup V} \subset \operatorname{int} \overline{r(U) \cup r(V)} \subset \operatorname{int} \overline{\overline{U} \cup \overline{V}} = \operatorname{int} \overline{U \cup V}.$$

Hence $r(U \cup V) = r(U) \vee r(V)$. Furthermore, $r(U \cap V) = \operatorname{int} \overline{U \cap V} \subset \operatorname{int} \overline{U} \cap \operatorname{int} \overline{V} = r(U) \wedge r(V)$. Thus it remains to verify that $\operatorname{int} \overline{U} \cap \operatorname{int} \overline{V} \subset \operatorname{int} \overline{U \cap V}$. This means that $W := \operatorname{int} \overline{U} \cap \operatorname{int} \overline{V} \smallsetminus \overline{U \cap V}$ is empty. Now W is open, and $W \cap U \cap V = \emptyset$. Hence $W \cap U \subset \overline{V}$ implies that $W \cap U = \emptyset$. Therefore, $W \subset \overline{U}$ yields $W = \emptyset$. Since $r(\emptyset) = \emptyset$ and $r(X) = X$, it follows that $r \in \mathbf{DL}_b$.

The restriction of r to $\mathcal{D}(X)$ gives a morphism

$$r_0\colon \mathcal{D}(X) \to \mathfrak{C}(X)$$

in $\mathbf{DL}_0$. Since $r(U) \supset U$ for all $U \in \mathcal{D}(X)$, we infer that r_0 is codense. The fact that $\mathcal{D}(X)$ is a basis of X implies that r_0 is essentially codense. Now let $f\colon \mathcal{D}(X) \to D$ be an essentially codense morphism in $\mathbf{DL}_0$. We define a map $g\colon D \to \mathfrak{C}(X)$ by

$$g(a) := \operatorname{int}\overline{\bigcup\{U \in \mathcal{D}(X) \mid f(U) \le a\}}. \tag{19}$$

Before we show that the image of g lies in $\mathfrak{C}(X)$ and not only in $\mathfrak{B}(X)$, we verify first that $gf = r_0$. Thus let $V \in \mathcal{D}(X)$ be given. Then

$$r(V) = \operatorname{int}\overline{V} \subset \operatorname{int}\overline{\bigcup\{U \in \mathcal{D}(X) \mid f(U) \le f(V)\}} = gf(V).$$

So we have to prove that $gf(V) \subset \overline{V}$. Let $W \in \mathcal{D}(X)$ be such that $W \subset gf(V) \smallsetminus \overline{V}$. Then $W \cap V = \emptyset$. Therefore, if $U \in \mathcal{D}(X)$ satisfies $f(U) \le f(V)$, then $f(W \cap U) \le f(W) \wedge f(V) = f(W \cap V) = 0$. Since f is codense, this gives $W \cap U = \emptyset$. Hence $W \cap gf(V) = \emptyset$, and thus $W = \emptyset$. So we have proved that $gf = r_0$. Now every $a \in D$ can be majorized by some $f(V)$ with $V \in \mathcal{D}(X)$. Therefore, we have $g(a) \le gf(V) = r(V) \subset \overline{V}$, which shows that $g(D) \subset \mathfrak{C}(X)$.

Next we prove that $g \in \mathbf{DL}_0$. For $a, b \in D$, we have $g(a) \vee g(b) \subset g(a \vee b)$ since g is monotone. The reverse inclusion is equivalent to

$$\operatorname{int}\overline{\bigcup\{U \in \mathcal{D}(X) \mid f(U) \le a \vee b\}} \subset \operatorname{int}\overline{\bigcup\{U \in \mathcal{D}(X) \mid f(U) \le a\}} \vee \operatorname{int}\overline{\bigcup\{U \in \mathcal{D}(X) \mid f(U) \le b\}}. \tag{20}$$

Suppose that $W \in \mathcal{D}(X)$ is contained in the left-hand side, but disjoint from the right-hand side of (20). Then $W \cap U = \emptyset$ for all $U \in \mathcal{D}(X)$ with $f(U) \le a$ or $f(U) \le b$. Hence if $f(W) \wedge a > 0$, the essential codensity of f implies that $f(W) \wedge a \ge f(U)$ for some non-empty $U \in \mathcal{D}(X)$. But this gives $f(U) \le a$ and $f(W \cap U) = f(W) \wedge f(U) \ge f(U) > 0$, a contradiction. So we get $f(W) \wedge a = 0$, and similarly, $f(W) \wedge b = 0$. Thus $f(W) \wedge (a \vee b) = 0$. For all $U \in \mathcal{D}(X)$ with $f(U) \le a \vee b$, this implies that $f(W \cap U) = f(W \cap U) \wedge (a \vee b) = 0$. Hence $W \cap U = \emptyset$ for all such U, and therefore $W = \emptyset$, which completes the proof of (20). Thus $g(a \vee b) = g(a) \vee g(b)$. Furthermore, we have

$$\begin{aligned}
g(a \wedge b) &= r\Big(\bigcup\{U \in \mathcal{D}(X) \mid f(U) \le a \wedge b\}\Big) \\
&= r\Big(\bigcup\{U \cap V \mid U, V \in \mathcal{D}(X),\ f(U) \le a,\ f(V) \le b\}\Big) \\
&= r\Big(\bigcup\{U \in \mathcal{D}(X) \mid f(U) \le a\} \cap \bigcup\{V \in \mathcal{D}(X) \mid f(V) \le b\}\Big) \\
&= r\Big(\bigcup\{U \in \mathcal{D}(X) \mid f(U) \le a\}\Big) \wedge r\Big(\bigcup\{V \in \mathcal{D}(X) \mid f(V) \le b\}\Big) \\
&= g(a) \wedge g(b).
\end{aligned}$$

Since $g(0) = \emptyset$ and $gf = r_0$, we have shown that $g \in \mathbf{DL}_0$. By Stone duality, the morphism $e := \operatorname{Spec} r_0$ satisfies the first part of the conclusion.

In particular,

$$\widetilde{X} = \operatorname{Spec} \mathfrak{C}(X). \tag{21}$$

To show that every essentially dense morphism $Y \to \widetilde{X}$ is a retraction, we apply Theorem 2 and Proposition 3. So we have to verify that $\mathcal{D}(\widetilde{X}) = \mathfrak{C}(\widetilde{X})$. We regard this equation as a property of the lattice $\mathfrak{C}(X) \cong \mathcal{D}(\widetilde{X})$. Denote the natural isomorphism $\mathfrak{C}(X) \xrightarrow{\sim} \mathcal{D}(\widetilde{X})$ by $U \mapsto U'$. Thus let $U \in \mathfrak{C}(X)$ be given. To show that $U' \subset \widetilde{X}$ is regular, let $V \in \mathfrak{C}(X)$ be such that $V' \subset \overline{U'}$. This means that the implication

$$U \cap W = \emptyset \;\Rightarrow\; V \cap W = \emptyset$$

holds for all $W \in \mathfrak{C}(X)$, hence for all $W \in \mathcal{D}(X)$. Thus $V' \subset \overline{U'}$ is equivalent to $V \subset \overline{U}$. Hence $V \subset U$, which shows that $U' \in \mathfrak{C}(\widetilde{X})$. This yields $\mathcal{D}(\widetilde{X}) \subset \mathfrak{C}(\widetilde{X})$. Conversely, a set $U_0 \in \mathfrak{C}(\widetilde{X})$ can be represented by $\mathfrak{U} := \{U \in \mathfrak{C}(X) \mid U' \subset U_0\}$. So there exists some $V \in \mathfrak{C}(X)$ with $U_0 \subset \overline{V'}$, i.e. $U' \subset \overline{V'}$ for all $U \in \mathfrak{U}$. By the above, this gives $U \subset \overline{V}$ for all $U \in \mathfrak{U}$. The regularity of U_0 implies that

$$V' \subset \overline{U}_0 \;\Rightarrow\; V' \subset U_0 \tag{22}$$

for all $V \in \mathfrak{C}(X)$. Here $V' \subset \overline{U}_0$ means that

$$U_0 \cap W' = \emptyset \;\Rightarrow\; V' \cap W' = \emptyset \tag{23}$$

for all $W \in \mathfrak{C}(X)$. Now we define

$$V := \operatorname{int} \overline{\bigcup \mathfrak{U}}.$$

Thus $V \in \mathfrak{C}(X)$, and $U_0 \subset V'$. Assume that $U_0 \cap W' = \emptyset$ for some $W \in \mathfrak{C}(X)$. Then $U \cap W = \emptyset$ for all $U \in \mathfrak{U}$. Hence $V \cap W = \emptyset$. So the implication (23) holds, i.e. $V' \subset \overline{U_0}$. By (22), this gives $V' \subset U_0$. Thus $U_0 = V' \in \mathcal{D}(\widetilde{X})$, which proves $\mathcal{D}(\widetilde{X}) = \mathfrak{C}(\widetilde{X})$.

It remains to prove the uniqueness statement. Thus assume that the essentially dense morphisms $e\colon \widetilde{X} \to X$ and $f\colon X' \to X$ in **GSp** satisfy the conditions of the theorem. Then $f = eg$ for some $g\colon X' \to \widetilde{X}$ in **GSp**. Since e is essentially dense, g is dense. Therefore, Theorem 2 implies that g is a retraction, say, $gs = 1$ for some $s\colon \widetilde{X} \to X'$ in **GSp**. Hence $e = egs = fs$. Since f is essentially dense, this implies that s is dense. Again by Theorem 2, we infer that s is a retraction. Thus s is a homeomorphism. ∎

Let us call $\widetilde{X}$, together with the map $e = e_X\colon \widetilde{X} \to X$ of Theorem 3, the *essential cover* of X.

Corollary 1. *For a schematic space X, the following are equivalent.*

(a) $X \cong \widetilde{X}$.
(b) *Every dense morphism $Y \to X$ in* **GSp** *is a retraction.*

(c) *For every essentially dense morphism* $f\colon Y \to X$ *in* **GSp**, *the restriction* $f|_{Y_{\min}}$ *is a homeomorphism.*

Proof. The implications (c)⇒(a)⇒(b) follow by Proposition 3, Theorem 1, and Theorem 2.

(b)⇒(c): By Theorem 2, every essentially dense morphism $f\colon Y \to X$ is a retraction. So there exists a morphism $s\colon X \to Y$ in **GSp** with $fs = 1$. Since f is essentially dense, s is dense. Furthermore, Proposition 3 and Theorem 2 imply that $\mathcal{D}(X) = \mathfrak{C}(X)$. Hence X is minimal by Theorem 1. This shows that $s(X) = Y_{\min}$. Since s maps X homeomorphically onto $Y_{\min}$, the corollary is proved. ∎

COROLLARY 2. *Let* $f\colon Y \to X$ *be an essentially dense morphism in* **GSp** *which factors through every essentially dense morphism* $Z \to X$ *in* **GSp**. *Then* Y *is extremally disconnected*, $Y_{\min} \hookrightarrow Y$ *is a section*, *and the composition* $Y_{\min} \hookrightarrow Y \xrightarrow{f} X$ *is an essential cover of* X.

Proof. Let $e_X\colon \widetilde{X} \to X$ be an essential cover. By assumption, $f = e_X g$ for some $g\colon Y \to \widetilde{X}$ in **GSp**. Since e_X and f are essentially dense, g is essentially dense. Now let $U \in \mathcal{O}(Y)$ be given. Then $g_*(Y \smallsetminus \overline{U})$ is open, and $g^{-1}(g_*(Y \smallsetminus \overline{U})) \cap U = \emptyset$. Suppose that $g^{-1}(\overline{g_*(Y \smallsetminus \overline{U})}) \cap U \neq \emptyset$. Since $\widetilde{X}$ is extremally disconnected and g essentially dense, there is some non-empty $V \in \mathcal{D}(\widetilde{X})$ such that $g^{-1}(V) \subset g^{-1}(\overline{g_*(Y \smallsetminus \overline{U})}) \cap U$. Since g is dense, this gives $V \subset \overline{g_*(Y \smallsetminus \overline{U})}$, whence $V \subset \overline{g_*(Y \smallsetminus \overline{U})} \smallsetminus g_*(Y \smallsetminus \overline{U})$, a contradiction. Thus $g^{-1}(\overline{g_*(Y \smallsetminus \overline{U})}) \cap \overline{U} = \emptyset$, which implies that $g_*(Y \smallsetminus \overline{U})$ is open and closed. Since g is essentially dense, we infer that $g^{-1}(g_*(Y \smallsetminus \overline{U})) = Y \smallsetminus \overline{U}$. Hence $Y \smallsetminus \overline{U}$ is closed, and thus $\overline{U}$ is open. This proves that Y is extremally disconnected. By Corollary 1, the restriction $g|_{Y_{\min}}$ is a homeomorphism. Now the remaining assertions follow immediately. ∎

4. The essential cover as a functor. To make the essential cover into a functor, we have to restrict the morphisms in **GSp**. For example, the continuous map $g\colon \mathbb{S} \to \mathbb{S}$ (cf. Example 1) with $g(\mathbb{S}) = \{0\}$ does not leave $\mathbb{S}_{\min} = \{1\}$ invariant.

DEFINITION 4. We call a morphism $f\colon X \to Y$ in **GSp** *regular* if

$$f^{-1}(r(V)) \subset r(f^{-1}(V)) \tag{24}$$

for all $V \in \mathcal{D}(V)$, where r is given by (18).

Note that spectral maps between locally Stone spaces are regular. We have the following characterization:

PROPOSITION 8. *A morphism* $f\colon X \to Y$ *in* **GSp** *is regular if and only if there exists a spectral map* $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ *such that the diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ \downarrow{\scriptstyle e_X} & & \downarrow{\scriptstyle e_Y} \\ X & \xrightarrow{f} & Y \end{array} \tag{25}$$

commutes.

Proof. By Definition 4, f is regular if and only if the open set $f^{-1}(r(V)) \setminus \overline{f^{-1}(V)}$ is empty for all $V \in \mathcal{D}(Y)$. This means that the implication

$$U \subset f^{-1}(r(V)),\ U \cap f^{-1}(V) = \emptyset \ \Rightarrow\ U = \emptyset \tag{26}$$

holds for all $U \in \mathcal{D}(X)$ and $V \in \mathcal{D}(Y)$. The left-hand side of (26) states that $f(U) \subset r(V)$ and $f(U) \cap V = \emptyset$, i.e. $f(U) \subset r(V) \setminus V$. Since U is quasi-compact, $f(U)$ is quasi-compact. Therefore, the inclusion $f(U) \subset r(V) \setminus V$ means that $f(U) \subset W \setminus V$ for some $W \in \mathcal{D}(Y)$ with $W \subset \overline{V}$.

Now assume that f is not regular. Then there is a non-empty $U \in \mathcal{D}(X)$ with $f(U) \subset Y \setminus Y_{\min}$. Since $X_{\min}$ is dense in X, there is an element $x \in U \cap X_{\min}$. By Proposition 4, the image of e_X is a dense patch, whence $X_{\min} \subset e_X(\widetilde{X})$ by Theorem 1. So there is an element $\widetilde{x} \in \widetilde{X}$ with $e_X(\widetilde{x}) = x$. Thus $fe_X(\widetilde{x}) \in f(U) \subset Y \setminus Y_{\min}$. On the other hand, e_Y factors through the essentially dense map $Y_{\min} \hookrightarrow Y$. Therefore, a commutative diagram (25) would give $fe_X(\widetilde{x}) = e_Y\widetilde{f}(\widetilde{x}) \in Y_{\min}$, a contradiction.

Conversely, assume that f is regular. For any $U \in \mathfrak{C}(Y)$ with $U \subset \overline{V}$ and $V \in \mathcal{D}(Y)$, this implies that $f^{-1}[U] = r(f^{-1}(U)) = rf^{-1}(r(V)) \subset rf^{-1}(V)$. Therefore, the map (6) defines a morphism $f^*\colon \mathfrak{C}(Y) \to \mathfrak{C}(X)$. For every $V \in \mathcal{D}(Y)$, we have $rf^{-1}(V) \subset rf^{-1}(r(V)) \subset rf^{-1}(V)$, which gives $rf^{-1}(V) = f^{-1}[r(V)]$, i.e. the diagram

$$\begin{array}{ccc} \mathcal{D}(Y) & \xrightarrow{\mathcal{D}(f)} & \mathcal{D}(X) \\ \downarrow{\scriptstyle r_0} & & \downarrow{\scriptstyle r_0} \\ \mathfrak{C}(Y) & \xrightarrow{f^*} & \mathfrak{C}(X) \end{array} \tag{27}$$

commutes. Hence $\widetilde{f} := \operatorname{Spec} f^*$ yields a commutative diagram (25). ∎

A continuous map $f\colon X \to Y$ is said to be *skeletal* [4] if the inverse image $f^{-1}(V)$ of a dense $V \in \mathcal{O}(Y)$ is dense in X. The subcategory of **GSp** with the same objects and skeletal maps as morphisms will be denoted by **SSp**. Let **ELSt** denote the full subcategory of **SSp** whose objects are extremally disconnected locally compact spaces. The proof of Proposition 8 shows that the essential cover provides a functor

$$\mathbf{SSp} \to \mathbf{ELSt} \tag{28}$$

which maps $f\colon X \to Y$ in **SSp** to $\widetilde{f} = \operatorname{Spec} f^*$, where $f^*\colon \mathfrak{C}(Y) \to \mathfrak{C}(X)$ is given by $f^*(V) := f^{-1}[V]$. In fact, the commutative diagram (25) shows that $\widetilde{f}$ is skeletal: If $W \in \mathcal{O}(\widetilde{Y})$ is dense, there is a dense $V \in \mathcal{O}(Y)$ with $e_Y^{-1}(V) \subset W$. Hence $\widetilde{f}^{-1}e_Y^{-1}(V) = e_X^{-1}f^{-1}(V)$ is dense in $\widetilde{X}$. Furthermore, a composite gf of skeletal maps f, g satisfies $f^*g^*(V) = rf^{-1}rg^{-1}(V) = rf^{-1}g^{-1}(V) = (gf)^*(V)$.

Proposition 9 (cf. [4, Lemma 4]). *Every essentially dense morphism in* **GSp** *is skeletal, and every skeletal morphism is regular. If $f \in$* **GSp** *is skeletal, the morphism $\widetilde{f}$ in the commutative diagram* (25) *is unique.*

Proof. Let $f\colon X \to Y$ be a morphism in **GSp**. Assume that f is essentially dense. If $V \in \mathcal{O}(Y)$ is dense in Y, then $f(f^{-1}(V)) = V \cap f(X)$ is dense in Y. Hence $f^{-1}(V) \subset X$ is dense by Proposition 1. Thus f is skeletal.

Assume now that f is skeletal. If $V \in \mathcal{D}(Y)$, then $V \cup (Y \smallsetminus \overline{V}) \in \mathcal{O}(Y)$ is dense. Hence $f^{-1}(V \cup (Y \smallsetminus \overline{V}))$ is dense in X, and thus $\operatorname{int} f^{-1}(\overline{V} \smallsetminus V) = \emptyset$. To verify (24), we have to show that $f^{-1}(r(V)) \subset \overline{f^{-1}(V)}$. Thus let $U \in \mathcal{D}(X)$ satisfy $U \subset f^{-1}(r(V)) \smallsetminus \overline{f^{-1}(V)}$. Then $f(U) \subset r(V)$ and $U \cap f^{-1}(V) = \emptyset$. Hence $f(U) \subset r(V) \smallsetminus V \subset \overline{V} \smallsetminus V$, which yields $U = \emptyset$. This proves that f is regular. By Proposition 8, there is a commutative diagram (27), and it remains to show that f^* is unique. Let $U \in \mathfrak{C}(Y)$ be given. For $U \in \mathcal{D}(Y)$ with $V \subset U$, this implies that $r(V) \subset U$, and thus $rf^{-1}(V) = f^*r(V) \subset f^*(U)$. Hence $f^{-1}[U] \subset f^*(U)$. To prove the reverse inclusion, we have to verify that $f^*(U) \subset \overline{f^{-1}(U)}$. Assume that $W \subset f^*(U) \smallsetminus \overline{f^{-1}(U)}$ for some $W \in \mathcal{D}(X)$. Then $W \cap f^{-1}(U) = \emptyset$. For every $V \in \mathcal{D}(Y)$ with $U \cap V = \emptyset$, we have $U \cap r(V) = \emptyset$, which gives $f^*(U) \cap rf^{-1}(V) = f^*(U) \cap f^*(r(V)) = \emptyset$. Hence $W \cap f^{-1}(V) = \emptyset$, and thus $f(W) \cap V = \emptyset$. So we get $f(W) \subset \overline{U}$, which yields $W \subset f^{-1}(\overline{U} \smallsetminus U)$. Since f is skeletal, this implies that $W = \emptyset$. Thus $f^*(U) = f^{-1}[U]$. ∎

Now we can give a functorial characterization of the essential cover. Recall that a subcategory of any category is said to be *coreflective* if the inclusion admits a right adjoint.

Theorem 4. *The full subcategory* **ELSt** *of extremally disconnected locally Stone spaces in* **SSp** *is coreflective.*

Proof. Let $f\colon X \to Y$ be a morphism in **SSp** with $X \in$ **ELSt**. By Corollary 1 of Theorem 3, the map e_X is a homeomorphism. Proposition 9 implies that the morphism $\widetilde{f}$ in (25) is unique. Therefore, the functor (28) is right adjoint to the inclusion **ELSt** $\hookrightarrow$ **SSp**. ∎

5. The absolute cover. Let $\mathcal{C}$ be an arbitrary category with a full subcategory $\mathcal{P}$. A morphism $p\colon P \to X$ is said to be a $\mathcal{P}$-*precover* if $P \in \mathcal{P}$ and every morphism $Q \to X$ with $Q \in \mathcal{P}$ factors through p. If, in addition, p is *minimal*, i.e. every morphism $f\colon P \to P$ with $pf = p$ is an automorphism, then p is called a $\mathcal{P}$-*cover*.

By Theorem 3, an essential cover $e\colon \widetilde{X} \to X$ in **GSp** is minimal. In fact, if $ef = e$, then f is essentially dense, hence a retraction. Thus $fg = 1$ for some $g\colon \widetilde{X} \to \widetilde{X}$, and $eg = efg = e$. Therefore, g is again a retraction, whence f is an automorphism with inverse g. In the subcategory **SSp**, the essential cover is also an **ELSt**-precover by Theorem 4, hence an **ELSt**-cover.

In [24] the first author has shown that in the category **Top** of topological spaces with continuous maps, every space X admits a cover $p\colon P \to X$ with respect to the full subcategory **Ed** of extremally disconnected spaces, and that p coincides with the *absolute* [23, 25, 27] of X. To distinguish p from the object P, we will also call P the *absolute*, and refer to p as the *absolute cover* of X. Let us briefly review the main features of the absolute, as far as needed for our present purpose.

Let $f\colon X \to Y$ be a continuous map between topological spaces. Then f is said to be *proper* [7] if $f \times 1\colon X \times Z \to Y \times Z$ is closed for each $Z \in$ **Top**. By [7, I.10.2, Theorem 1], f is proper if and only if f is closed and has quasi-compact fibers. The map f is said to be *separated* [5, 27] if the diagonal map $X \to X \times_Y X$ is closed, i.e. if any two points $x \neq y$ in X with $f(x) = f(y)$ have disjoint neighbourhoods. A separated proper map is said to be *perfect*. We call f an *absolute* (*cover*) of Y if f is essentially dense and perfect, and X is extremally disconnected. By [24, Theorem 1], an absolute is an **Ed**-cover, hence unique up to homeomorphism.

Recall [8] that every topological space X is the union of its *irreducible components*, i.e. maximal irreducible subspaces. We will show that the absolute leads to a refined version of irreducible components which may have a multiplicity and need not be maximal.

Proposition 10. *Let $f\colon X \to Y$ be a perfect map in* **Top**. *Every irreducible component of X is mapped bijectively onto a closed irreducible subset of Y. If f is surjective, each irreducible component of Y arises in this way.*

Proof. By [8, II.4.1, Proposition 4], the image of an irreducible component C of X is irreducible. Since f is perfect, $f(C)$ is closed. Suppose that there are different points $x, y \in C$ with $f(x) = f(y)$. Since f is separated, there are disjoint $U, V \in \mathcal{O}(X)$ with $x \in U$ and $y \in V$. As C is irreducible, we have $U \cap V \neq \emptyset$, a contradiction. Thus $f|_C$ is injective.

Now let f be surjective, and let D be an irreducible component of Y. Then the closed set $f^{-1}(D)$ is mapped onto D. Let $\mathfrak{A}$ be a chain of closed subsets $A \subset f^{-1}(D)$ with $f(A) = D$. With $C := \bigcap \mathfrak{A}$, suppose that $f(C) \neq D$.

Then there is a point $y \in D \smallsetminus f(C)$, and $\bigcap\{f^{-1}(y) \cap A \mid A \in \mathfrak{A}\} = \emptyset$. Since $f^{-1}(y)$ is compact, this is impossible. Hence $f(C) = D$, and the Kuratowski–Zorn lemma yields a minimal closed subset A of $f^{-1}(D)$ with $f(A) = D$. If $A = A_1 \cup A_2$ with A_1, A_2 closed, then $D = f(A_1) \cup f(A_2)$. Since f is closed and D irreducible, this implies that $D = f(A_i)$ for some $i \in \{1, 2\}$. By the minimality of A, this gives $A_i = A$. Hence A is irreducible. By [8, II.4.1, Proposition 5], there is an irreducible component A' of X with $A \subset A'$. As $f(A')$ is irreducible, we have $f(A') = D$. Since $f|_{A'}$ is injective, this implies that $A' = A$. ∎

Recall that the *quasicomponent* of a point x of a topological space is defined to be the intersection of all closed open sets containing x.

PROPOSITION 11. *The irreducible components of an extremally disconnected space X coincide with its quasicomponents.*

Proof. Let C be an irreducible component of X. Then every closed open set which intersects C non-trivially contains all of C. Hence C is contained in a quasicomponent C'. If $U \in \mathcal{O}(X)$ satisfies $U \cap C' \neq \emptyset$, then $\overline{U}$ is open, whence $C' \subset \overline{U}$. Thus C' is irreducible, which gives $C' = C$. ∎

DEFINITION 5. Let X be a topological space, and $p\colon P \to X$ a fixed absolute of X. We define the *irreducible components* of X to be the restrictions of p to the quasicomponents of P.

In other words, the quasicomponents of the absolute P of X parametrize the irreducible components of X. The following example shows that multiple components as well as embedded components actually occur.

EXAMPLE 3. Let $X := X' \sqcup \{0\}$ be the spectral space of Example 2. Then the Stone-Čech compactification $\beta(X')$ of X' is extremally disconnected (see [14, 6M]). Let $i\colon X' \hookrightarrow \beta(X')$ be the natural inclusion. Then i is an open map. Consider the disjoint union $P := X' \sqcup \beta(X')$ with the topology given by the open sets $U \sqcup V$ such that $V \in \mathcal{O}(\beta(X'))$ and $i^{-1}(V) \subset U \subset X'$. Then $i^{-1}(\overline{V}) \subset U$, which implies that $\overline{U \sqcup V} = U \sqcup \overline{i(U) \cup V}$. Hence P is extremally disconnected. Define a map $p\colon P \to X$ by $p(x) := x$ for all $x \in X'$, and $p(\beta(X')) = \{0\}$. Thus p is a continuous surjection. By definition, the closed sets in P are of the form $U \sqcup V$ with $V \subset \beta(X')$ closed and $U \subset i^{-1}(V)$. Hence p is closed. Since $\beta(X')$ is a compact subspace of P, the map p is perfect. Finally, every non-empty $U \sqcup V \in \mathcal{O}(P)$ satisfies $U \neq \emptyset$. Hence $p^{-1}(U) \subset U \sqcup V$, which shows that p is essentially dense. Thus p is the absolute cover of X.

The closed open sets of P are the sets $i^{-1}(V) \sqcup V$ with $V = \overline{V} \in \mathcal{O}(\beta(X'))$. Therefore, the quasicomponents of P are the singletons in $\beta(X') \smallsetminus i(X')$ and the two-point sets $\{x, i(x)\}$ with $x \in X'$. So the irreducible components of X are $\{0\}$, with a big multiplicity, and the sets $\{0, x\}$ with $x \in X'$.

Now we turn our attention to schematic spaces.

PROPOSITION 12. *Let $p\colon P \twoheadrightarrow X$ be the absolute of a schematic space X. Then P is again schematic, and p is a spectral map.*

Proof. Since p is proper, [7, I.10.2, Proposition 6] implies that $p^{-1}(V)$ is quasi-compact for each $V \in \mathcal{D}(X)$. As p is separated, P is a T_0-space. By [24, Proposition 5], the fact that p is perfect implies that the sets $U \cap p^{-1}(V)$ with $U \in \mathfrak{B}(P)$ and $V \in \mathcal{O}(X)$ form an open basis of P. By the corollary of Proposition 2, the regular open sets of P are of the form $p^{-1}[W]$ with $W \in \mathcal{O}(X)$. Therefore, since P is extremally disconnected, an open basis of P is given by the sets

$$p^{-1}(V) \cap \overline{p^{-1}(W)} \tag{29}$$

with $V \in \mathcal{D}(X)$ and $W \in \mathcal{O}(X)$. Furthermore, the sets (29) are quasi-compact. In fact, if $p^{-1}(V) \cap \overline{p^{-1}(W)} \subset \bigcup_{i\in I} U_i$ with $U_i \in \mathcal{O}(P)$, then $p^{-1}(V) \subset (P \smallsetminus \overline{p^{-1}(W)}) \cup \bigcup_{i\in I} U_i$. As $p^{-1}(V)$ is quasi-compact, this implies that $p^{-1}(V) \subset (P \smallsetminus \overline{p^{-1}(W)}) \cup \bigcup_{j\in J} U_j$ for a finite subset $J \subset I$. Hence $p^{-1}(V) \cap \overline{p^{-1}(W)} \subset \bigcup_{j\in J} U_j$. So we have shown that P admits a basis (29) of quasi-compact open sets. Therefore, every $U \in \mathcal{D}(P)$ is a finite union of sets (29).

Since P is extremally disconnected, the equality

$$\overline{U_1 \cap U_2} = \bar{U}_1 \cap \bar{U}_2 \tag{30}$$

holds for all $U_1, U_2 \in \mathcal{O}(P)$. Therefore, the intersection of two sets (29) is again of the same form.

Finally, let $C \subset P$ be closed and irreducible. Then $p(C)$ is closed and irreducible. So there is a generic point $y \in p(C)$. Choose $x \in C$ with $p(x) = y$. Then $p(\overline{\{x\}})$ is closed and contains y. Thus $p(\overline{\{x\}}) = p(C)$. Since $p|_C$ is injective by Proposition 10, we infer that $\overline{\{x\}} = C$. ∎

For a topological space X, let $Q(X)$ denote the space of quasicomponents (see [22, 46.Va]). An open basis for the topology of $Q(X)$ is given by the sets $U \subset Q(X)$ with $c^{-1}(U)$ closed and open. Thus $Q(X)$ is 0-dimensional, hence a Tikhonov space, and we have a continuous surjection

$$q\colon X \twoheadrightarrow Q(X) \tag{31}$$

with the quasicomponents of X as fibers. Our final theorem characterizes the essential cover of a schematic space in terms of the absolute.

THEOREM 5. *Let X be a schematic space with absolute $p\colon P \twoheadrightarrow X$. For each quasicomponent C of P, let $e(C)$ denote the generic point of $p(C)$. Then $e\colon Q(P) \to X$ is an essential cover of X.*

Proof. Consider the natural map $q\colon P \twoheadrightarrow Q(P)$, and the map $s\colon Q(P) \to P$ which associates the generic point to each quasicomponent of P. Then

$e = ps$ and $qs = 1$. By Proposition 11, the image of s coincides with the subspace μP of maximal points. Proposition 6 and Theorem 1 imply that μP is a locally Stone space. By the definition of $Q(P)$, the map q preserves closed open sets. Since μP is 0-dimensional, it follows that $q|_{\mu P}$ is a homeomorphism. Hence s is a continuous embedding. Furthermore, s is dense, and thus essentially dense. So the composition $e = ps$ is essentially dense, and $Q(P)$ is extremally disconnected. By Proposition 9, e is skeletal. Thus Theorem 4 implies that e factors through the essential cover of X. Hence e is an essential cover by Theorems 2 and 3. ∎

Remark. By Theorem 5, the essential cover $\widetilde{X}$ of a schematic space X can be regarded as the space of irreducible components of X in the sense of Definition 5. Since μX need not coincide with $X_{\min}$, this gives a general reason why embedded components of X are possible.

Corollary. *For an extremally disconnected schematic space* X, *the map* (31) *is a retraction.*

Proof. Since the identity $1_X \colon X \to X$ is an absolute cover, the continuous map e of Theorem 5 satisfies $qe = 1$. ∎

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