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WEAK MULTIPLICATION MODULES OVER A PULLBACK OF DEDEKIND DOMAINS

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Abstract. Let R be the pullback, in the sense of Levy [J. Algebra 71 (1981)], of two local Dedekind domains. We classify all those indecomposable weak multiplication R-modules M with finite-dimensional top, that is, such that $M/\mathrm{Rad}(R)M$ is finite-dimensional over $R/\mathrm{Rad}(R)$. We also establish a connection between the weak multiplication modules and the pure-injective modules over such domains.

1. Introduction. One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary ring R. The reader is referred to [1], [28, Chapters 1 and 14], [30, Chapter 19] and [28] for a detailed discussion of classification problems, representation types (finite, tame, or wild), and useful computational reduction procedures; see also a recent paper [29] for a discussion of the notion of wild representation type for module classification problems.

Nazarova and Roĭter [21] (also see [5], [22] and [25]) described all finitely generated modules over the pullback of two local Dedekind rings R_1, R_2 for which the residue fields are isomorphic, say to $\bar{R}$. Nazarova and Roĭter used this to describe all finitely generated $\mathbb{Z}_pG_p$-modules (p prime, $\mathbb{Z}_p$ the p-localization of $\mathbb{Z}$), hence all finite $\mathbb{Z}G_p$-modules. Their method was to reduce the problem to a matrix problem over $\bar{R}$ and then solve the matrix problem (see [27, Chapters 1 and 16]). Their results were extended in [16], by allowing R_1 and R_2 to be arbitrary Dedekind domains, with $\bar{R}$ still a field (but the reduction to a matrix problem was done differently). This permitted the classification of all finitely generated (rather than just finite) $\mathbb{Z}G_p$-modules, as well as modules over many subrings of $\mathbb{Z}\oplus\mathbb{Z}$. Dedekind-like rings (see [15]) have cyclic index in their integral closure (see [15, Lemma 1.1]). Equivalently, every ideal of R is generated by two elements. Thus there is some overlap with results of Bass [5]. However, he only studied torsion-free modules.

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Indecomposable weak multiplication modules over a Dedekind domain have been classified in [9]. Let R be the pullback of two local Dedekind domains over a common factor field. Indecomposable pure-injective modules (and also indecomposable representable and indecomposable prime modules) with finite-dimensional top (for any module M we define its *top* as $M/\mathrm{Rad}(R)M$) have been classifed in [6–8], and here we follow the idea of those papers. More precisely, our purpose is to give a complete description of the indecomposable weak multiplication R-modules with finite-dimensional top over R. The classification is divided into two stages: first, we describe all indecomposable separated weak multiplication R-modules, and then we show that non-separated indecomposable weak multiplication R-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable weak multiplication R-modules. Then we use the classification of separated indecomposable weak multiplication modules from Section 2, together with results of Levy [15], [16] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable weak multiplication modules M with finite-dimensional top (see Theorem 3.12). We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable weak multiplication modules (where infinite length weak multiplication modules can occur only at the ends) and where adjacency corresponds to amalgamation in the socles of these separated weak multiplication modules.

It is well-known that, for a vast majority of rings, the classification of arbitrary modules is impossible. In the present paper we introduce a new class of R-modules, called weak multiplication modules (see Definition 1.2), and we study them in detail from the classification point of view. We are mainly interested in the case where R is a pullback of two local Dedekind domains. For any field k, the infinite-dimensional k-algebra $T = k[x, y : xy = 0]_{(x,y)}$ is the pullback $(k[x]_{(x)} \to k \leftarrow k[y]_{(y)})$ of the local Dedekind domains $k[x]_{(x)}$ and $k[y]_{(y)}$. This paper includes the classification of indecomposable weak multiplication modules with finite-dimensional top over T. The above example illustrates the difficulties in extending the classification to arbitrary weak multiplication modules over T: the k-algebra T has, among its factor algebras, the "Gelfand–Ponomarev" algebras $k[x, y : xy = 0 = x^n = y^m]$. These are algebras of tame, non-domestic (for $n + m \geq 5$) representation type (see [1], [27, Chapter 14] and [30, Chapter XIX]) and the classification of the indecomposable weak multiplication modules over these has not yet been achieved (at least as far as we are aware). We show that every indecomposable non-separated weak multiplication R-module is pure-injective (Corollary 3.13). It seems that the classification of those indecomposable pure-injectives over a pullback ring which have infinite-dimensional top is a very difficult problem (see e.g. [3], [24]).

For the sake of completeness, we state some definitions and notations used throughout. In this paper all rings are commutative with identity and all modules unitary. Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two local Dedekind domains R_i onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinatewise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [14]).

Definition 1.1. An R-module S is defined to be *separated* if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2)$).

Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [14, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also S is separated if and only if $P_1S \cap P_2S = 0$ [14, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module; indeed, every R-module has a "minimal" such representation: a separated representation of an R-module M is an epimorphism $\varphi : S \to M$ of R-modules where S is separated and, if φ admits a factorization $\varphi : S \xrightarrow{f} S' \to M$ with S' separated, then f is one-to-one. The module $K = \mathrm{Ker}(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [14, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [14, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [14, Theorem 2.8]. Moreover, R-homomorphisms lift to separated representations, preserving epimorphisms and monomorphisms [14, Theorem 2.6].

If R is a ring and N is a submodule of an R-module M, then the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Thus $(0 : M)$ is the annihilator of M. A proper submodule N of a module M over a ring R is said to be a *prime submodule* if whenever $rm \in N$ for some $r \in R$, $m \in M$, then $m \in N$ or $r \in (N : M)$, so $(N : M) = P$ is a prime ideal of R, and N is said to be a *P-prime submodule*. The set of all prime submodules in an R-module M is denoted $\mathrm{Spec}(M)$.

Definition 1.2.

(a) An R-module M is defined to be a *weak multiplication module* if $\mathrm{Spec}(M) = \emptyset$ or for every prime submodule N of M, $N = IM$ for some ideal I of R (note that we can take $I = (N : M)$).

(b) An R-module M is defined to be a *multiplication module* if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M)$.
(c) We say that an R-module M is *prime* if the zero submodule of M is a prime submodule of M (so if N is a prime R-submodule of M, then M/N is a prime R-module).

Let M be an R-module and N a submodule of M. Call N a *pure submodule* of M if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called *relatively divisible* (or an *RD-submodule*) in M if $rN = N \cap rM$ for all $r \in R$. A module M is *pure-injective* if it has the injective property relative to all pure exact sequences [23]. An important property of modules N, M over a Dedekind domain is that N is pure in M if and only if N is an RD-submodule of M (see [13] and [31] for more details). In particular, by [13] and [31], an R-module is pure-injective if and only if it is algebraically compact (see also [26] and [12]). The indecomposable weak multiplication modules over discrete valuation domains are known.

Proposition 1.3 (see [9, Proposition 3.3]). *If R is a discrete valuation domain with a unique maximal ideal P then the indecomposable weak multiplication R-modules are*: R, R/P^n $(n \geq 1)$, $E(R/P)$, *the injective hull of* R/P, *and* $Q(R)$, *the field of fractions of* R.

Theorem 1.4 (see [9, Theorem 3.5]). *Let R be a discrete valuation domain. Then the following hold*:

(i) *If $M \neq R$ is a torsion-free weak multiplication R-module, then M is a direct sum of copies of $Q(R)$.*
(ii) *If M is a torsion weak multiplication R-module, then M is a direct sum of copies of R/P^n $(n \geq 1)$ and $E(R/P)$.*

Throughout this paper we shall assume, unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \tag{1}$$

is the pullback of two local Dedekind domain R_1, R_2 with maximal ideals P_1, P_2 generated respectively by p_1, p_2; moreover, P denotes $P_1 \oplus P_2$, and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field.

In particular, R is a commutative noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is, $P_1 \oplus 0$) and P_2 (that is, $0 \oplus P_2$).

2. The separated case. In this section we determine the indecomposable weak multiplication separated R-modules where R is the pullback of two local Dedekind domains (we do not need the a priori assumption of finite-dimensional top for this classification). We need the following lemma proved in [18, Lemma 4.1].

LEMMA 2.1. *Let R be a ring, and let M be an R-module. Let $K \subseteq N$ be submodules of M. Then N is a prime submodule of M if and only if N/K is a prime submodule of M/K.*

LEMMA 2.2. *Let I be an ideal of a commutative ring R, M a weak multiplication R-module, and N a non-zero R-submodule of M with $I \subseteq (N : M)$. Then M/N is a weak multiplication R/I-module.*

Proof. Let L be a prime submodule of M/N. Then $L = K/N$ for some prime submodule K of M by 2.1. So $K = (K : M)M$, since M is a weak multiplication module. An inspection will show that $(L :_{R/I} M/N)M/N = L$. ■

LEMMA 2.3. *Let M be a weak multiplication module over a commutative ring R. Then the following hold:*

(i) *If N is a submodule of M, then M/N is a weak multiplication R-module.*
(ii) *Every direct summand of M is a weak multiplication R-module.*

Proof. (i) Take $I = 0$ in Lemma 2.2. (ii) follows from (i). ■

LEMMA 2.4. *Let M be an R-module, N a P-prime submodule of M, and I an ideal of R with $I \subseteq (0 : M)$. Then N is a P/I-prime submodule of M as an R/I-module.*

Proof. Since N is a P-prime submodule of M, it follows that M/N is a P-prime R-module. Then M/N is P/I-prime as an R/I-module by [8, Lemma 2.2]; hence $(0 :_{R/I} M/N) = (N :_{R/I} M) = P/I$, as needed. ■

LEMMA 2.5. *Let R and R' be any commutative rings, $f : R \to R'$ a surjective homomorphism, and M an R'-module. Then the following hold:*

(i) *If N is a prime R-submodule of M, then N is a prime R'-submodule of M.*
(ii) *If M is a weak multiplication R'-module, then M is a weak multiplication R-module.*

Proof. (i) Since M/N is a prime R-module, [8, Lemma 2.1] shows M/N is a prime R'-module; hence N is a prime R'-submodule of M.

(ii) Let N be a prime R-submodule of M. Then N is a prime R'-submodule of M, so $N = I'M$ for some ideal I' of R'. Set $I = f^{-1}(I')$. Then I is an ideal of R and $f(I) = f(f^{-1}(I')) = I' \cap f(R) = I'$; hence $IM = f(I)M = N$. ■

PROPOSITION 2.6. *Let R be the pullback ring as in* (1), *and let*

$$S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$$

be any separated R-module. Then the following hold:

(i) *If S has a P-prime R-submodule T, then T/P_2S is a P_1-prime submodule of S_1 and T/P_1S is a P_2-prime submodule of S_2. In particular, $S_i \neq 0$ for $i = 1, 2$.*

(ii) *If S has a $(P_1 \oplus 0)$-prime R-submodule T, then T/P_1S is a 0-prime submodule of S_2 and $S_1 = 0$.*

(iii) *If S has a $(0 \oplus P_2)$-prime R-submodule T, then T/P_2S is a 0-prime submodule of S_1 and $S_2 = 0$.*

Proof. (i) Since $(P_1 \oplus 0)S \subseteq PS \subseteq T$, Lemma 2.1 implies $T/(P_1 \oplus 0)S$ is a P-prime R-submodule of $S/(P_1 \oplus 0)S$ (note that $(T/(P_1 \oplus 0)S : S/(P_1 \oplus 0)S) = (T : S) = P$). As $(P_1 \oplus 0) \subseteq (0 : S/(P_1 \oplus 0)S)$, $T/(P_1 \oplus 0)S$ is a $P/(P_1 \oplus 0) \cong P_2$-prime submodule of $R/(P_1 \oplus 0) \cong R_2$-module $S_2 \cong S/(P_1 \oplus 0)S$, by Lemma 2.4. Similarly, T/P_2S is a P_1-prime submodule of S_1. Finally, it is clear that $S_i \neq 0$.

(ii) If T is a $(P_1 \oplus 0)$-prime submodule of S, then $(P_1 \oplus 0)S \subseteq T$ and $(0 \oplus P_2)S \nsubseteq T$ since $(P_1 \oplus 0) \cap (0 \oplus P_2) = 0$. We show that $S_1 = 0$. Suppose not. Let $s_1 \in S_1$. Then there is an element $s_2 \in S_2$ such that $f_1(s_1) = f_2(s_2)$, so $(s_1, s_2) \in S$; hence $(P_1 \oplus 0)(s_1, s_2) \subseteq T$ gives $P_1s_1 \subseteq T_1 = T/P_2T$. It follows that $P(s_1, 0) \subseteq T$, so $s_1 \in T_1$; hence $T_1 = T/P_2S = S_1$, which is a contradiction. Moreover, by Lemma 2.1, $T/(P_1 \oplus 0)S$ is a $(P_1 \oplus 0)$-prime submodule of R-module $S/(P_1 \oplus 0)S$; hence Lemma 2.4 shows $T/(P_1 \oplus 0)S$ is a 0-prime submodule of $R/(P_1 \oplus 0) \cong R_2$-module $S_2 \cong S/(P_1 \oplus 0)S$. The proof of (iii) is similar to that of (ii). ∎

Theorem 2.7. *Let R be the pullback ring as in* (1), *and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a separated R-module. Then* $\mathrm{Spec}(S) = \emptyset$ *if and only if* $\mathrm{Spec}(S_i) = \emptyset$ *for $i = 1, 2$.*

Proof. For the necessity, assume that $\mathrm{Spec}(S) = \emptyset$ and let π be the projection map of R onto R_i. Suppose that $\mathrm{Spec}(S_1) \neq \emptyset$ and let N_1 be a prime submodule of S_1, so N_1 is a prime R-submodule of $S/(0 \oplus P_2)S$; hence $\mathrm{Spec}(S) \neq \emptyset$, by Lemma 2.1, which is a contradiction. Similarly, $\mathrm{Spec}(S_2) = \emptyset$. The sufficiency is clear by Proposition 2.6. ∎

Theorem 2.8. *Let R be the pullback ring as in* (1), *and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a separated R-module. Then S is a weak multiplication R-module if and only if S_i is a weak multiplication R_i-module for $i = 1, 2$.*

Proof. By Theorem 2.7, we may assume that $\mathrm{Spec}(S) \neq \emptyset$. Assume that S is a weak multiplication R-module. Since $(0 \oplus P_2) \subseteq ((0 \oplus P_2)S :_R S)$, Lemma 2.3 shows $S_1 \cong S/(0 \oplus P_2)S$ is a weak multiplication $R/(0 \oplus P_2) \cong R_1$-module. Similarly, S_2 is a weak multiplication R_2-module. Coversely, assume that each S_i is a weak multiplication R_i-module. Let T be a prime submodule of S. We split the proof into two cases.

CASE 1: $(T : S) = P$. By Proposition 2.6(i), $S_i \neq 0$ for $i = 1, 2$. By a similar argument to that in Proposition 2.6, we find that T/P_1S is a P_2-prime submodule of R_2-module S_2, so $T/P_1S = P_2S_2$, since S_2 is weak multiplication. Similarly, $T/P_2S = P_1S_1$. Therefore, $T = PS$.

CASE 2: $(T : S) = P_1 \oplus 0$. By Proposition 2.6(ii), we must have $S_1 = 0$ and $T/(P_1 \oplus 0)S$ is a 0-prime R_2-submodule of S_2; hence $T/(P_1 \oplus 0)S = (T/(P_1 \oplus 0)S : S_2)S_2 = 0$. Therefore, $T = (P_1 \oplus 0)S$. For $(T : S) = 0 \oplus P_2$, we get $T = (0 \oplus P_2)S$, and the proof is complete. ∎

LEMMA 2.9. *Let R be the pullback ring as in* (1). *The following separated R-modules are indecomposable and weak multiplication*:

(I) $R = (R_1 \to \bar{R} \leftarrow R_2)$;
(II) $S = (E(R_1/P_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow E(R_2/P_2))$ *where* $E(R_i/P_i)$ *is the* R_i*-injective hull of* R_i/P_i *for* $i = 1, 2$;
(III) $S = (Q(R_1) \to 0 \leftarrow 0)$ *where* $Q(R_1)$ *is the field of fractions of* R_1;
(IV) $(0 \to 0 \leftarrow Q(R_2))$ *where* $Q(R_2)$ *is the field of fractions of* R_2;
(V) $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$ *for all positive integers* n, m.

Proof. By [6, Lemma 2.8], these modules are indecomposable. Weak multiplication follows from Proposition 1.3 and Theorem 2.8 (note that R is weak multiplication since for each $i = 1, 2$, R_i is multiplication). ∎

We refer to modules of type (II) in Lemma 2.9 as P_1-*Prüfer* and P_2-*Prüfer* respectively.

PROPOSITION 2.10. *Let R be the pullback ring as in* (1), *and let $S \neq R$ be a separated weak multiplication R-module. Then S is of the form $S = M \oplus N \oplus K$, where M is a direct sum of copies of modules as in* (II), *N is a direct sum of copies of modules as in* (III)–(IV), *and K is a direct sum of copies of modules as in* (V) *of Lemma* 2.9. *In particular*, *every separated weak multiplication R-module not isomorphic to R is pure-injective.*

Proof. Let T denote an indecomposable summand of S. Then we can write $T = (T_1 \to \bar{T} \leftarrow T_2)$, and T is a weak multiplication R-module by Lemma 2.3. We split the proof into three cases.

CASE 1. If $\mathrm{Spec}(T) = \emptyset$, then $\mathrm{Spec}(T_i) = \emptyset$ by Theorem 2.7, so $T_i = P_iT_i$ for each $i = 1, 2$ by Proposition 1.1; hence $T = PT = P_1T_1 \oplus P_2T_2 = T_1 \oplus T_2$. Therefore, $T = T_1$ or T_2 and so T is of type (II) by Proposition 1.3.

CASE 2. If T has a $(P_1 \oplus 0)$-prime R-submodule N, then N/P_1T is a 0-prime R_2-submodule of the weak multiplication module T_2 and $T_1 = 0$ (so $\bar{T} = 0$) by Proposition 2.6 and Theorem 2.8; hence T is of type (III). Similarly, if T has a $(0 \oplus P_2)$-prime R-submodule, then T is of type (IV).

Case 3. If T has a P-prime R-submodule $N = (N_1 \to \bar{N} \leftarrow N_2)$, then $PT \subseteq N \neq T$, so $PT \neq T$ (that is, $\bar{T} \neq 0$). Then by Proposition 2.6 and Theorem 2.8, we must have $P_1T_1 = N_1 \neq T_1$ and $P_2T_2 = N_2 \neq T_2$; hence for each $i = 1, 2$, T_i is torsion and it is not a divisible R_i-module (see [9, Proposition 3.3]). Then there are positive integers m, n and k such that $P_1^m T_1 = 0$, $P_2^k T_2 = 0$ and $P^n T = 0$. For $t \in T$, let $o(t)$ denote the least positive integer m such that $P^m t = 0$. Now choose $t \in T_1 \cup T_2$ with $\bar{t} \neq 0$ and $o(t)$ maximal. There exists a $t = (t_1, t_2)$ such that $o(t) = n$, $o(t_1) = m$ and $o(t_2) = k$. Then $R_i t_i$ is pure in T_i for $i = 1, 2$ (see [6, Theorem 2.9]). Thus, $R_1t_1 \cong R_1/(0 : t_1) \cong R_1/P_1^m$ is a direct summand of T_1, since R_1t_1 is pure-injective. Similarly, $R_2t_2 \cong R_2/P_2^k$ is a direct summand of T_2. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{T}$ generated by $\bar{t}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1t_1 = M_1 \to \bar{M} \leftarrow M_2 = R_2t_2)$. Then M is a direct summand of T; this implies that $T = M$, and T is as in (V) (see [6, Theorem 2.9]). ∎

Theorem 2.11. *Let R be the pullback ring as in* (1), *and let $S \neq R$ be an indecomposable separated weak multiplication R-module. Then S is isomorphic to one of the modules listed in Lemma* 2.9.

Proof. Apply Proposition 2.10 and Lemma 2.9. ∎

3. The non-separated case. We continue to use the notation already established, so R is a pullback ring as in (1).

In this section we find the indecomposable non-separated weak multiplication modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable weak multiplication modules. We begin by describing one indecomposable non-separated weak multiplication module, namely the injective hull of the unique simple module.

For $i = 1, 2$, let E_i be the R_i-injective hull of R_i/P_i, regarded as an R-module (so E_1, E_2 are as in (II) of Lemma 2.9). Set $A_n = \mathrm{Ann}_{E_1}(P_1^n)$, and $B_n = \mathrm{Ann}_{E_2}(P_2^n)$ ($n \geq 1$). Then A_n is a cyclic R_1-module, say $A_n = R_1a_n$, and we may choose a_n so that $a_n = p_1a_{n+1}$ for each $n \geq 0$. Also $p_1a_0 = 0$ and $R_1a_0 \cong R/P$. Similarly, B_n is a cyclic R_2-module with $B_n = R_2b_n$, where we may suppose that $b_n = p_2b_{n+1}$, $p_2b_0 = 0$ and $R_2b_0 \cong R/P$. Then $F = (E_1 \oplus E_2)/\langle a_0 - b_0\rangle$ is the injective hull of $Ra_0 = Rb_0 \cong R/P$ and is a non-separated R-module (see [6, p. 4053]). Consider the R-module F with $a_0 = b_0$ and let $C_n = \mathrm{Ann}_F(P^n)$. Moreover, we identify A_n (resp. B_n) with the submodule A'_n (resp. B'_n) of F, consisting of all elements of the form $a + \langle a_0, b_0\rangle$ (resp. $b + \langle a_0, b_0\rangle$), where $a \in A_n$ (resp. $b \in B_n$). The above notation will be kept in the first two results.

Proposition 3.1. *Let R be the pullback ring as in* (1). *Then the following hold*:

(i) *For each* n, $C_n = A_n + B_n$, $C_0 = R/P = Ra_0 = Rb_0$, $C_n \subseteq C_{n+1}$ *and* $F = \bigcup C_n$.

(ii) *The non-zero proper* R*-submodules of* F *are*: E_1, E_2, A_n, B_m, $E_1 + B_n$, $A_m + E_2$ *and* $A_m + B_n$ *for all* $n, m \geq 1$.

Proof. (i) If $x \in C_n$, then $P^n x = 0$, and $x = x_1 + x_2$, where $x_i \in E_i$ for $i = 1, 2$. Therefore, $P_1^n x = 0 = P_2^n x$, so $0 = P_1^n x = P_1^n x_1 + P_1^n x_2 = P_1^n x_1$. Similarly, $P_2^n x_2 = 0$. Thus, $C_n \subseteq A_n + B_n$. The proof of the other implication is similar. Moreover, $C_n \subseteq A_{n+1} + B_{n+1} = C_{n+1}$ for each n. Finally, if $z = z_1 + z_2 \in F$ ($z_i \in E_i$), then $z \in C_k$ for some k, and so $F = \bigcup C_n$.

(ii) Let L be a non-zero proper submodule of F. Then either $L \cap E_1 = E_1$ or $L \cap E_1 \neq E_1$. Similarly for E_2. If $L \cap E_i = E_i$ for each i, then $E_1 + E_2 \subseteq L$, so $F = L$, which is a contradiction. If $L \cap E_1 = E_1$ and $L \cap E_2 \neq E_2$, then by [10, Lemma 2.6], there is an integer m such that $L \cap B_m = B_{m-1}$, $B_m \nsubseteq L$ and $E_1 + B_{m-1} \subseteq L$. Let $c \in L$. Then there are integers s, t such that $c = c_1 a_s + d_1 b_t$ for some $c_1, d_1 \in R - P$; hence $p_2 c = d_1 p_2 b_t = d_1 b_{t-1}$. It follows that $B_{t-1} \subseteq p_2 L$. Therefore, $t - 1 \leq m - 1$, which implies $L \subseteq E_1 + B_m$. So $L = L \cap (E_1 + B_m) = E_1 + B_m \cap L = E_1 + B_{m-1}$. Similarly for E_2.

Now it suffices to show that if $L \neq E_1, E_2, E_1 + B_m, A_n + E_2, A_n, B_m$ ($n, m \geq 1$), then $L = A_m + B_t$ for some m and t. Clearly, $(E_1 \cap L) + (E_2 \cap L) \subseteq L$. There exist integers m and t such that $E_1 \cap L = A_{m-1}$ and $E_2 \cap L = B_{t-1}$, and so $A_{m-1} + B_{t-1} \subseteq L$, $A_m \nsubseteq L$, and $B_t \nsubseteq L$. Let $c \in L$. Then there are integers n, s such that $c = c_1 a_n + d_1 b_s$ for some $c_1, d_1 \in R - P$. So $p_1 c = c_1 p_1 a_n = c_1 a_{n-1}$ and $p_2 c = d_1 p_2 b_s = d_1 b_{s-1}$; hence $A_{n-1} \subseteq p_1 L \subseteq p_1 A_{m-1} = A_{m-2}$ and $B_{s-1} \subseteq p_2 L \subseteq B_{t-2}$. Therefore, $n - 1 \leq m - 2$ and $s - 1 \leq t - 2$, which implies $L \subseteq A_{m-1} + B_{t-1}$, and the proof is complete. ∎

Theorem 3.2. *Let* R *be the pullback ring as in* (1). *Then* F, *the injective hull of* R/P, *is a non-separated weak multiplication* R*-module.*

Proof. It suffices to show that $\mathrm{Spec}(F) = \emptyset$. Let L be any submodule of F as described in Proposition 3.1(ii). We claim that $(L : F) = 0$. Suppose that $r \in (L : F)$ with $r \neq 0$. Then $rF \subseteq L$ and for all $x \in F$, we must have $x = ry$ for some $y \in F$, since F is divisible. Thus $x \in L$; hence $L = F$, which is a contradiction. Set $P = \langle p_1, p_2 \rangle = \langle p \rangle$. However, no L, say $E_1 + A_n$, is a prime submodule of F, for if m is any positive integer, then $p^m \notin (L : F) = 0$ and $x_1 + a_{n+m} \notin E_1 + A_n$ ($x_1 \in E_1$), but $p^m(x_1 + a_{n+m}) = p_1^m x_1 + a_n \in E_1 + A_n$, as required. ∎

We need the following lemma proved in [19, Result 4.1].

Lemma 3.3. *Let* M *be a module over a commutative ring* R, N *an* R*-submodule of* M, *and* K *a prime* R*-submodule of* M *with* $N \nsubseteq K$. *Then* $(K : M) = (K : N)$.

Proposition 3.4. *Let R be the pullback ring as in* (1), *and let M be any weak multiplication R-module. Then the following hold*:

(i) *If M has a $P_1 \oplus 0$-prime submodule N, then M is separated.*
(ii) *If M has a $0 \oplus P_2$-prime submodule N, then M is separated.*

Proof. (i) Since M/N is a prime R-module, it is a separated R-module by [8, Proposition 3.1]. By assumption, $(0 \oplus P_2)N = (0 \oplus P_2)(P_1 \oplus 0)M = 0$; hence $P_1N \cap P_2N = 0$. Therefore, N is separated. Now we show that M is separated. It suffices to show that $p_1M \cap p_2M = 0$. Let $x = p_1a = p_2b$ for some $a, b \in M$. Then $p_1(a+N) = p_2(b+N) \in (P_1(M/N)) \cap (P_2(M/N)) = 0$, so $p_1a = p_2b \in N$. Then N prime gives $b \in N$; hence $x = 0$, and the proof is complete. The proof of (ii) is similar and we omit it. ∎

Lemma 3.5. *Let R be the pullback ring as in* (1) *and let M be any R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. If T is a prime submodule of S, then $K \subseteq T$.*

Proof. If $(T : S) = P$, then [14, Proposition 2.3] shows that $K \subseteq PS = (T : S)S \subseteq T$. So suppose that $(T : S) = P_1 \oplus 0$ and $K \nsubseteq T$. Then Lemma 3.3 gives $(T : S) = (T : K)$. Since $PK = 0$ by [14, Proposition 2.4], we must have $P \subseteq (T : K) = (T : S) = P_1 \oplus 0$, which is a contradiction. Likewise, if $(T : S) = 0 \oplus P_2$, then $K \subseteq T$. ∎

Proposition 3.6. *Let R be the pullback ring as in* (1) *and let M be any R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then $\mathrm{Spec}_R(S) = \emptyset$ if and only if $\mathrm{Spec}_R(M) = \emptyset$.*

Proof. First suppose that $\mathrm{Spec}_R(S) = \emptyset$ and $\mathrm{Spec}_R(M) \neq \emptyset$. So $M \cong S/K$ has a prime submodule, say T/K where T is a prime submodule of S, which is a contradiction. Next suppose that $\mathrm{Spec}_R(M) = \emptyset$ and $\mathrm{Spec}_R(S) \neq \emptyset$. Let T be a prime submodule of S. Then by Lemma 3.5, $K \subseteq T$; hence T/K is a prime submodule of M, which is a contradiction. ∎

Proposition 3.7. *Let R be the pullback ring as in* (1) *and let M be any non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is weak multiplication if and only if M is weak multiplication.*

Proof. By Proposition 3.6, we may assume that $\mathrm{Spec}(S) \neq \emptyset$. If S is weak multiplication, then $M \cong S/K$ is weak multiplication, by Lemma 2.3. Conversely, assume that M is a weak multiplication R-module and let N be a prime submodule of S. First suppose that $(N : S) = P$. Then by Lemma 3.5, $K \subseteq N$, and N/K is a prime submodule of $S/K \cong M$ by Lemma 2.1, so $(N/K : S/K) = (N : S) = P$; hence $P(S/K) = N/K$, since S/K is weak multiplication. As $K \subseteq PS$, we must have $N = PS$. If $(N : S) = P_1 \oplus 0$ (resp. $(N : S) = 0 \oplus P_2$), then N/K is a $(P_1 \oplus 0)$-prime (resp. $(0 \oplus P_2)$-prime) submodule of M, which is a contradiction by Proposition 3.4. Thus S is weak multiplication. ∎

PROPOSITION 3.8. *Let R be the pullback ring as in* (1) *and let M be an indecomposable weak multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is pure-injective.*

Proof. By [6, Proposition 2.6(i)], $S/PS \cong M/PM$, so S has finite-dimensional top. Now the assertion follows from Propositions 3.7 and 2.10. ∎

Let R be the pullback ring as in (1) and let M be an indecomposable weak multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$. Consider the separated representation $0 \to K \to S \to M \to 0$. By Proposition 3.8, S is pure-injective. So in the proofs of [6, Lemma 3.1, Proposition 3.2 and Proposition 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [6, Proposition 2.6(ii)]) we can replace the statement "M is an indecomposable pure-injective non-separated R-module" by "M is an indecomposable weak multiplication non-separated R-module", because the main keys to those results are the pure-injectivity of S, and the indecomposability and non-separability of M. So we have the following results:

COROLLARY 3.9. *Under the assumptions of Proposition* 3.8, *the quotient fields $Q(R_1)$ and $Q(R_2)$ do not occur among the direct summands of S.*

COROLLARY 3.10. *Under the assumptions of Proposition* 3.8, *S is a direct sum of finitely many indecomposable weak multiplication modules.*

COROLLARY 3.11. *Under the assumptions of Proposition* 3.8, *at most two copies of modules of infinite length can occur among the indecomposable summands of S.*

Before we state the main theorem of this section, let us explain the idea of proof. Let M be an indecomposable weak multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then by Corollary 3.10, S is a direct sum of finitely many indecomposable separated weak multiplication modules, and these are known by Theorem 2.11. In any separated representation $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ the kernel of the map φ to M is annihilated by P, hence is contained in the socle of the separated module S. Thus M is obtained by amalgamation in the socle of the various direct summands of S. This explains Corollary 3.9: the modules $Q(R_1)$ and $Q(R_2)$ have zero socle and so cannot occur in a separated (hence "minimal") representation. So the questions are: does this provide any further condition on the possible direct summands of S? How can these summands be amalgamated in order to form M?

In [16], Levy shows that the indecomposable finitely generated R-modules are of two non-overlapping types which he calls deleted cycle and block cycle types. It is the modules of deleted cycle type that are most relevant to us. Such a module is obtained from a direct sum, S, of indecomposable separated modules by amalgamating the direct summands of S in pairs to form a chain but leaving the two ends unamalgamated [16] (see also [15, Section 11]).

Since every "block cycle" type R-module is a quotient of a weak multiplication R-module, it is weak multiplication. So by Corollary 3.11, the infinite length non-separated indecomposable weak multiplication modules are obtained in just the same way as the deleted cycle type indecomposable ones are, except that at least one of the two "end" modules must be a separated indecomposable weak multiplication module of infinite length (that is, P_1-Prüfer or P_2-Prüfer). Note that one cannot have, for instance, a P_1-Prüfer module at each end (consider the alternation of primes P_1, P_2 along the amalgamation chain). So, apart from any finite length modules we have amalgamations involving two Prüfer modules as well as modules of finite length (the injective hull $E(R/P)$ is the simplest module of this type), a P_1-Prüfer module and a P_2-Prüfer module. If the P_1-Prüfer and the P_2-Prüfer modules are direct summands of S then we will describe these modules as *doubly infinite*. Those where S has just one infinite length summand will be called *singly infinite* (see [6, Section 3]). It remains to show that the modules obtained by these amalgamations are, indeed, indecomposable weak multiplication modules.

THEOREM 3.12. *Let $R = (R_1 \to \overline{R} \leftarrow R_2)$ be the pullback of two discrete valuation domains R_1, R_2 with common factor field $\overline{R}$. Then the indecomposable non-separated weak multiplication modules with finite-dimensional top are the following*:

(i) *the indecomposable modules of finite length* (*apart from R/P which is separated*),
(ii) *the doubly infinite weak multiplication modules as described above*,
(iii) *the singly infinite weak multiplication modules as described above*, *apart from the two Prüfer modules* (II) *in Lemma* 2.9.

Proof. We already know that every indecomposable weak multiplication non-separated module with finite-dimensional top has one of these forms so it remains to show that the modules obtained by these amalgamations are, indeed, indecomposable weak multiplication modules. Let M be an indecomposable non-separated weak multiplication R-module with finite-dimensional top and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M.

(i) Since M is a quotient of a weak multiplication R-module, it is weak multiplication. The indecomposability follows from [16, 1.9].

(ii) & (iii) (one or two Prüfer modules), Since a quotient of any weak multiplication R-module is weak multiplication, M is weak multiplication and the indecomposability follows from [6, Theorem 3.5]. ∎

COROLLARY 3.13. *Let R be the pullback ring as described in Theorem* 3.12. *Then every indecomposable non-separated weak multiplication R-module with finite-dimensional top is pure-injective.*

Proof. Apply [6, Theorem 3.5] and Theorem 3.12. ∎

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