C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 99 2004 NO. 1

ON TOPOLOGICAL PROPERTIES OF THE SPACES OF DARBOUX BAIRE 1 FUNCTIONS AND BOUNDED DERIVATIVES

BY

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Abstract. We investigate the topological structure of the space $\mathcal{DB}_1$ of bounded Darboux Baire 1 functions on $[0,1]$ with the metric of uniform convergence and with the p^*-topology. We also investigate some properties of the set Δ of bounded derivatives.

The class $\mathcal{DB}_1$ of bounded Darboux Baire 1 functions on $[0,1]$ contains subclasses of functions important for differentiation theory such as derivatives. For that reason many mathematicians have investigated this class. In [2], [8], [3] "typical" properties in this class were considered, where a property Φ is called *typical in* $\mathcal{DB}_1$ if the class of all functions satisfying Φ is residual in $\mathcal{DB}_1$. Therefore, the topological structure of $\mathcal{DB}_1$ is worth investigating, and this is one of the purposes of this article. First we shall consider some properties of the set Δ of all bounded derivatives on $[0,1]$. One of these properties (superporosity at each point of $\mathcal{DB}_1$) plays an important role in further considerations connected with $\mathcal{DB}_1$.

We apply the classical terminology and notation. We adopt the following definition of a Darboux function ([9], [5]):

A function $F : X \to Y$ (where X, Y are topological spaces) is called a *Darboux function* if $F(C)$ is a connected set for each connected set $C \subset X$.

By $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$, $\mathbb{I}$ we denote the sets of real numbers, rational numbers, natural numbers, and the segment $[0,1]$ respectively. The symbol m_1 stands for the Lebesgue measure on the real line. By C_f (resp. D_f) we denote the set of all points of continuity (resp. discontinuity) of a function $f : X \to Y$. For $x_0 \in Y$, we denote by $\mathrm{const}_{x_0} : X \to Y$ the constantly x_0 function.

A subset $L \subset X$ is called an *arc* if there exists a homeomorphism h from $\mathbb{I}$ onto L. The elements $h(0)$ and $h(1)$ are called the endpoints of L. The arc with endpoints a and b is denoted by $L(a,b)$.

We say that a set $A \subset \mathbb{I}$ is *bilaterally* $\mathbf{c}$-*dense in itself* if $\operatorname{card} A \cap (x, x+\delta) = \operatorname{card} A \cap (x-\delta, x) = \mathbf{c}$ for all $x \in A$ and $\delta > 0$.

2000 *Mathematics Subject Classification*: 26A21, 54C30, 54C35, 26A24.

Key words and phrases: Darboux Baire 1 function, p^*-topology, superporosity, derivative.

By $\mathcal{D}$ (resp. $\mathcal{B}_1$) we denote the set of bounded Darboux (resp. Baire 1) functions $f : \mathbb{I} \to \mathbb{R}$. By ϱ we denote the metric of uniform convergence.

We say that a function $f : \mathbb{I} \to \mathbb{R}$ satisfies the *Young condition* if

- for every $x \in (0,1)$ there exist sequences $x_n \searrow x$ and $y_n \nearrow x$ such that both $f(x_n)$ and $f(y_n)$ converge to $f(x)$,
- there exists a sequence $x_n \searrow 0$ such that $f(x_n)$ converges to $f(0)$,
- there exists a sequence $y_n \nearrow 1$ such that $f(y_n)$ converges to $f(1)$.

We say that $A \subset X$ is a *stationary set* for the class $\mathcal{F}$ of functions from X to Y provided that, for each $f \in \mathcal{F}$, if f is constant on A, then f must be constant on the whole domain.

If (X,d) is a metric space, then we denote by $B(x,R)$ the open ball with center at x and radius $r > 0$. Let $M \subset X$, $x \in X$, $R > 0$. Then $\gamma(x,R,M)$ denotes the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,R) \setminus M$. The set M is called *porous* at x if $\limsup_{R\to 0^+} \gamma(x,R,M)/R > 0$. We say that $E \subset X$ is *superporous* at $x \in X$ if $E \cup F$ is porous at x whenever F is porous at x. A set $G \subset X$ is said to be *p-open* if $X \setminus G$ is superporous at each point of G. The system of all superporous sets at x forms an ideal. Therefore the system of all p-open sets forms a topology, called the *p-topology* ([12]). A set $H \subset X$ is said to be *p^*-open* if $H = G \setminus N$, where G is p-open and N is p-meager. The system of all p^*-open sets forms a topology, called the *p^*-topology*. Clearly the p^*-topology is stronger than the p-topology, and the p-topology is stronger than the topology generated by the metric d ([12]).

The notion of an abstract density topology (in the category sense) is understood as in [6].

It is known that $\Delta \subset \mathcal{DB}_1$ ([1], [10]). It is easy to see that $\text{card}(\Delta) = \text{card}(\mathcal{DB}_1) = \mathbf{c}$. But it turns out that Δ is a “small” subset of $\mathcal{DB}_1$ in the topological sense. To prove this we need two lemmas.

First from [7, Theorem 1.1.9(3) and Corollary 1.7.12] we infer

LEMMA 1. *If $f : [a,b] \to \mathbb{R}$ $(a < b)$ is a Darboux (resp. Baire 1) function, then for every $\alpha \in \mathbb{R}$, the functions $f^* = \max(f, \text{const}_\alpha)$ and $f_* = \min(f, \text{const}_\alpha)$ are Darboux (resp. Baire 1) functions.* ∎

Let $\{a_i\}_{i\in K}$ $(K = \{1,\ldots,n\})$ be a finite increasing sequence of real numbers from an interval (a,b). Put $F_1 = [a,a_1]$, $F_i = [a_{i-1},a_i]$ for $i \in K \setminus \{1,n\}$, $F_n = [a_{n-1},b]$. Obviously the family $\{F_i\}_{i\in K}$ is a closed covering of $[a,b]$.

It is easy to check

LEMMA 2. *Let $\{F_i\}_{i\in K}$ be the sequence of sets defined above and let $f_i : F_i \to \mathbb{R}$, where $i \in K$, be a family of compatible Darboux (resp. Baire 1)* *functions. Then the common extension* $f = \triangledown_{i\in K} f_i$ *is a Darboux (resp. Baire* 1) *function.* ■

THEOREM 3. *The set* Δ *is superporous at each point of the space* $(\mathcal{DB}_1, \varrho)$.

Proof. Let $f \in \mathcal{DB}_1$ and let $\Phi \subset \mathcal{DB}_1$ be porous at f. Let $R > 0$. Put $r_1' = \gamma(f, R, \Phi)/2 > 0$. Then there exist $r_1 > r_1'$ and $g \in \mathcal{DB}_1$ such that

$$B(g, r_1) \subset B(f, R) \setminus \Phi. \tag{1}$$

We shall show that there exists $h \in \mathcal{DB}_1$ such that

$$B(h, r_1/8) \subset B(g, r_1) \setminus \Delta. \tag{2}$$

Since $g \in \mathcal{B}_1$, there exists a point $x_0 \in (0,1)$ of continuity of g. Consequently, there exists $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset (0,1)$ and

$$g([x_0 - \delta, x_0 + \delta]) \subset (g(x_0) - r_1/4, g(x_0) + r_1/4).$$

Let $C_\delta \subset (x_0 - \delta/2, x_0 + \delta/2)$ be a bilaterally **c**-dense in itself F_σ set of null Lebesgue measure. Then ([1, Theorem II.2.4]) there exists a Darboux Baire 1 function $s : [x_0 - \delta/2, x_0 + \delta/2] \to \mathbb{I}$ such that $s(x) = 0$ for $x \notin C_\delta$ and $0 < s(x) \le 1$ for $x \in C_\delta$.

Fix $\alpha \in (0,1] \cap s(C_\delta)$. Put $s_1(x) = \min(1, \alpha^{-1}s(x))$ for $x \in [x_0 - \delta/2, x_0 + \delta/2]$. Obviously $s_1 : [x_0 - \delta/2, x_0 + \delta/2] \to \mathbb{I}$ is a bounded Darboux Baire 1 function ([1, Theorem II.3.2] and Lemma 1). Note that $1 \in s_1(C_\delta)$.

We define a function $\mu : [x_0 - \delta/2, x_0 + \delta/2] \to \mathbb{R}$ as follows:

$$\mu(x) = \frac{r_1}{4} s_1(x) + g(x_0).$$

Then $\mu : [x_0 - \delta/2, x_0 + \delta/2] \to [g(x_0), g(x_0) + r_1/4]$ is a bounded Darboux Baire 1 function ([1, Theorem II.3.2]). Note that $r_1/4 + g(x_0) \in \mu(C_\delta)$.

We define a function $h : \mathbb{I} \to \mathbb{R}$ as follows:

$$h(x) = \begin{cases} g(x) & \text{if } x \in \mathbb{I} \setminus (x_0 - \delta, x_0 + \delta), \\ l_1(x) & \text{if } x \in [x_0 - \delta, x_0 - \delta/2], \\ \mu(x) & \text{if } x \in [x_0 - \delta/2, x_0 + \delta/2], \\ l_2(x) & \text{if } x \in [x_0 + \delta/2, x_0 + \delta], \end{cases}$$

where l_1 and l_2 are linear functions such that $l_1(x_0 - \delta) = g(x_0 - \delta)$, $l_1(x_0 - \delta/2) = g(x_0)$, $l_2(x_0 + \delta) = g(x_0 + \delta)$ and $l_2(x_0 + \delta/2) = g(x_0)$. Then $h \in \mathcal{DB}_1$ (Lemma 2). Note that $r_1/4 + g(x_0) \in h(C_\delta)$.

Notice that $\varrho(h, g) \le r_1/2$, so

$$B(h, r_1/8) \subset B(g, r_1). \tag{3}$$

Now, assume that there exists a function $\xi \in B(h, r_1/8) \cap \Delta$. Then

$$(x_0 - \delta/2, x_0 + \delta/2) \setminus C_\delta \subset \xi^{-1}((-\infty, g(x_0) + r_1/8)). \tag{4}$$

Let $z_0 \in C_\delta$ be such that $h(z_0) = r_1/4 + g(x_0)$. Hence $\xi(z_0) > g(x_0) + r_1/8$. Therefore $z_0 \in \xi^{-1}((g(x_0) + r_1/8, \infty)) \cap C_\delta$. Let $U_0 \subset (x_0 - \delta/2, x_0 + \delta/2)$ be a unilateral neighbourhoood of z_0. Note that by (4),

$$U_0 \cap \xi^{-1}((g(x_0) + r_1/8, \infty)) \subset C_\delta,$$

so

$$m_1(U_0 \cap \xi^{-1}((g(x_0) + r_1/8, \infty))) \le m_1(C_\delta) = 0.$$

Thus $\xi \notin \mathcal{M}_2$, which contradicts the fact that $\Delta \subset \mathcal{M}_2$ ([1]). Hence $B(h, r_1/8) \cap \Delta = \emptyset$. This equality and (3) finish the proof of (2). From (1) and (2) we infer that

$$\gamma(f, R, \Delta \cup \Phi) \ge r_1/8.$$

Therefore

$$\limsup_{R \to 0^+} \frac{\gamma(f, R, \Delta \cup \Phi)}{R} \ge \frac{1}{8} > 0. \ \blacksquare$$

It is easy to observe that Δ is a nowhere dense and perfect subset of $\mathcal{DB}_1$. So its topological structure is similar to that of the Cantor set. There are several constructions of Darboux functions from $[0, 1]$ to $\mathbb{R}$ in which the Cantor set plays an important role. This suggests that Δ can play a similar role in constructions of Darboux functions from $\mathcal{DB}_1$ to $\mathbb{R}$. It turns out that in some cases we can obtain analogous results (Theorem 4), in others it is impossible (Corollary 6).

THEOREM 4. *There exists a Darboux function $F : \mathcal{DB}_1 \to \mathbb{R}$ such that $D_F = \Delta$ and $F(B(g, \varepsilon) \cap \Delta) = \mathbb{R}$ for any $g \in \Delta$ and $\varepsilon > 0$.*

Proof. In $\mathbb{R}$ we define an equivalence relation $\star$ in the following way: $x \star y \Leftrightarrow x - y \in \mathbb{Q}$. Denote by $\mathcal{E}$ the set of equivalence classes of this relation and let $\xi : \mathcal{E} \to \mathbb{R}$ be a bijection. Define a function $\chi : \mathbb{R} \to \mathbb{R}$ by $\chi(x) = \xi([x]_\star)$. Then χ is a Darboux function such that $\chi((a, b)) = \mathbb{R}$ for all $a < b$. Let $\varphi(x) = (1/x)\sin(1/x)$ for $x \in (0, \infty)$. We define $F : \mathcal{DB}_1 \to \mathbb{R}$ by

$$F(f) = \begin{cases} \chi(f(0)) & \text{if } f \in \Delta, \\ \varphi(\varrho_\Delta(f)) & \text{if } f \in \mathcal{DB}_1 \setminus \Delta. \end{cases}$$

First we shall show that

(5) $$F \text{ is a Darboux function.}$$

Let $C \subset \mathcal{DB}_1$ be a connected set. Consider the following three cases.

CASE 1: $C \subset \Delta$. Suppose that $F(C)$ is disconnected. Then there exist $r_1 < r_0 < r_2$ and $f_1, f_2 \in C$ such that $F(f_1) = r_1$, $F(f_2) = r_2$ and $F(f) \ne r_0$

[1] A function $f : \mathbb{I} \to \mathbb{R}$ is said to be of class $\mathcal{M}_2$ if for each $a \in \mathbb{R}$ the set $E = \{x \in \mathbb{I} : f(x) > a\}$ is either empty or an F_σ and $m_1(E \cap (x - \delta, x)) > 0$ and $m_1(E \cap (x, x + \delta)) > 0$ for each $x \in E$ and each $\delta > 0$. Zahorski proved that every bounded derivative is of class $\mathcal{M}_2$ ([10]).

for each $f \in C$. Consequently, there exists $E^\star \in \mathcal{E}$ such that $\xi(E^\star) = r_0$. Then

(6) $$f(0) \notin E^\star \quad \text{for each } f \in C.$$

As $r_1 \neq r_2$, we have $F(f_1) \neq F(f_2)$. Then $[f_1(0)]_\star \neq [f_2(0)]_\star$, so $f_1(0) \neq f_2(0)$. Let, for instance, $f_1(0) < f_2(0)$. Then there exists $y^\star \in E^\star \cap (f_1(0), f_2(0))$. By (6),

$$C = \{f \in C : f(0) < y^\star\} \cup \{f \in C : f(0) > y^\star\},$$

where the sets $\{f \in C : f(0) < y^\star\}$ and $\{f \in C : f(0) > y^\star\}$ are nonempty (they contain f_1, f_2 respectively) and separated, which contradicts the connectedness of C.

Case 2: $C \subset \mathcal{DB}_1 \setminus \Delta$. If $\varrho_\Delta(C)$ is a singleton, so is $F(C)$. In the opposite case, let $r_1 = \inf\{r > 0 : \exists_{f \in C}\ \varrho_\Delta(f) = r\}$ and $r_2 = \sup\{r > 0 : \exists_{f \in C}\ \varrho_\Delta(f) = r\}$. It is evident that $r_1 \neq r_2$ and $r_1 \geq 0$, $r_2 > 0$. Note that (by the conectedness of C)

$$\forall_{r \in (r_1, r_2)} \quad C \cap \{f \in \mathcal{DB}_1 : \varrho_\Delta(f) = r\} \neq \emptyset.$$

Consider the following subcases:

(a) $\forall_{f \in C}\ (\varrho_\Delta(f) \neq r_1 \wedge \varrho_\Delta(f) \neq r_2)$. Then $F(C) = \varphi((r_1, r_2))$ is connected because φ is continuous on $(0, \infty)$.

(b) $(\forall_{f \in C}\ \varrho_\Delta(f) \neq r_1) \wedge (\exists_{f^0 \in C}\ \varrho_\Delta(f^0) = r_2)$. Then $F(C) = \varphi((r_1, r_2])$ is connected.

(c) $(\exists_{f_0 \in C}\ \varrho_\Delta(f_0) = r_1) \wedge (\forall_{f \in C}\ \varrho_\Delta(f) \neq r_2)$. Since $f_0 \in C \subset \mathcal{DB}_1 \setminus \Delta$ and Δ is a closed set, we have $r_1 = \varrho_\Delta(f_0) > 0$. Hence $[r_1, r_2) \subset (0, \infty)$ and $F(C) = \varphi([r_1, r_2))$ is connected.

(d) $(\exists_{f_0 \in C}\ \varrho_\Delta(f_0) = r_1) \wedge (\exists_{f^0 \in C}\ \varrho_\Delta(f^0) = r_2)$. As in (c) we can show that $[r_1, r_2] \subset (0, \infty)$. Hence $F(C) = \varphi([r_1, r_2])$ is connected.

Case 3: $C \cap \Delta \neq \emptyset$ and $C \setminus \Delta \neq \emptyset$. Then there exists a function $\hat{f} \in C \setminus \Delta$. Let $\hat{r} = \varrho_\Delta(\hat{f}) > 0$. Since C is connected, we have

$$\forall_{r \in (0, \hat{r})} \quad C \cap \{f \in \mathcal{DB}_1 : \varrho_\Delta(f) = r\} \neq \emptyset.$$

Hence $F(C) \supset \varphi((0, \hat{r})) = \mathbb{R}$ and $F(C) = \mathbb{R}$ is connected. This ends the proof of (5).

Now we shall show that

(7) $$\forall_{g \in \Delta}\ \forall_{\varepsilon > 0} \quad F(K(g, \varepsilon) \cap \Delta) = \mathbb{R}.$$

Indeed, if $g \in \Delta$ and $\varepsilon > 0$, then

$$F(K(g, \varepsilon) \cap \Delta) \supset F(\{g + \alpha : \alpha \in (-\varepsilon, \varepsilon)\}) = \chi((g(0) - \varepsilon, g(0) + \varepsilon)) = \mathbb{R}.$$

It is easy to see that $\mathcal{DB}_1 \setminus \Delta \subset C_F$. From (7) we infer that $\Delta \subset D_F$, so $D_F = \Delta$, which ends the proof. ∎

It is known that for each perfect set $P \subset \mathbb{I}$ there exists a bounded Darboux Baire 1 function $h : \mathbb{I} \to \mathbb{R}$ such that h vanishes off P but does not vanish identically ([1, Theorem II.2.4]). This fact leads to the question: Does there exist a Darboux function $F : \mathcal{DB}_1 \to \mathbb{R}$ which vanishes off Δ but does not vanish identically? The answer is negative (Corollary 6).

The above question is connected with the theory of stationary sets. It is known that E is a stationary set for the family of Darboux functions $f : \mathbb{I} \to \mathbb{R}$ if and only if $\operatorname{card}(\mathbb{I} \setminus E) < \mathbf{c}$ ([1, Theorem XII.1.1]). But it turns out that for the family of real Darboux functions defined on $\mathcal{DB}_1$ (with the metric of uniform convergence) this characterization of stationary sets fails.

THEOREM 5. *In the space* $(\mathcal{DB}_1, \varrho)$ *the set* $\Delta' = \mathcal{DB}_1 \setminus \Delta$ *is stationary for the class of real Darboux functions.*

Proof. Let $F : \mathcal{DB}_1 \to \mathbb{R}$ be a Darboux function such that $F(\Delta') = \{\alpha_0\}$ (where $\alpha_0 \in \mathbb{R}$). Let $g \in \Delta$. To prove the theorem it is sufficient to construct an arc $L = L(g, h)$ such that $L \setminus \{g\} \subset \Delta'$.

Since $g \in \mathcal{B}_1$, there exists a point $x_0 \in (0,1)$ of continuity of g. For $r \in \mathbb{I}$ we define $t_r : \mathbb{I} \to \mathbb{R}$ in the following way:

$$t_r(x) = \begin{cases} g(x_0) + r & \text{if } x = x_0, \\ g(x) + r \sin \frac{1}{x - x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$

Obviously, t_r $(r \in \mathbb{I})$ is a bounded Baire 1 function. It is not difficult to see that it satisfies the Young condition, so it is a Darboux function ([1]). Hence $t_r \in \mathcal{DB}_1$ for $r \in \mathbb{I}$.

Note that $t_r = g + d_r$ $(r \in \mathbb{I})$, where

$$d_r(x) = \begin{cases} r & \text{if } x = x_0, \\ r \sin \frac{1}{x - x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$

We shall prove that

$$d_r \notin \Delta \quad \text{for } r \in (0,1]. \tag{8}$$

Indeed, for a fixed $r^* \in (0,1]$, define $k : \mathbb{I} \to \mathbb{R}$ by

$$k(x) = \begin{cases} 0 & \text{if } x = x_0, \\ r^*(x - x_0)^2 \cos \frac{1}{x - x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$

Then

$$k'(x) = \begin{cases} 0 & \text{if } x = x_0, \\ 2r^*(x - x_0) \cos \frac{1}{x - x_0} + r^* \sin \frac{1}{x - x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$

Consider a function $h : \mathbb{I} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} 0 & \text{if } x = x_0, \\ 2r^*(x - x_0) \cos \frac{1}{x - x_0} & \text{if } x \in \mathbb{I} \setminus \{x_0\}. \end{cases}$$

Then h is continuous and bounded on $\mathbb{I}$, so $h \in \Delta$. Therefore also $f = k' - h \in \Delta$. Hence $d_{r^*} \notin \Delta$, because the difference

$$d_{r^*}(x) - f(x) = \begin{cases} r^* & \text{if } x = x_0, \\ 0 & \text{if } x \in \mathbb{I} \setminus \{x_0\}, \end{cases}$$

is not a derivative (it does not have the Darboux property). In this way we have proved condition (8).

Since $d_r = t_r - g$ for $r \in (0,1]$ and $g \in \Delta$, it follows that t_r is not a derivative for each $r \in (0,1]$.

Note that for any $r_1, r_2 \in \mathbb{I}$,

$$\varrho(t_{r_1}, t_{r_2}) = |r_1 - r_2|. \tag{9}$$

Therefore the function $\zeta : \mathbb{I} \to \{t_r : r \in \mathbb{I}\}$ given by $\zeta(r) = t_r$ is a homeomorphism. Hence $L = \{t_r : r \in \mathbb{I}\}$ is an arc in $\mathcal{DB}_1$ such that $L = L(g, t_1)$ and $L \setminus \{g\} = \{t_r : r \in (0,1]\} \subset \Delta'$. ∎

Corollary 6. *There does not exist a Darboux function $F : \mathcal{DB}_1 \to \mathbb{R}$ which is zero for $t \in \Delta'$, but not identically zero.* ∎

Now we shall investigate the topological structure of the space $\mathcal{DB}_1$ with the metric of uniform convergence and the p^*-topology.

It is easy to see that $(\mathcal{DB}_1, \varrho)$ is a Baire space. So from [12, Theorem 2], we infer

Corollary 7. *The p^*-topology is an abstract density topology* (*in the category sense*) *on* $(\mathcal{DB}_1, \varrho)$. ∎

Obviously $(\mathcal{DB}_1, \varrho)$ is a perfectly normal space. For the p^*-topology we have

Theorem 8. $(\mathcal{DB}_1, p^*)$ *is a Hausdorff space but it is not regular.*

Proof. Since the p^*-topology is stronger than the ϱ-topology ([12]), we deduce that $(\mathcal{DB}_1, p^*)$ is a Hausdorff space.

Suppose that $(\mathcal{DB}_1, p^*)$ is a regular space. For $q \in \mathbb{Q}$ put

$$A_q = \{f \in \mathcal{DB}_1 : f(0) = q\}, \qquad F = \bigcup_{q \in \mathbb{Q}} A_q.$$

Obviously $F \subset \mathcal{DB}_1$. Note that

$$F \text{ is } \varrho\text{-meager.} \tag{10}$$

To see this, it suffices to prove that A_q is ϱ-nowhere dense for each $q \in \mathbb{Q}$. So fix $q_0 \in \mathbb{Q}$ and let $B(g, \varepsilon)$ be an arbitrary open ball in $(\mathcal{DB}_1, \varrho)$. We shall show that there exists a ϱ-open set $U \subset B(g, \varepsilon) \setminus A_{q_0}$. If $B(g, \varepsilon) \cap A_{q_0} = \emptyset$, we obviously put $U = B(g, \varepsilon)$. Hence, we may assume that there exists $f_0 \in B(g, \varepsilon) \cap A_{q_0}$. Put $\delta = \varepsilon - \varrho(g, f_0) > 0$ and $U = B(f_0 + \delta/2, \delta/4)$.

Clearly $U \subset B(g, \varepsilon)$. If $h \in U$, then $h(0) > q_0 + \delta/4$. Hence $h \notin A_{q_0}$, so $U \cap A_{q_0} = \emptyset$. The proof of (10) is thus finished. Hence ([12, Theorem 2])

$$F \text{ is } p^*\text{-closed.}$$

Now we shall show that

$$F \text{ is } \varrho\text{-dense.} \tag{11}$$

Indeed, let $B(g, \varepsilon)$ be an arbitrary open ball in $\mathcal{DB}_1$. Let $q^* \in (g(0) - \varepsilon, g(0) + \varepsilon) \cap \mathbb{Q}$. Put $h^*(x) = g(x) - g(0) + q^*$ for $x \in \mathbb{I}$. Clearly $h^* \in \mathcal{DB}_1$ ([1, Theorem II.3.2]) and $h^*(0) = q^* \in \mathbb{Q}$, so $h^* \in A_{q^*} \subset F$. Of course $\varrho(g, h^*) < \varepsilon$, so $h^* \in B(g, \varepsilon)$. Hence $F \cap B(g, \varepsilon) \neq \emptyset$, which proves (11).

Now, let $f^* \in \mathcal{DB}_1 \setminus F$. Since $(\mathcal{DB}_1, p^*)$ is (by assumption) a regular space, there exist p^*-open and disjoint sets U_1, U_2 such that $F \subset U_1$ and $f^* \in U_2$. Then ([12, Theorem 2])

$$U_1 = H_1 \setminus N_1, \quad U_2 = H_2 \setminus N_2,$$

where H_1, H_2 are p-open and N_1, N_2 are p-meager.

From $U_1 \cap U_2 = \emptyset$ we conclude that $(H_1 \cap H_2) \setminus (N_1 \cup N_2) = \emptyset$. Since $(\mathcal{DB}_1, \varrho)$ (and hence $(\mathcal{DB}_1, p)$, see e.g. [12]) is a Baire space, we deduce that $H_1 \cap H_2 = \emptyset$. Therefore

$$F \subset U_1 \subset H_1 \subset \mathcal{DB}_1 \setminus H_2.$$

Since H_2 is a p-open set, we conclude that F is porous at f^* (in the space $(\mathcal{DB}_1, \varrho)$). This contradicts (11). ∎

In the proof of Theorem 5 we have constructed an arc in $\mathcal{DB}_1$. This leads to the question: Are the spaces considered arcwise connected? Theorems 9 and 12 give an answer to this question.

THEOREM 9. *The space $(\mathcal{DB}_1, \varrho)$ is arcwise connected.*

Proof. Let $f_1, f_2 \in \mathcal{DB}_1$ and $f_1 \neq f_2$. Consider the following cases:

CASE 1: $f_1 = \text{const}_0$ or $f_2 = \text{const}_0$. Assume, for instance, that $f_1 = \text{const}_0$. Put $L = \{af_2 : a \in \mathbb{I}\}$. Then L is an arc in $(\mathcal{DB}_1, \varrho)$ such that $L = L(f_1, f_2)$.

CASE 2: $f_1 \neq \text{const}_0$ and $f_2 \neq \text{const}_0$. Then there are two possibilities:

- There exists $r^* \in \mathbb{R}$ such that $f_1 = r^* f_2$. Since $f_1 \neq f_2$, we have $r^* \neq 1$. Assume, for instance, that $r^* > 1$ (the other case is similar). Put $L = \{af_2 : a \in [1, r^*]\}$. Then L is an arc in $(\mathcal{DB}_1, \varrho)$ such that $L = L(f_2, f_1)$.
- There is no $r \in \mathbb{R}$ such that $f_1 = rf_2$. Put $L_1 = \{af_1 : a \in \mathbb{I}\}$ and $L_2 = \{af_2 : a \in \mathbb{I}\}$. Then L_1 and L_2 are arcs in $(\mathcal{DB}_1, \varrho)$ such that $L_1 = L(\text{const}_0, f_1)$, $L_2 = L(\text{const}_0, f_2)$ and $L_1 \cap L_2 = \{\text{const}_0\}$. Thus $L_1 \cup L_2$ is an arc in $(\mathcal{DB}_1, \varrho)$ such that $L_1 \cup L_2 = L(f_1, f_2)$. ∎

LEMMA 10. *For each* $f \in \mathcal{DB}_1$ *and each* $r > 0$, *there exist arcs* L_1, L_2, L_3 *in the space* $(\mathcal{DB}_1, \varrho)$ *contained in* $B(f, r)$ *such that* $L_i \cap L_j = \{f\}$ *for* $i, j \in \{1, 2, 3\}$, $i \neq j$.

Proof. Let $f \in \mathcal{DB}_1$ and let $r > 0$. Consider the following cases:

• f is not a constant function. Put $M = \sup\{|f(x)| : x \in \mathbb{I}\} > 0$. Then $L_1 = \{f + a : a \in [0, r/2]\}$, $L_2 = \{f + a : a \in [-r/2, 0]\}$ and $L_3 = \{af : a \in [1, 1 + r/(2M)]\}$ satisfy the required conditions.

• f is a constant function. Put $L_1 = \{f + a : a \in [0, r/2]\}$, $L_2 = \{f + a : a \in [-r/2, 0]\}$ and $L_3 = \{l_a : a \in [0, r/2]\}$, where $l_a(x) = ax + f(x)$, $x \in \mathbb{I}$. Then L_1, L_2 and L_3 satisfy the required conditions.

LEMMA 11. *Let* X *be an arbitrary set and let* $\mathcal{T}$ *and* $\mathcal{T}'$ *be two topologies on* X *such that* $\mathcal{T} \subset \mathcal{T}'$ *and* $(X, \mathcal{T})$ *is a Hausdorff space. Then, if* L *is an arc in* $(X, \mathcal{T}')$, *it is also an arc in* $(X, \mathcal{T})$. ■

THEOREM 12. *The space* $(\mathcal{DB}_1, p^*)$ *is not arcwise connected. Moreover, there exists no arc in* $(\mathcal{DB}_1, p^*)$.

Proof. Suppose that there exists an arc L in $(\mathcal{DB}_1, p^*)$. From Lemma 11, we infer that L is an arc in $(\mathcal{DB}_1, \varrho)$. Now we show that

$$L \text{ has empty interior in } (\mathcal{DB}_1, \varrho). \tag{12}$$

Indeed, suppose that there exists an open ball $B(f, r) \subset L$. By Lemma 10 there exist arcs L_1, L_2, L_3 in $(\mathcal{DB}_1, \varrho)$, contained in $B(f, r)$, such that $L_i \cap L_j = \{f\}$ for $i \neq j$. Then L_1, L_2, L_3 are arcs in (L, ϱ). It is not difficult to check that this is impossible.

Clearly L is ϱ-closed. Hence (by (12)) L is ϱ-nowhere dense, so L is ϱ-meager. Then each subset of L is p^*-closed. It follows that L is disconnected in $(\mathcal{DB}_1, p^*)$, which contradicts the fact that L is an arc. ■

REFERENCES

[1] A. M. Bruckner, *Differentiation of Real Functions*, Springer, 1978.

[2] J. G. Ceder and T. L. Pearson, *On typical bounded Darboux Baire 1 functions*, Acta Math. Acad. Sci. Hungar. 37 (1981), 339–348.

[3] —, —, *A survey of Darboux Baire 1 functions*, Real Anal. Exchange 9 (1983–84), 179–194.

[4] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[5] J. Jędrzejewski, *Properties of functions relating to connectedness*, Acta Univ., Łódź, 1984 (in Polish).

[6] J. Lukeš, J. Maly and L. Zajíček, *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Math. 1183, Springer, 1986.

[7] T. Natkaniec, *Almost Continuity*, WSP, Bydgoszcz, 1992.

[8] G. Petruska, *An extension of the Darboux property and some typical properties of Baire 1 functions*, Real Anal. Exchange 8 (1982–83), 62–64.

[9] F. Roush, R. G. Gibson and K. R. Kellum, *Darboux retracts*, Proc. Amer. Math. Soc. 79 (1980), 491–494.
[10] Z. Zahorski, *Sur la première dérivée*, Trans. Amer. Math. Soc. 69 (1950), 1–54.
[11] L. Zajíček, *Porosity and σ-porosity*, Real Anal. Exchange 13 (1987–88), 314–350.
[12] —, *Porosity, I-density topology and abstract density topologies*, ibid. 12 (1986–87), 313–326.

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Received 19 November 1998;
revised 7 October 2003 (3663)