COLLOQUIUM MATHEMATICUM

VOL. 99 2004 NO. 1

# *A NOTE ON CHARACTERIZATIONS OF RINGS OF CONSTANTS WITH RESPECT TO DERIVATIONS*

BY

PIOTR JĘDRZEJEWICZ (Toruń)

**Abstract.** Let $A$ be a commutative algebra without zero divisors over a field $k$. If $A$ is finitely generated over $k$, then there exist well known characterizations of all $k$-subalgebras of $A$ which are rings of constants with respect to $k$-derivations of $A$. We show that these characterizations are not valid in the case when the algebra $A$ is not finitely generated over $k$.

**1. Introduction.** Let $k$ be a field and let $A$ be a $k$-*domain* (that is, a commutative $k$-algebra without zero divisors). We denote by $A_0$ the field of fractions of $A$. If $\operatorname{char}(k) = p > 0$, then we denote by $A^p$ the set $\{a^p;\ a \in A\}$. A $k$-linear mapping $d : A \to A$ is called a $k$-*derivation of* $A$ if $d(ab) = ad(b) + bd(a)$ for all $a, b \in A$. If $d$ is a $k$-derivation of $A$, then we denote by $A^d$ the *ring of constants* of $d$, that is,

$$A^d = \{a \in A;\ d(a) = 0\}.$$

The following two known theorems describe all $k$-subalgebras of $A$ which are rings of constants with respect to derivations of $A$.

THEOREM 1 ([2], [3]). *Let $A$ be a finitely generated $k$-domain, where $k$ is a field of characteristic zero. Let $B$ be a $k$-subalgebra of $A$. The following conditions are equivalent*:

(1) *There exists a $k$-derivation $d$ of $A$ such that $B = A^d$.*
(2) *The ring $B$ is integrally closed in $A$ and $B_0 \cap A = B$.* ■

THEOREM 2 ([1]). *Let $A$ be a finitely generated $k$-domain, where $k$ is a field of characteristic $p > 0$. Let $B$ be a $k$-subalgebra of $A$. The following conditions are equivalent*:

(1) *There exists a $k$-derivation $d$ of $A$ such that $B = A^d$.*
(2) *$k[A^p] \subseteq B$ and $B_0 \cap A = B$.* ■

It is clear that in the above theorems the implications (1)⇒(2) hold for any, not necessarily finitely generated, $k$-domain $A$. There is a natural question if there exists an infinitely generated $k$-domain $A$ such that some

2000 *Mathematics Subject Classification*: Primary 12H05; Secondary 13N05.

$k$-subalgebra $B$ of $A$ satisfies (2) and is not the ring of constants of any $k$-derivation of $A$. In this note we will give a positive answer to this question.

**2. The case of characteristic zero.** Let us start from the following proposition.

PROPOSITION 3. *Let $A$ be a $k$-domain, where $k$ is a field of characteristic zero. Let $\delta : A_0 \to A_0$ be a $k$-derivation and let $B = (A_0)^\delta \cap A$. Then $B$ is a $k$-subalgebra of $A$ such that $B$ is integrally closed in $A$ and $B_0 \cap A = B$. In other words, $B$ satisfies condition* (2) *of Theorem* 1.

*Proof.* Obviously $B \subset B_0 \cap A$. On the other hand, $B \subset (A_0)^\delta$, so $B_0 \subset (A_0)^\delta$, and then $B_0 \cap A \subset (A_0)^\delta \cap A = B$.

If $x \in A$ is integral over $B$, then $x$ is algebraic over $(A_0)^\delta$, because $B \subset (A_0)^\delta$. The subfield $(A_0)^\delta$, as the field of constants of a $k$-derivation of $A_0$, is algebraically closed in $A_0$ (see for example [2]), so $x$ belongs to $(A_0)^\delta$. Thus $x \in (A_0)^\delta \cap A = B$. This shows that $B$ is integrally closed in $A$. ∎

EXAMPLE 4. *Let $A$ be the polynomial ring $k[x_0, x_1, x_2, \ldots]$, where $k$ is a field of characteristic zero. Consider the $k$-derivation $\delta$ of $A_0$ defined by*

$$\delta(x_0) = 1 \quad \textit{and} \quad \delta(x_i) = 1/x_i \quad \textit{for } i = 1, 2, \ldots$$

*Then the ring $B = (A_0)^\delta \cap A$ satisfies condition* (2) *of Theorem* 1 *and is not the ring of constants of any $k$-derivation of $A$.*

*Proof.* We already know (by Proposition 3) that $B$ satisfies condition (2) of Theorem 1.

Suppose that $B = A^d$, where $d$ is a $k$-derivation of $A$. Then $d(x_i^2 - 2x_0) = 0$ for $i = 1, 2, \ldots$, because every polynomial of the form $x_i^2 - 2x_0$, with $i \geq 1$, belongs to $B$. So, $x_i d(x_i) = d(x_0)$ for all $i \geq 1$, and we see that each variable $x_i$, for $i \geq 1$, divides the polynomial $d(x_0)$. Hence, $d(x_0) = 0$, and consequently $d(x_i) = 0$ for $i = 1, 2, \ldots$ This implies that $d = 0$, that is, $B = A$. But this is a contradiction, because $x_0 \notin B$. ∎

**3. The case of positive characteristic.** In this case we have the following evident proposition.

PROPOSITION 5. *Let $A$ be a $k$-domain, where $k$ is a field of characteristic $p > 0$. Let $\delta : A_0 \to A_0$ be a $k$-derivation and let $B = (A_0)^\delta \cap A$. Then $B$ is a $k$-subalgebra of $A$ such that $k[A^p] \subseteq B$ and $B_0 \cap A = B$. In other words, $B$ satisfies condition* (2) *of Theorem* 2. ∎

Using the above proposition and repeating the proof of Example 4 we obtain the following two examples.

EXAMPLE 6. *Let $A$ be the polynomial ring $k[x_0, x_1, x_2, \ldots]$, where $k$ is a field of characteristic* 2. *Consider the $k$-derivation $\delta$ of $A_0$ defined by*

$$\delta(x_0) = 1 \quad \textit{and} \quad \delta(x_i) = 1/x_i^2 \quad \textit{for } i = 1, 2, \ldots$$

*Then the ring $B = (A_0)^\delta \cap A$ satisfies condition* (2) *of Theorem* 2 *and is not the ring of constants of any $k$-derivation of $A$.* ■

EXAMPLE 7. *Let $A$ be the polynomial ring $k[x_0, x_1, x_2, \ldots]$, where $k$ is a field of characteristic $p > 2$. Consider the $k$-derivation $\delta$ of $A_0$ defined by*

$$\delta(x_0) = 1 \quad \textit{and} \quad \delta(x_i) = 1/x_i \quad \textit{for } i = 1, 2, \ldots$$

*Then the ring $B = (A_0)^\delta \cap A$ satisfies condition* (2) *of Theorem* 2 *and is not the ring of constants of any $k$-derivation of $A$.* ■

**4. A question.** Let $k$ be an arbitrary field and let $A$ be a $k$-domain. If $D$ is a family of $k$-derivations of $A$, then we denote by $A^D$ the ring of constants of $A$ with respect to $D$, that is, $A^D = \bigcap_{d \in D} A^d$. Repeating the proof of Example 4 it is easy to deduce that no algebra $B$ in the above examples is of the form $A^D$, where $D$ is a family of $k$-derivations of $A$. Let us end this note with the following question.

QUESTION 8. *Let $A$ be a $k$-domain, where $k$ is a field, and let $D$ be a family of $k$-derivations of $A$. Is it true that there exists a $k$-derivation $d$ of $A$ such that $A^d = A^D$?*

If the algebra $A$ is finitely generated over $k$, then of course the answer to this question is affirmative (this is an easy consequence of Theorems 1 and 2). If $A$ is not finitely generated, then we do not know the answer even in the case when the family $D$ has only two derivations.

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Faculty of Mathematics and Computer Science
N. Copernicus University
87-100 Toruń, Poland
E-mail: pjedrzej@mat.uni.torun.pl


*Received 2 December 2003* (4402)