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ON MARCZEWSKI–BURSTIN REPRESENTABLE ALGEBRAS

BY

MAREK BALCERZAK (Łódź), ARTUR BARTOSZEWICZ (Łódź) and PIOTR KOSZMIDER (São Paulo)

Abstract. We construct algebras of sets which are not MB-representable. The existence of such algebras was previously known under additional set-theoretic assumptions. On the other hand, we prove that every Boolean algebra is isomorphic to an MB-representable algebra of sets.

Introduction. Marczewski in [Sz] introduced operations S and S_0, and applied them to the family of all perfect subsets of a Polish space X. The results of these operations yielded an interesting pair of a σ-algebra and a σ-ideal of sets, investigated by several authors (see e.g. [Mo], [Mi]). As observed in [P] (see also [BBRW]), if the same operations are applied to an arbitrary family $\mathcal{F}$ of nonempty subsets of a set $X \neq \emptyset$, that is,

$$S(\mathcal{F}) = \{E \subseteq X\colon (\forall A \in \mathcal{F})(\exists B \in \mathcal{F}) B \subseteq A \cap E \vee B \subseteq A \setminus E\},$$
$$S_0(\mathcal{F}) = \{E \subseteq X\colon (\forall A \in \mathcal{F})(\exists B \in \mathcal{F}) B \subseteq A \setminus E\},$$

then $S(\mathcal{F})$ and $S_0(\mathcal{F})$ constitute an algebra and an ideal of subsets of X, respectively. An old result of Burstin [Bu] states that the pair consisting of the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$ and the σ-ideal of Lebesgue null sets in $\mathbb{R}$ is of the form $(S(\mathcal{F}), S_0(\mathcal{F}))$, where $\mathcal{F}$ consists of the perfect sets of positive measure. Note that Burstin worked earlier than and independently of Marczewski, and he did not use operations S and S_0 explicitly.

A new direction of study that appeared in [R] and [BR] was devoted to characterization of the families $S(\mathcal{F})$, $S_0(\mathcal{F})$ when $\mathcal{F}$ denotes the collection of perfect sets in various known topologies. Another trend initiated in [BBRW], [BBC], [BET] was concerned with the problem of how to express known algebras and/or ideals of sets in the form $S(\mathcal{F})$, $S_0(\mathcal{F})$ where sometimes $\mathcal{F}$ is required to be a "good" family in an appropriate sense. This is the question of *Marczewski–Burstin representability* (for short, MB-representability) of a given algebra of sets, or of a pair consisting of an algebra and an ideal of sets.

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Recently, the following algebras have been proved MB-representable:

- the algebra of sets with the Baire property in a Polish space [BET],
- the interval algebra on $[0,1)$ [BBRW],
- the algebra of Borel sets in an uncountable Polish space (under $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$) [BBC],
- the algebra of clopen sets in some topological spaces [BR1].

The latest result of Baldwin [Bd] states that if a pair $(\mathcal{A}, \mathcal{I})$ of an algebra and an ideal has the hull property then $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$ and $\mathcal{I} = S_0(\mathcal{A} \setminus \mathcal{I})$, which (by the argument based on mutual coinitiality of generating families [BBRW]) easily recovers the results of Burstin [Bu] and Brown–Elalaoui Talibi [BET]. For other recent results concerning operations S, S_0 and MB-representability, see [Sch], [ET], [BC]. Interestingly, Wroński [W] proved that the maximal number of different algebras which can be obtained by the successive repetitions of the operation $S((\cdot) \setminus \{\emptyset\})$ is three.

In [BBC] it was shown that if $2^\kappa = \kappa^+$ (a part of GCH) and $|X| = \kappa \geq \omega$ then there is a non-MB-representable algebra on X. In the present paper, we propose another method of constructing non-MB-representable algebras. In particular we obtain one such algebra on ω in ZFC. Surprisingly, we show that every Boolean algebra is isomorphic to an MB-representable algebra of sets.

Results. Let $X \neq \emptyset$. For an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ we put

$$\mathcal{H}(\mathcal{A}) = \{A \in \mathcal{A}\colon (\forall B \subseteq A) B \in \mathcal{A}\};$$

this is the ideal of sets that are hereditary in $\mathcal{A}$.

THEOREM 1. *If an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$, $\mathcal{A} \neq \mathcal{P}(X)$, satisfies the condition*

$$(*) \qquad (\forall B \in \mathcal{P}(X) \setminus \mathcal{H}(\mathcal{A}))(\exists A \in \mathcal{A} \setminus \mathcal{H}(\mathcal{A})) A \subseteq B,$$

then $\mathcal{A}$ is not MB-representable.

Proof. Suppose $\mathcal{A}$ is MB-representable and $\mathcal{A} = S(\mathcal{F})$ for some $\mathcal{F} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. First observe that for each $B \in \mathcal{P}(X) \setminus \mathcal{H}(\mathcal{A})$ there is an $F \in \mathcal{F}$ such that $F \subseteq B$. Indeed, suppose there is a $B \in \mathcal{P}(X) \setminus \mathcal{H}(\mathcal{A})$ with $F \setminus B \neq \emptyset$ for each $F \in \mathcal{F}$. By $(*)$, we pick an $A \in \mathcal{A} \setminus \mathcal{H}(\mathcal{A})$ contained in B. Since $A \in S(\mathcal{F})$ and $F \setminus A \neq \emptyset$ for each $F \in \mathcal{F}$, we have $A \in S_0(\mathcal{F}) \subseteq \mathcal{H}(S(\mathcal{F})) = \mathcal{H}(\mathcal{A})$, a contradiction.

Next, we show that $\mathcal{P}(X) = S(\mathcal{F})$, which is also a contradiction. Let $Y \in \mathcal{P}(X)$. Take an $F \in \mathcal{F}$. If $F \notin \mathcal{H}(\mathcal{A})$ then at least one of the sets $F \cap Y$, $F \setminus Y$ is not in $\mathcal{H}(\mathcal{A})$. Thus, by our first observation, there is an $F_1 \in \mathcal{F}$ such that either $F_1 \subseteq F \cap Y$ or $F_1 \subseteq F \setminus Y$. Hence $Y \in S(\mathcal{F})$. If $F \in \mathcal{H}(\mathcal{A})$ then for $Z = Y \cap F$ we have $Z \in \mathcal{H}(\mathcal{A}) \subseteq \mathcal{A}$. Thus there is an $F_2 \in \mathcal{F}$ such that either $F_2 \subseteq Z \cap F = Y \cap F$ or $F_2 \subseteq F \setminus Z = F \setminus (Y \cap F) = F \setminus Y$. Hence $Y \in S(\mathcal{F})$. ∎

For an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$, a family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called $\mathcal{I}$-*almost disjoint* if $F_1 \cap F_2 \in \mathcal{I}$ for any distinct $F_1, F_2 \in \mathcal{F}$.

THEOREM 2. *Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be an ideal such that $|\mathcal{P}(X) \setminus \mathcal{I}| = \kappa$, and*

(**) *for each $Y \in \mathcal{P}(X) \setminus \mathcal{I}$ there is an $\mathcal{I}$-almost disjoint family $\mathcal{G} \subseteq \mathcal{P}(Y) \setminus \mathcal{I}$ of cardinality κ.*

Then there is an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ such that $\mathcal{H}(\mathcal{A}) = \mathcal{I}$ and $\mathcal{A}$ satisfies (*).

Proof. Our argument is motivated by [Ha]. Let X_α, $\alpha < \kappa$, be an enumeration of sets from $\mathcal{P}(X) \setminus \mathcal{I}$. Denote by $\mathcal{A}_0$ the algebra generated by $\mathcal{I}$. We will construct sequences $\langle A_\alpha\colon \alpha < \kappa\rangle$ and $\langle B_\alpha\colon \alpha < \kappa\rangle$ of sets from $\mathcal{P}(X) \setminus \mathcal{I}$ such that for all $\alpha < \kappa$ we have:

(1) $A_\alpha, B_\alpha \subseteq X_\alpha$,
(2) $(\forall \beta < \alpha) B_\beta \notin \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha$ stands for the algebra generated by $\mathcal{A}_0 \cup \{A_\gamma\colon \gamma < \alpha\}$.

Suppose the construction is finished; then we define $\mathcal{A} = \bigcup_{\alpha<\kappa} \mathcal{A}_\alpha$. Let us check that $\mathcal{A}$ is as desired. Obviously $\mathcal{I} \subseteq \mathcal{H}(\mathcal{A})$. To check the converse, take any $X_\alpha \in \mathcal{P}(X) \setminus \mathcal{I}$. Thus $X_\alpha \notin \mathcal{H}(\mathcal{A})$ since $B_\alpha \subseteq X_\alpha$ and $B_\alpha \notin \mathcal{A}$ by (1) and (2). To show (*) consider $X_\alpha \in \mathcal{P}(X) \setminus \mathcal{I}$ and observe that $A_\alpha \subseteq X_\alpha$ and $A_\alpha \in \mathcal{A}$.

Now, let us describe the construction. Let $\alpha < \kappa$ and assume that A_β, B_β for $\beta < \alpha$ have been defined. We will find a set $A_\alpha \subseteq X_\alpha$ with $B_\beta \notin \mathcal{A}_{\alpha+1}$ for all $\beta < \alpha$, and a set $B_\alpha \subseteq X_\alpha$ with $B_\alpha \notin \mathcal{A}_{\alpha+1}$. By (**) there is an $\mathcal{I}$-almost disjoint family $\mathcal{G} = \{C_\xi\colon \xi < \kappa\} \subseteq \mathcal{P}(X_\alpha) \setminus \mathcal{I}$. We claim that we may take for A_α one of the sets C_ξ. If not, for each $\xi < \kappa$ there is a $\beta < \alpha$, $\beta = \beta(\xi)$, such that B_β is in the algebra generated by $\mathcal{A}_\alpha \cup \{C_\xi\}$, that is,

$$B_\beta = (E_\xi \cap C_\xi) \cup (F_\xi \setminus C_\xi) \cup G_\xi,$$

where $E_\xi, F_\xi, G_\xi \in \mathcal{A}_\alpha$ are pairwise disjoint. Recall that $\mathcal{A}_\alpha$ is generated by $\mathcal{A}_0 \cup \{A_\gamma\colon \gamma < \alpha\}$, which means that the quotient algebra $\mathcal{A}_\alpha/\mathcal{I}$ is generated by $\{[A_\gamma]\colon \gamma < \alpha\}$, where $[A_\gamma]$ stands for the corresponding equivalence class. Since $|\mathcal{G}| = \kappa > \alpha$, there are two distinct ordinals $\xi, \xi' < \kappa$ such that $\beta(\xi) = \beta(\xi')$ $(:= \beta)$ and $[E_\xi] = [E_{\xi'}]$, $[F_\xi] = [F_{\xi'}]$, $[G_\xi] = [G_{\xi'}]$ for the corresponding representations of B_β. Write $C \sim D$ whenever $(C \setminus D) \cup (D \setminus C) \in \mathcal{I}$. We have

$$\begin{aligned} B_\beta \cap E_\xi &\sim B_\beta \cap E_{\xi'} \\ &\sim (B_\beta \cap E_\xi) \cap (B_\beta \cap E_{\xi'}) = (E_\xi \cap C_\xi) \cap (E_{\xi'} \cap C_{\xi'}) \subseteq (C_\xi \cap C_{\xi'}) \in \mathcal{I}, \end{aligned}$$

and

$$\begin{aligned} B_\beta \cap F_\xi &\sim B_\beta \cap F_{\xi'} \\ &\sim (B_\beta \cap F_\xi) \cup (B_\beta \cap F_{\xi'}) = (F_\xi \setminus C_\xi) \cup (F_{\xi'} \setminus C_{\xi'}) \\ &\sim (F_\xi \setminus C_\xi) \cup (F_\xi \setminus C_{\xi'}) = F_\xi \setminus (C_\xi \cap C_{\xi'}) \sim F_\xi. \end{aligned}$$

Hence

$$B_\beta = (B_\beta \cap E_\xi) \cup (B_\beta \cap F_\xi) \cup G_\xi \in \mathcal{A}_\alpha,$$

which contradicts the assumption.

Now, if $\mathcal{A}_{\alpha+1}$ is defined, we choose a set $B_\alpha \subseteq X_\alpha$ with $B_\alpha \notin \mathcal{A}_{\alpha+1}$ by a similar argument (we consider an $\mathcal{I}$-almost disjoint subfamily of $\mathcal{P}(X_\alpha) \setminus \mathcal{I}$; at least one set in this family is not in $\mathcal{A}_{\alpha+1}$). This ends the construction. ■

Recall the following theorem from [Ku]:

THEOREM 3. *If $|X| = \kappa \geq \omega$ and $2^{<\kappa} = \kappa$, then there is a family $\mathcal{A} \subseteq [X]^\kappa$ with $|\mathcal{A}| = 2^\kappa$ and $|A \cap B| < \kappa$ for all $A, B \in \mathcal{A}$, $A \neq B$.*

Let us apply Theorem 2 to $X = \omega$ and $\mathcal{I} = [\omega]^{<\omega}$. Then by Theorem 3, condition $(**)$ is satisfied. Thus we obtain the following corollary:

COROLLARY 1. *There is an algebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{A}$ is not MB-representable and $\mathcal{H}(\mathcal{A}) = [\omega]^{<\omega}$.*

Next, we apply Theorem 2 to the case when $|X| = \mathfrak{c}$, so we may assume that $X = \mathbb{R}$. The equality $2^{<\mathfrak{c}} = \mathfrak{c}$ is not provable in ZFC but it is ensured by MA (Martin's Axiom). Assume MA, and let either $\mathcal{I} = \mathcal{M}$, the ideal of meager sets in $\mathbb{R}$, or $\mathcal{I} = \mathcal{N}$, the ideal of Lebesgue null sets in $\mathbb{R}$. Thus for any $Y \in \mathcal{P}(\mathbb{R}) \setminus \mathcal{I}$ we have $|Y| = \mathfrak{c}$ since, under MA, if $Y \subseteq \mathbb{R}$ and $|Y| < \mathfrak{c}$ then $Y \in \mathcal{I}$. Consequently, from Theorems 1–3 we derive

COROLLARY 2. *Assume MA. Let $\mathcal{I} = \mathcal{M}$ or $\mathcal{I} = \mathcal{N}$. Then there is an algebra $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$ such that $\mathcal{A}$ is not MB-representable and $\mathcal{H}(\mathcal{A}) = \mathcal{I}$.*

Although, as we have seen, not every algebra of sets is MB-representable, we have the following positive result:

THEOREM 4. *For every Boolean algebra $\mathcal{A}$ there is a set $X \neq \emptyset$ and a family $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $S(\mathcal{F})$ is isomorphic to $\mathcal{A}$ and $S_0(\mathcal{F}) = \{\emptyset\}$.*

Proof. Let Y be the Stone space of $\mathcal{A}$, that is, the unique zero-dimensional compact Hausdorff space whose algebra of clopen sets is isomorphic to $\mathcal{A}$. Let $X = Y \times Y$. Define $\mathcal{A}^+$ to be the algebra of all subsets of X of the form $A \times Y$, where $A \in \mathcal{A}$. It is clear that $\mathcal{A}^+$ and $\mathcal{A}$ are isomorphic. For each $y \in Y$ and clopen A such that $y \in A$, define

$$F(y, A) = (\{y\} \times Y) \cup ((A \setminus \{y\}) \times (Y \setminus \{y\})).$$

Put $\mathcal{F} = \{F(y, A)\colon y \in A,\ A \text{ is clopen}\}$. Note that for any $F(x, A), F(y, B) \in \mathcal{F}$ we have

$$(***) \qquad F(x, A) \subseteq F(y, B) \ \Rightarrow\ x = y.$$

This is because $\{x\} \times Y \subseteq F(x', A)$ if and only if $x = x'$.

First, let us show that $\mathcal{A}^+ \subseteq S(\mathcal{F})$. Given $A \times Y \in \mathcal{A}^+$ and $F(y, B) \in \mathcal{F}$, if $y \in A$ then $F(y, A \cap B) = F(y, B) \cap (A \times Y)$. If $y \notin A$ then $F(y, B \setminus A) \subseteq F(y, B)$ and $F(y, B \setminus A) \cap (A \times Y) = \emptyset$.

Now, let us prove that $S(\mathcal{F}) \subseteq \mathcal{A}^+$. So, let $B \in \mathcal{P}(X) \setminus \mathcal{A}^+$. Consider two cases.

Case 1: There is $y \in Y$ such that neither $\{y\} \times Y \subseteq B$ nor $(\{y\} \times Y) \cap B = \emptyset$. Take $F(y, Y)$. By $(***)$, any subset of $F(y, Y)$ in $\mathcal{F}$ is of the form $F(y, A)$ for $A \in \mathcal{A}$, but such a set includes $\{y\} \times Y$, so it cannot be disjoint from nor include B. Thus $B \notin S(\mathcal{F})$.

Case 2: Case 1 is false. Put $A = \{y \in Y\colon \{y\} \times Y \subseteq B\}$. We have $B = A \times Y$. Since $B \notin \mathcal{A}^+$, the set A is not clopen, that is, either A or $Y \setminus A$ is not closed. Since $S(\mathcal{F})$ is an algebra of sets, we may assume without loss of generality that A is not closed. Let $y \in (\operatorname{cl} A) \setminus A$ and consider $F(y, Y)$. Again by $(***)$ any element of $\mathcal{F}$ included in $F(y, Y)$ is of the form $F(y, C)$ for a clopen $C \subseteq Y$ with $y \in C$. Then from $y \in (\mathrm{cl}A) \cap C$ it follows that $A \cap C \neq \emptyset$. Hence $F(y, C) \cap B \neq \emptyset$. Also since $y \notin A$, we never have $F(y, C) \subseteq B$. So $B \notin S(\mathcal{F})$.

Observe that $S_0(\mathcal{F}) \subseteq \mathcal{H}(S(\mathcal{F})) = \mathcal{H}(\mathcal{A}^+) = \{\emptyset\}$. Thus $S_0(\mathcal{F}) = \{\emptyset\}$. ∎

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M. Balcerzak
Institute of Mathematics
Łódź Technical University
Al. Politechniki 11, I-2
90-924 Łódź, Poland
E-mail: mbalce@p.lodz.pl
and
Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland

A. Bartoszewicz
Institite of Mathematics
Łódź Technical University
Al. Politechniki 11, I-2
90-924 Łódź, Poland
E-mail: arturbar@p.lodz.pl

P. Koszmider
Departamento de Matemática
Universidade de São Paulo
Caixa Postal 66281
São Paulo, SP, CEP: 05315-970, Brasil
E-mail: piotr@ime.usp.br

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