COLLOQUIUM MATHEMATICUM

VOL. 99 2004 NO. 1

ASYMPTOTICS OF PARABOLIC EQUATIONS WITH POSSIBLE BLOW-UP

BY

RADOSŁAW CZAJA (Katowice)

Abstract. We describe the long-time behaviour of solutions of parabolic equations in the case when some solutions may blow up in a finite or infinite time. This is done by providing a maximal compact invariant set attracting any initial data for which the corresponding solution does not blow up. The abstract result is applied to the Frank-Kamenetskii equation and the N-dimensional Navier–Stokes system with small external force.

1. Introduction. In this paper we study the asymptotic behaviour of parabolic equations when some solutions may blow up in a finite or infinite time. We consider X^α solutions as in [C-D1] and earlier in [HE] with the modification given in [MI]. We make use of the theory of semilinear abstract parabolic equations given e.g. in [HE], [HA] or [CZ].

The situation that for some initial data the corresponding solution blows up often occurs in physical applications. For a detailed mathematical description of this phenomenon we refer the reader to [G-V]. Here we only mention a particular problem with a parameter $\lambda > 0$,

$$\begin{cases} u_t = \Delta u + \lambda e^u, & t > 0, \\ u(0) = u_0, \end{cases}$$

which comes from combustion theory and is known under the name of *solid fuel ignition model*, *exponential reaction-diffusion equation* or *Frank-Kamenetskii equation*, the latter name being used from now on in this paper. Among many other results concerning this problem (see e.g. [GE], [J-L], [B-E]) it was shown in [FU] that for special initial data u_0 the corresponding solution blows up. On the other hand, there also exist parabolic problems for which global solvability for all initial data remains unknown. A typical example is the famous N-dimensional Navier–Stokes system, $N \geq 3$, that has now been investigated for more than a century.

2000 *Mathematics Subject Classification*: Primary 35B40; Secondary 35K90, 35K55.

Key words and phrases: parabolic equation, attracting set, blow-up, Navier–Stokes system, Frank-Kamenetskii equation.

The above mentioned specific circumstances make it impossible to describe the asymptotics by using the notion of a global attractor in the large space of initial data as in [HA], [LA], [C-D1]. Instead, we consider the maximal compact invariant set attracting each initial condition for which the corresponding solution does not blow up. We first describe all these ideas abstractly in Section 2 considering a Cauchy problem for a semilinear sectorial equation in a Banach space. Next in Section 3 we discuss two particular examples: the one-dimensional version of the Frank-Kamenetskii equation and the N-dimensional Navier–Stokes system with small external force.

2. Abstract parabolic problems with possible blow-up. Consider an abstract parabolic problem

$$\begin{cases} u_t + Au = F(u), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where X is a Banach space and A: $X \supset \mathrm{dom}(A) \to X$ is a positive sectorial operator with compact resolvent. Moreover, assume that F: $X^\alpha \to X$ ($\alpha \in [0,1)$ is fixed from now on) is Lipschitz continuous on bounded subsets of the fractional power space $X^\alpha = \mathrm{dom}(A^\alpha)$ (cf. [HE], [AM]).

Under the above assumptions the theory of semilinear parabolic equations given e.g. in [HE] ensures for each $u_0 \in X^\alpha$ the existence of a unique local X^α solution defined on a maximal interval of existence $[0, \tau_{u_0})$, where $\tau_{u_0} \le \infty$. Thus we know that

$$u \in C([0,\tau_{u_0}), X^\alpha) \cap C((0,\tau_{u_0}), X^1) \cap C^1((0,\tau_{u_0}), X)$$

and (2.1) is satisfied in X. Moreover, we have either $\tau_{u_0} = \infty$, or $\tau_{u_0} < \infty$ and

$$\limsup_{t \to \tau_{u_0}} \|u(t,u_0)\|_{X^\alpha} = \infty.$$

Since the problem (2.1) is autonomous, the uniqueness of solutions allows us to construct a local semiflow on X^α. We thus have

$$u(s, u(t,u_0)) = u(s+t, u_0), \qquad u_0 \in X^\alpha,\ s,t \ge 0,\ s+t < \tau_{u_0},$$

and the solutions of (2.1) are continuous with respect to their initial data on compact time intervals (cf. [HE, Theorem 3.4.1] or [C-D1, Proposition 2.3.2]).

For our further investigations let us define a metric space

$$V = \{u_0 \in X^\alpha\colon \sup_{t \in [0,\tau_{u_0})} \|u(t,u_0)\|_{X^\alpha} < \infty\} \tag{2.2}$$

and assume that V is nonvoid.

Consider a C^0-semigroup $T(t)$: $V \to V$ defined by

$$T(t)u_0 = u(t,u_0), \qquad t \ge 0,\ u_0 \in V. \tag{2.3}$$

Note that we do not know in advance whether V is a closed subset of X^α. Thus it is unknown if V, which is the natural phase space for the problem (2.1), is a complete metric space or not. Therefore the assumption of the compact resolvent does not necessarily imply the compactness of the semigroup $\{T(t)\colon t \geq 0\}$ on V. Nevertheless, we shall show below that this semigroup is *asymptotically smooth* in the sense of [HA].

For $B \subset V$ we denote its *orbit* by $\gamma^+(B) = \bigcup_{t\geq 0} T(t)B$, while its *ω-limit set* is given by

$$\omega(B) = \bigcap_{t\geq 0} \mathrm{cl}_{X^\alpha} T(t)\gamma^+(B).$$

We also abbreviate $\gamma^+(\{v\})$ and $\omega(\{v\})$ to $\gamma^+(v)$ and $\omega(v)$, respectively.

REMARK 2.1. *If $B \subset V$ and $\gamma^+(B)$ is bounded, then $\mathrm{cl}_{X^\alpha} \gamma^+(B) \subset V$ and $T(t)B$ is precompact in X^α for any $t > 0$.*

Proof. Let $v_0 \in \mathrm{cl}_{X^\alpha} \gamma^+(B)$. Then there exist $v_n \in B$ and $t_n \geq 0$ such that $T(t_n)v_n \to v_0$ in X^α. Since

$$\forall_{s\geq 0} \quad \|u(s, T(t_n)v_n)\|_{X^\alpha} = \|u(s + t_n, v_n)\|_{X^\alpha} \leq R_{\gamma^+(B)},$$

the norm $\|u(s, v_0)\|_{X^\alpha}$ cannot blow up so that $v_0 \in V$. Also, if $\gamma^+(B)$ is bounded in X^α, then $T(t)B$ with $t > 0$ is in fact bounded in $X^{\alpha+\varepsilon}$, which, via compactness of the embeddings $X^\beta \subset X^\alpha$, $\beta > \alpha$, ensures that it is precompact in X^α. ∎

PROPOSITION 2.2. *The C^0-semigroup $\{T(t)\colon t \geq 0\}$ on V is asymptotically smooth, i.e. each nonvoid closed (in V) bounded and positively invariant set $W \subset V$ contains a nonvoid compact subset $\omega(W)$ which attracts W.*

Proof. Since $\gamma^+(W) \subset W$, we infer from Remark 2.1 that $\mathrm{cl}_{X^\alpha} T(1)\gamma^+(W)$ is a compact subset of V. Thus

$$\omega(W) = \bigcap_{t\geq 0} \mathrm{cl}_{X^\alpha} T(t)\gamma^+(W) \subset \mathrm{cl}_{X^\alpha} T(1)\gamma^+(W) \subset V$$

is compact and nonvoid as the intersection of a centered family of closed sets in a compact space. Also $\omega(W) \subset W$, since by our assumptions on W and Remark 2.1 we have

$$\omega(W) \subset \mathrm{cl}_{X^\alpha} W \cap V = \mathrm{cl}_V W = W.$$

We now prove that $\omega(W)$ attracts W, that is,

$$d(T(t)W, \omega(W)) \underset{t\to\infty}{\longrightarrow} 0,$$

where d denotes the Hausdorff semidistance. Contrary to our claim suppose that

$$\exists_{\varepsilon>0}\ \exists_{t_n\to\infty}\ \forall_{n\in\mathbb{N}}\ \exists_{x_n\in W}\ \forall_{v\in\omega(W)} \quad \|T(t_n)x_n - v\|_{X^\alpha} > \varepsilon. \tag{2.4}$$

Since almost all elements of the sequence $\{T(t_n)x_n\}$ belong to the compact set $\mathrm{cl}_{X^\alpha} T(1)\gamma^+(W)$ it contains a subsequence convergent to an element of $\omega(W)$, which contradicts (2.4). ∎

Remark 2.3. It is worth noting that Remark 2.1 implies immediately that if $u_0 \in V$, then $\mathrm{cl}_{X^\alpha} \gamma^+(u_0)$ is a compact positively invariant subset of V and the ω-limit set $\omega(u_0)$ is a nonvoid compact connected and invariant subset of V which attracts u_0 (cf. [LA, Theorem 2.1]). More generally, since by Proposition 2.2 the semigroup is asymptotically smooth, we infer that for $\emptyset \neq B \subset V$ with $\gamma^+(B)$ bounded, the ω-limit set $\omega(B)$ is a nonvoid compact and invariant subset of V attracting B (see [C-D1, Proposition 1.1.1]).

Under an additional assumption of the *point dissipativeness* of the semigroup $\{T(t): t \geq 0\}$ we now prove the existence of a nonvoid compact and invariant set which attracts each point of V. Indeed, if B_0 is a nonvoid bounded subset of V attracting points of V, then any ε-neighbourhood $\mathcal{N}(B_0) = \bigcup_{u \in B_0} B(u,\varepsilon)$ of B_0 in V absorbs each point of V. Consequently,

$$\widetilde{B}_0 = \{v \in \mathcal{N}(B_0): \gamma^+(v) \subset \mathcal{N}(B_0)\}$$

is a nonvoid bounded positively invariant subset of V absorbing each point of V. From this and Remark 2.3 it follows that $\omega(\widetilde{B}_0)$ is a nonvoid compact invariant subset of V attracting each point of V.

The required dissipativeness property can be easily controlled if there exists a Lyapunov function. Recall that by a *Lyapunov function* on V we mean a continuous function $\mathcal{L}: V \to \mathbb{R}$ such that for any $u_0 \in V$,

(i) the function $t \mapsto \mathcal{L}(u(t,u_0))$ is nonincreasing in $(0,\infty)$,
(ii) if $\mathcal{L}(u(\cdot,u_0)) \equiv \mathcal{L}(u_0)$, then $u_0 \in \mathcal{E}$,

where $\mathcal{E}$ denotes the set of all stationary solutions of (2.1). Recall also that the existence of a Lyapunov function on V implies that $\omega(u_0) \subset \mathcal{E}$ for each $u_0 \in V$. Therefore, if there exists a Lyapunov function on the metric space V given in (2.2), then the set $\mathcal{E}$ is nonvoid, and if it is bounded, it is also compact and attracts each point of V. Thus we get

Corollary 2.4. *Suppose that $\{T(t): t \geq 0\}$ is defined on V given in* (2.2). *Assume further that $\{T(t): t \geq 0\}$ is point dissipative* (*for example*, *there exists a Lyapunov function on V and $\mathcal{E}$ is bounded*). *Then for any $u_0 \in X^\alpha$ the X^α solution $u(\cdot,u_0)$ of* (2.1) *either blows up* (*in a finite or infinite time*) *or stays bounded and approaches a nonvoid compact and invariant set.*

We have shown above that the semigroup is asymptotically smooth and we have also stated natural conditions ensuring its point dissipativeness. However, these two properties do not guarantee the existence of a compact global attractor in V. It would exist if we knew the semigroup on V was compact (cf. [C-D1, Corollary 1.1.6]), which, as we have already observed, may not be the case, or if the orbits of bounded sets were bounded (cf. [C-D1, Theorem 1.1.2]). Unfortunately, the latter condition may be difficult to check in specific examples. As will be seen in Section 3 it is much easier to examine the boundedness of the set of all (hypothetical) bounded complete orbits of points.

Following [LA], we recall that a *complete trajectory* of a point $v \in V$ for the semigroup $\{T(t)\colon t \geq 0\}$ is the curve $\phi\colon \mathbb{R} \to V$ satisfying the following conditions:

(i) $\phi(0) = v$,
(ii) $T(t)\phi(s) = \phi(s+t)$, $s \in \mathbb{R}$, $t \geq 0$.

If $\mathcal{S}$ denotes the set of all points in V for which there exists at least one *bounded* complete trajectory for the semigroup $\{T(t)\colon t \geq 0\}$, then $T(t)\mathcal{S} = \mathcal{S}$ for each $t > 0$. Also the following result holds.

THEOREM 2.5. *Suppose that the semigroup $\{T(t)\colon t \geq 0\}$ is defined by* (2.3) *on the nonvoid metric space V given in* (2.2). *Then $\mathcal{S}$ is a nonvoid invariant subset of V which attracts each subset of V with bounded orbit. If $\mathcal{S}$ is bounded, then it is a compact and maximal bounded invariant subset of V. If, additionally, the orbits of bounded subsets of V are bounded, then $\mathcal{S}$ is a compact global attractor in V.*

Proof. As a consequence of Remark 2.3, $\mathcal{S}$ is nonvoid whenever V is nonvoid. Moreover, $\omega(B) \subset \mathcal{S}$ for any $\emptyset \neq B \subset V$ such that $\gamma^+(B)$ is bounded. It is next sufficient to note that if $\mathcal{S}$ is bounded, then—since it is also invariant—$\mathrm{cl}_{X^\alpha} \mathcal{S}$ is a compact subset of V (see Remark 2.1). The proof is thus complete. ■

COROLLARY 2.6. *Suppose A is a sectorial operator in a Banach space X and A has compact resolvent. Assume that $F\colon X^\alpha \to X$, with $\alpha \in [0,1)$ fixed, is Lipschitz continuous on bounded subsets of X^α. Let all* (*hypothetical*) *bounded complete orbits of points be uniformly bounded in X^α. Then for any $u_0 \in X^\alpha$ the X^α solution $u(\cdot, u_0)$ of* (2.1) *either blows up* (*in a finite or infinite time*) *or stays bounded and approaches a maximal compact invariant set.*

We recall that similarly to the case of an ω-limit set, the *α-limit set* of $u_0 \in \mathcal{S}$ along a bounded complete trajectory ϕ of u_0,

$$\alpha_\phi(u_0) = \bigcap_{t \leq 0} \mathrm{cl}_{X^\alpha} \bigcup_{s \leq t} \{\phi(s)\},$$

is a nonvoid compact subset of V. If, in addition, there exists a Lyapunov function on V, then $\alpha_\phi(u_0) \subset \mathcal{E}$. In the latter case abstract conditions for the boundedness of $\mathcal{S}$ can be formulated.

Proposition 2.7. *Assume that there exists a Lyapunov function* $\mathcal{L}$ *on* V. *Then the following conditions are equivalent*:

(a) $\mathcal{S}$ *is a bounded subset of* V,

(b) $\mathcal{E}$ *is a bounded subset of* V *and one of the equivalent conditions holds*:

  (i) *if* $v_n \in \mathcal{S}$ *and* $\|v_n\|_{X^\alpha} \to \infty$, *then* $|\mathcal{L}(v_n)| \to \infty$ (cf. [HA, Definition 3.8.1]),

  (ii) *if* $B \subset \mathcal{S}$ *and* $\mathcal{L}(B)$ *is a bounded subset of* $\mathbb{R}$, *then* B *is bounded.*

Proof. We shall prove that (b) implies (a). Therefore we assume (b) and suppose contrary to our claim that there exist $v_n = \phi_n(t_n)$, where $t_n \in \mathbb{R}$, ϕ_n are bounded complete trajectories, say of some u_n, and $\|v_n\|_{X^\alpha} \to \infty$ as $n \to \infty$. We choose $\alpha_n \in \alpha_{\phi_n}(u_n)$ and $\omega_n \in \omega(u_n)$, both in $\mathcal{E}$. Since the Lyapunov function $\mathcal{L}$ is nonincreasing along each complete trajectory, we see that

$$\max\{\mathcal{L}(e)\colon e \in \mathcal{E}\} \geq \mathcal{L}(\alpha_n) \geq \mathcal{L}(v_n) \geq \mathcal{L}(\omega_n) \geq \min\{\mathcal{L}(e)\colon e \in \mathcal{E}\},$$

where the maximum and minimum exist due to the compactness of $\mathcal{E}$. But this is impossible, because of (i). This shows that $\mathcal{S}$ is bounded. ∎

3. Examples

Example 3.1. Consider the Dirichlet problem for the *Frank-Kamenetskii equation*

$$\begin{cases} u_t = \Delta u + \lambda e^u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where $\lambda > 0$ is a parameter and $\Omega = B(0,1) \subset \mathbb{R}^N$. This problem occurs in models of thermal explosions, especially in the description of thermal self-ignition of a chemically active mixture contained in some vessel. We refer the reader to [FK], [GE, §15] and [B-E] for more details.

Rewriting the problem (3.1) in an abstract setting we consider

$$\begin{cases} u_t + Au = F(u), & t > 0, \\ u(0) = u_0, \end{cases} \tag{3.2}$$

in the Hilbert space $X = L^2(\Omega)$, where $A = -\Delta_D\colon X \supset \operatorname{dom}(A) \to X$ with $\operatorname{dom}(A) = H^2(\Omega) \cap H^1_0(\Omega)$. It is well known that this operator is a positive sectorial operator in X with compact resolvent. Although the Frank-Kamenetskii equation is also interesting and complex for dimensions $3 \leq N \leq 9$ (especially because of an infinite number of stationary solutions for $\lambda = 2(N-2)$, see e.g. [J-L], [B-E, Theorem 2.19], [F-P], [N-S]), we restrict our attention to $N = 1$. Nevertheless, we still use the general notation.

Fix $3/4 < \alpha < 1$ so that $X^\alpha \subset C^1(\overline{\Omega})$. Evidently, if $u \in X^\alpha$, then $u \in C^1(\overline{\Omega})$, $F(u) = \lambda e^u \in C^1(\overline{\Omega})$, and $F\colon X^\alpha \to X$ is Lipschitz continuous on bounded sets. Consequently, (3.1) generates a local semiflow of X^α solutions. Denoting then by V the set of all initial data for which the solution stays bounded in X^α we see that

$$\mathcal{L}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \lambda e^u\,dx \tag{3.3}$$

is a Lyapunov function on V.

Stationary solutions and their Morse indices. If $w \in \operatorname{dom}(A)$ is a stationary solution, then w satisfies

$$\begin{cases} -\Delta w = \lambda e^w, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

which is also known in the literature under the names of the *Emden–Fowler equation* or *Gelfand problem*. From the regularity theory of elliptic operators (cf. [TR, Theorem 5.4.1]) it follows that the stationary solutions are smooth and in particular they belong to $C^2([-1,1])$. Also, all solutions of (3.4) are positive, and thus radially symmetric by the result of [G-N-N]. We recall (see [B-E, Theorem 2.19]) that there exists $\lambda^* > 0$, $\lambda^* \approx 0.878$, such that

(a) for each $\lambda \in (0,\lambda^*)$ there are two solutions,
(b) for $\lambda = \lambda^*$ there is a unique solution,
(c) for $\lambda > \lambda^*$ there are no solutions.

Moreover, each solution w has to satisfy

$$w(x) = w(0) - 2\ln\cosh\left(\frac{1}{2}\sqrt{2\lambda e^{w(0)}}\,x\right), \quad -1 \le x \le 1. \tag{3.5}$$

If $\lambda \in (0,\lambda^*)$, then from [FU] we infer that there exists the minimal solution. Let us denote it by w^+ and the maximal solution by w^-. We know that there exists $\gamma > 0$ such that

$$\gamma\varrho(x) \le w^-(x) - w^+(x), \qquad x \in \overline{\Omega},$$

where $\varrho(x)$ is the distance from x to $\partial\Omega$. Consequently, the curve shown in Figure 1 describes the set of solution curves

$$\{(\lambda,w) \in (0,\infty) \times C(\overline{\Omega})\colon (\lambda,w) \text{ satisfies } (3.4)\}.$$

We remark that, as shown in [FU, Theorem 6] (see also the refinement in [FI, Remark 2.5]), if $\lambda \in (0,\lambda^*)$, $w^-(x) \le u_0(x)$ for $x \in \overline{\Omega}$ and $u_0 \not\equiv w^-$, then the solution u of (3.1) blows up in a finite time.

Consider now the linearization of (3.4) at w,

$$\begin{cases} \Delta v + \lambda e^w v = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.6}$$

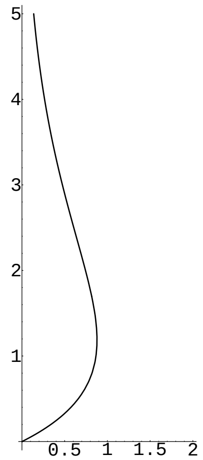

Fig. 1. Dependence between $w(0)$ and λ

and observe that $\sigma(-\Delta-\lambda e^{w})$ consists only of real eigenvalues. We say that w is a *hyperbolic stationary solution* if $0\notin\sigma(-\Delta-\lambda e^{w})$. Furthermore, the number of negative eigenvalues of $-\Delta-\lambda e^{w}$ is called the *Morse index* $\operatorname{ind}(w)$ of the stationary solution w. Although $0\in\sigma(-\Delta-\lambda^{*}e^{w})$, it is known that if $\lambda\in(0,\lambda^{*})$, then w^{-} and w^{+} are both hyperbolic stationary solutions. Additionally, we have $\operatorname{ind}(w^{+})=0$ and $\operatorname{ind}(w^{-})=1$ (for details see [C-R, Proposition 2.15] and [N-S, Section 2]).

Unstable manifold of w^{-} and description of $\mathcal{S}$. Let $\lambda\in(0,\lambda^{*})$. Note that w^{-} is a hyperbolic fixed point in the sense of [C-C-H, p. 357]. Then the *unstable set* $W^{\mathrm{u}}(w^{-})$ is a C^{1} submanifold of X^{α} with

$$\dim W^{\mathrm{u}}(w^{-})=\operatorname{ind}(w^{-})=1$$

(see [HE, Theorem 6.1.9], [C-C-H, Appendix C]) and by [B-F1, Theorem 2.1] we have

$$\forall_{v\in W^{\mathrm{u}}(w^{-})}\quad z(v-w^{-})<\dim W^{\mathrm{u}}(w^{-})=1, \tag{3.7}$$

where $z(g)$ denotes the number of sign changes of a continuous function g.

The existence of a Lyapunov function excludes the existence of (nonconstant) homoclinic orbits. Thus we restrict our attention to heteroclinic orbits. Let ϕ be a (hypothetical) complete trajectory of $u_0\in X^{\alpha}$ such that $\phi(t)\to w^{-}$ as $t\to-\infty$ and $\phi(t)\to w^{+}$ as $t\to\infty$ in $X^{\alpha}\subset C^{1}(\overline{\Omega})$. Since the complete trajectory ϕ does not blow up, from (3.7) we obtain

$$\forall_{t\in\mathbb{R}}\ \forall_{x\in\overline{\Omega}}\quad \phi(t)(x)\le w^{-}(x).$$

If we show that

$$\exists_{t^{*}<0}\ \forall_{t\le t^{*}}\ \forall_{x\in\overline{\Omega}}\quad w^{+}(x)\le\phi(t)(x), \tag{3.8}$$

by monotonicity we will then have

$$\forall_{t\in\mathbb{R}}\ \forall_{x\in\overline{\Omega}}\quad w^+(x) \le \phi(t)(x) \le w^-(x).$$

Hence for the boundedness of $\mathcal{S}$ it is sufficient to prove (3.8).

Contrary to our claim, assume that there exist $t_n \to -\infty$, $y_n \in \overline{\Omega}$ such that

$$\phi(t_n)(y_n) < w^+(y_n). \tag{3.9}$$

Then there exist $x_n \to x_0$ with $x_0 \in \overline{\Omega}$ and

$$\phi(t_n)(x_n) = w^+(x_n), \qquad n \in \mathbb{N}. \tag{3.10}$$

Indeed, there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ we have $\phi(t_n)(0) > w^+(0)$. Since (3.9) holds, the Darboux property ensures the existence of $x_n \in \overline{\Omega}$ such that $\phi(t_n)(x_n) = w^+(x_n)$. By the compactness of $\overline{\Omega}$ we may choose a convergent subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, which is as required.

We consider two cases. If $x_0 \in (-1,1)$, then $\phi(t_n)(x_n)$ tends to $w^-(x_0)$ and $w^-(x_0) = w^+(x_0)$, which is impossible. Therefore $x_0 \in \{-1,1\}$. From (3.5) we get

$$(w^+)'(1) > (w^-)'(1), \qquad (w^+)'(-1) < (w^-)'(-1).$$

Let us consider $x_0 = 1$. By (3.10) and the mean value theorem we have

$$\frac{w^+(x_n) - w^+(1)}{x_n - 1} = \frac{\phi(t_n)(x_n) - \phi(t_n)(1)}{x_n - 1} = [\phi(t_n)]'(\xi_n).$$

Since the left hand side tends to $(w^+)'(1)$ and the right hand side to $(w^-)'(1)$, we get $(w^+)'(1) = (w^-)'(1)$, a contradiction. The same reasoning applies to $x_0 = -1$. This ends the proof of the boundedness of $\mathcal{S}$.

Observe that the solution semigroup for $u \in X^\alpha$ satisfying $w^+(x) \le u(x) \le w^-(x)$ for all $x \in \overline{\Omega}$ is compact. Hence it has a compact *connected* global attractor. In particular there must exist a heteroclinic orbit connecting w^- to w^+.

In fact there exists a unique heteroclinic orbit connecting these two equilibria. This can be established using the argument in [B-F3, Lemma 3.5]. Thus we obtain the description of $\mathcal{S} = \{w^-, w^+, \phi(\mathbb{R})\}$, where ϕ is the only complete trajectory connecting w^- to w^+.

We conclude that $\mathcal{S}$ is a maximal compact invariant set attracting any subset of V with bounded orbit. In particular, if $u_0 \in X^\alpha$, then the corresponding X^α solution either blows up or stays bounded and approaches a maximal compact invariant set.

EXAMPLE 3.2. Consider the N-dimensional Navier–Stokes system for incompressible viscous fluid flow subject to a small perturbation. Following [C-D2, Section 4], we shall show that in this case there exists a Lyapunov function. We consider the problem

$$(3.11)\qquad \begin{cases} u_t = \nu \Delta u - \nabla p - (u \cdot \nabla) u + f, & t > 0,\ x \in \Omega, \\ \operatorname{div} u = 0, & t > 0,\ x \in \Omega, \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where $N \geq 2$, $\nu > 0$ is a viscosity constant and Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $C^{2+\varepsilon}$.

For any $f \in [L^p(\Omega)]^N$, $p > N$, the system can be viewed as an abstract Cauchy problem

$$(3.12)\qquad \begin{cases} u_t + Au = F(u), & t > 0, \\ u(0) = u_0, \end{cases}$$

in the space

$$X = \mathrm{cl}_{[L^p(\Omega)]^N} \{\phi \in [C_0^\infty(\Omega)]^N \colon \operatorname{div} \phi = 0 \text{ in } \Omega\}$$

using the projector P given by the decomposition of $[L^p(\Omega)]^N$ into the spaces of divergence free vector fields and scalar function gradients (see [F-M] and [G-M]). Namely, we define $A = -\nu P \Delta \colon X \supset \mathrm{dom}(A) \to X$ with

$$\mathrm{dom}(A) = X \cap \{\phi \in [W^{2,p}(\Omega)]^N \colon \phi = 0 \text{ on } \partial\Omega\},$$

which is a sectorial operator with compact resolvent, and $F \colon X^\alpha \to X$ by

$$(3.13)\qquad F(u) = -P(u \cdot \nabla)u + Pf, \quad u \in X^\alpha.$$

Restricting further α to the interval $[1/2, 1)$ we observe that F in (3.13) is well defined and is Lipschitz continuous on bounded subsets of X^α.

Recall that for sufficiently small $f \in [L^p(\Omega)]^N$ (especially if the external force f is zero)

(3.14) there exists a stationary solution $w \in \mathrm{dom}(A)$ of the Navier–Stokes system such that $\|w\|_{[W^{1,\infty}(\Omega)]^N} < \nu / C_\Omega^2$,

where C_Ω is the constant in the Poincaré inequality.

Lyapunov function on V and description of $\mathcal{S}$. Assuming (3.14) we define V as in (2.2) and consider the functional

$$(3.15)\qquad \mathcal{L}(u) = \tfrac{1}{2} \|u - w\|^2_{[L^2(\Omega)]^N}, \quad u \in V.$$

We shall show that $\mathcal{L}$ is a Lyapunov function on V. Since $p > N \geq 2$, it follows that $\mathcal{L}$ is continuous on V. Fix $u_0 \in V$. Letting $u(t) = u(t, u_0)$, $t \geq 0$, we have

$$(3.16)\quad (u - w)_t = -A(u - w) - P((u - w) \cdot \nabla) w - P(u \cdot \nabla)(u - w)$$

for $t > 0$. From [F-M] it follows that

$$Pv = P_2 v, \quad v \in [L^p(\Omega)]^N,$$

where P_2 is a selfadjoint bounded projection operator on $[L^2(\Omega)]^N$. Hence for $v_1, v_2, v_3 \in \mathrm{dom}(A)$ we have

$$\begin{aligned}\langle P(v_1\cdot\nabla)v_2, v_3\rangle_{[L^2(\Omega)]^N} &= \langle P_2(v_1\cdot\nabla)v_2, v_3\rangle_{[L^2(\Omega)]^N}\\ &= \langle (v_1\cdot\nabla)v_2, v_3\rangle_{[L^2(\Omega)]^N}\end{aligned}$$

and

$$\langle Av_1, v_1\rangle_{[L^2(\Omega)]^N} = -\nu\langle \Delta v_1, v_1\rangle_{[L^2(\Omega)]^N}.$$

Multiplying (3.16) by $u-w$ in $[L^2(\Omega)]^N$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u-w\|^2_{[L^2(\Omega)]^N} &\le -\nu\sum_{i=1}^N \|\nabla(u_i-w_i)\|^2_{L^2(\Omega)}\\ &\quad + \|w\|_{[W^{1,\infty}(\Omega)]^N}\|u-w\|^2_{[L^2(\Omega)]^N}\\ &\le (-\nu/C_\Omega^2 + \|w\|_{[W^{1,\infty}(\Omega)]^N})\|u-w\|^2_{[L^2(\Omega)]^N}.\end{aligned} \tag{3.17}$$

The above inequality proves that $t\mapsto \mathcal{L}(u(t,u_0))$ is nonincreasing for $t>0$. Moreover, there exists $a>0$ such that

$$\|u-w\|_{[L^2(\Omega)]^N} \le \|u_0-w\|_{[L^2(\Omega)]^N}e^{-at}, \qquad t\ge 0.$$

This shows in particular that

$$\lim_{t\to\infty} u(t,u_0) = w \quad \text{in } [L^2(\Omega)]^N. \tag{3.18}$$

Our task will be, however, to justify the convergence in X^α. Meanwhile, assume that $\mathcal{L}(u(\cdot,u_0)) \equiv \mathcal{L}(u_0)$. Then as in (3.17) we have

$$0 = \frac{d}{dt}\mathcal{L}(u(t,u_0)) \le (-\nu/C_\Omega^2 + \|w\|_{[W^{1,\infty}(\Omega)]^N})\|u(t,u_0)-w\|^2_{[L^2(\Omega)]^N} \le 0.$$

This implies $u_0 = w \in \mathcal{E}$ and consequently $\mathcal{E} = \{w\}$.

Suppose that ϕ is a (hypothetical) bounded complete trajectory of u_0. Since $\alpha_\phi(u_0) = \{w\}$ and $\omega(u_0) = \{w\}$, we have

$$\mathcal{L}(w) \ge \mathcal{L}(\phi(s)) \ge \mathcal{L}(w), \qquad s\in\mathbb{R},$$

so that $\phi(\mathbb{R}) = \{w\}$. Hence we obtain $\mathcal{S} = \{w\}$.

We thus infer that if (3.14) holds, then for any $u_0\in X^\alpha$, $\alpha\in[1/2,1)$, the X^α solution $u(\cdot,u_0)$ of the Navier–Stokes system either blows up in X^α or stays bounded and approaches in X^α the maximal compact invariant set $\{w\}$.

Acknowledgements. This paper was inspired by the note of Franz Rothe [RO]. The author wishes to thank Professor Marek Fila for the information concerning the existence of a single heteroclinic orbit in one-dimensional Frank-Kamenetskii problem with small λ. He would also like to thank the referee for improvements to the exposition.

REFERENCES

[AM] H. Amann, *Linear and Quasilinear Parabolic Problems*, *Vol. I*, Birkhäuser, Basel, 1995.

[B-E] J. Bebernes and D. Eberly, *Mathematical Problems from Combustion Theory*, Springer, New York, 1989.

[B-F1] P. Brunovský and B. Fiedler, *Numbers of zeros on invariant manifolds in reaction-diffusion equations*, Nonlinear Anal. 10 (1986), 179–193.

[B-F2] —, —, *Connecting orbits in scalar reaction diffusion equations*, Dynamics Reported 1 (1988), 57–89.

[B-F3] —, —, *Connecting orbits in scalar reaction diffusion equations II. The complete solution,* J. Differential Equations 81 (1989), 106–135.

[C-C-H] M. Chen, X.-Y. Chen and J. K. Hale, *Structural stability for time-periodic one-dimensional parabolic equations*, ibid. 96 (1992), 355–418.

[C-D1] J. W. Cholewa and T. Dłotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge Univ. Press, Cambridge, 2000.

[C-D2] —, —, *Parabolic equations with critical nonlinearities*, Topol. Methods Nonlinear Anal. 21 (2003), 311–324.

[C-R] M. G. Crandall and P. H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rat. Mech. Anal. 58 (1975), 207–218.

[CZ] R. Czaja, *Differential Equations with Sectorial Operator*, Silesian Univ., Katowice, 2002.

[FI] M. Fila, *Boundedness of global solutions of nonlinear diffusion equations*, J. Differential Equations 98 (1992), 226–240.

[F-P] M. Fila and P. Poláčik, *Global solutions of a semilinear parabolic equation*, Adv. Differential Equations 4 (1999), 163–196.

[FK] D. A. Frank-Kamenetskii, *Diffusion and Heat Transfer in Chemical Kinetics*, Plenum Press, New York, 1969.

[FU] H. Fujita, *On the nonlinear equations* $\Delta u + e^u = 0$ *and* $\partial v/\partial t = \Delta v + e^v$, Bull. Amer. Math. Soc. 75 (1969), 132–135.

[F-M] D. Fujiwara and H. Morimoto, *An* L_r*-theorem of the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo 24 (1977), 685–700.

[G-V] V. A. Galaktionov and J. L. Vázquez, *The problem of blow-up in nonlinear parabolic equations*, Discrete Contin. Dynam. Systems 8 (2002), 399–433.

[GE] I. M. Gelfand, *Some problems in the theory of quasilinear equations*, Uspekhi Mat. Nauk 14 (1959), no. 2, 87–158 (in Russian); English transl.: Amer. Math. Soc. Transl. 29 (1963), 295–381.

[G-N-N] B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68 (1979), 209–243.

[G-M] Y. Giga and T. Miyakawa, *Solutions in* L_r *of the Navier–Stokes initial value problem*, Arch. Rat. Mech. Anal. 89 (1985), 267–281.

[HA] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.

[HE] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, Berlin, 1981.

[J-L] D. Joseph and T. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rat. Mech. Anal. 49 (1973), 241–269.

[LA] O. A. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge Univ. Press, Cambridge, 1991.

[MI] M. Miklavčič, *Stability for semilinear equations with noninvertible linear operator*, Pacific J. Math. 118 (1985), 199–214.

[N-S] K. Nagasaki and T. Suzuki, *Spectral and related properties about the Emden–Fowler equation $-\Delta u = \lambda e^u$ on circular domains*, Math. Ann. 299 (1994), 1–15.

[RO] F. Rothe, *Dynamics of a scalar nonlinear diffusion equation*, Semesterberichte Funktionalanalysis Universität Tübingen, 1987.

[TR] H. Triebel, *Interpolation Theory*, *Function Spaces*, *Differential Operators*, Deutscher Verlag der Wiss., Berlin, 1978.

Institute of Mathematics
Silesian University
Bankowa 14
40-007 Katowice, Poland
E-mail: rczaja@ux2.math.us.edu.pl

Received 6 November 2003;
revised 8 January 2004 (4393)