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ON DIFFEOMORPHISMS WITH POLYNOMIAL GROWTH OF THE DERIVATIVE ON SURFACES

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Abstract. We consider zero entropy C^∞-diffeomorphisms on compact connected C^∞-manifolds. We introduce the notion of polynomial growth of the derivative for such diffeomorphisms, and study it for diffeomorphisms which additionally preserve a smooth measure. We show that if a manifold M admits an ergodic diffeomorphism with polynomial growth of the derivative then there exists a smooth flow with no fixed point on M. Moreover, if $\dim M = 2$, then necessarily $M = \mathbb{T}^2$ and the diffeomorphism is C^∞-conjugate to a skew product on the 2-torus.

1. Introduction. Let $f : M \to M$ be a smooth diffeomorphism of a compact connected smooth manifold M. An important question of the theory of smooth dynamical systems is whether asymptotic properties of the sequence $\{Df^n\}$ affect the dynamical properties of the diffeomorphism $f : M \to M$. There are classical results describing this phenomenon in some special cases.

For example, suppose that $f : \mathbb{T}^d \to \mathbb{T}^d$ is a diffeomorphism of the d-dimensional torus, homotopic to the identity. If the sequence $\{Df^n\}$ is uniformly bounded and the coordinates of the rotation vector of f are rationally independent then f is C^0-conjugate to an ergodic rotation (see [8, p. 181]). On the other hand, suppose that M is a surface (2-dimensional case). If the sequence $\{Df^n\}$ has an "exponential growth", more precisely, if f is an Anosov diffeomorphism then f is C^0-conjugate to a hyperbolic automorphism of the 2-torus (see [7]).

Let us first define the notion of polynomial growth of the derivative. Let M be a k-dimensional compact connected C^∞-manifold. There is a natural collection of sets of measure zero on M. This is the collection of sets A such that for any local chart (U, φ) the set $\varphi(A \cap U) \subset \mathbb{R}^k$ has Lebesgue measure zero. Let $f : M \to M$ be a C^∞-diffeomorphism and let $\{(U_i, \varphi_i)\}_{i\in I}$ be a C^∞-atlas of M.

Definition 1. We say that the pair $(f, \{(U_i, \varphi_i)\}_{i\in I})$ has *β-polynomial growth of the derivative* if for every $i, j \in I$ there exists a measurable function

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$A_{ji} : U_i \to M_k(\mathbb{R})$ non-zero at a.e. point such that for a.e. $x \in U_i$ we have

$$\frac{1}{n^\beta} D(\varphi_j \circ f^n \circ \varphi_i^{-1})(\varphi_i(x)) \to A_{ji}(x)$$

whenever n tends to infinity through values such that $f^n(x) \in U_j$. We say that the diffeomorphism $f : M \to M$ has *β-polynomial growth of the derivative* if there exists an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ for which $(f, \{(U_i, \varphi_i)\}_{i\in I})$ has such growth.

Suppose that M is the d-torus and $\{(U_i, \varphi_i)\}_{i\in I}$ is an atlas which comes from the natural projection of $\mathbb{R}^d$ on $\mathbb{T}^d$. Then the notion of β-polynomial growth of the derivative of $f : \mathbb{T}^d \to \mathbb{T}^d$ with respect to $\{(U_i, \varphi_i)\}$ coincides with the definition presented in [4], i.e. $n^{-\beta} D\bar{f}^n$ converges a.e. to a non-zero function, where $\bar{f} : \mathbb{R}^d \to \mathbb{R}^d$ is a lift of f. In [4], it is shown that if $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an area-preserving ergodic C^2-diffeomorphism with β-polynomial growth of the derivative (in the above sense) then $\beta = 1$ and f is algebraically conjugate (i.e. via a group automorphism) to a skew product

$$T_{\alpha,\varphi}(x_1, x_2) = (x_1 + \alpha, x_2 + \varphi(x_1)),$$

where α is irrational and the topological degree $d(\varphi)$ of φ is non-zero. Further versions of this result can be found in [5] and [6].

The aim of this paper is a further study of diffeomorphisms with polynomial growth of the derivative on general smooth manifolds, more precisely, we will consider such diffeomorphisms which possess an ergodic positive smooth invariant measure. In Section 2 we show that if a manifold M admits such a diffeomorphism then there exists a smooth flow with no fixed point on M. In particular, the Euler characteristic of M equals 0, by the Poincaré–Hopf Index Formula. It follows that if $\dim M = 2$, then M is diffeomorphic either to the 2-torus or to the Klein bottle. In Section 3, roughly speaking, we prove that every diffeomorphism with polynomial growth of the derivative on the 2-torus is diffeomorphic to a skew product $T_{\alpha,\varphi}$, where $d(\varphi) \neq 0$. In particular, the matrix of the algebraic action of such a diffeomorphism on the 1-homology group $H_1(\mathbb{T}^2, \mathbb{Z})$ cannot be conjugate to a diagonal matrix. Next we will apply this result to eliminate the Klein bottle. More precisely, we lift diffeomorphisms with polynomial growth of the derivative on the Klein bottle to diffeomorphisms with such growth on the torus. However it will turn out that the matrices of the algebraic actions of such lifts on the 1-homology group are diagonal. It follows that the two-dimensional torus is the only two-dimensional compact smooth surface which admits ergodic diffeomorphisms with polynomial growth of the derivatives.

2. Fundamental properties. Let $f : M \to M$ be a C^∞-diffeomorphism of a compact connected C^∞-manifold M. Let μ be an f-invariant positive probability C^∞-measure on M (see [10, Ch. 5] for the definition of positive smooth measures).

LEMMA 1. *Suppose that $f : (M,\mu) \to (M,\mu)$ is ergodic and has β-polynomial growth of the derivative. Then there exists an atlas $\{(U_i,\widetilde{\varphi}_i)\}_{i\in I}$ such that $(f,\{(U_i,\widetilde{\varphi}_i)\}_{i\in I})$ has β-polynomial growth of the derivative and $A_{ji}(x)$ is independent of $j\in I$ for each $i\in I$ and for a.e. $x\in U_i$.*

Proof. Let $\{(U_i,\varphi_i)\}_{i\in I}$ be an atlas such that $(f,\{(U_i,\varphi_i)\}_{i\in I})$ has β-polynomial growth of the derivative. Take $i,j,k\in I$. Suppose that $y\in U_j\cap U_k$. By the ergodicity of f, for a.e. $x\in U_i$ the orbit $\{f^n x\}_{n=0}^\infty$ is dense in M. Therefore for a.e. $x\in U_i$ there exists an increasing sequence $\{n_l\}_{l=1}^\infty$ of natural numbers such that

$$f^{n_l}x\in U_j\cap U_k \quad\text{and}\quad f^{n_l}x\to y.$$

Since

$$\begin{aligned}\frac{1}{n_l^\beta}\,&D(\varphi_j\circ f^{n_l}\circ\varphi_i^{-1})(\varphi_i(x))\\ &= D(\varphi_j\circ\varphi_k^{-1})(\varphi_k(f^{n_l}x))\cdot\frac{1}{n_l^\beta}\,D(\varphi_k\circ f^{n_l}\circ\varphi_i^{-1})(\varphi_i(x)),\end{aligned}$$

letting $l\to\infty$ we obtain

$$A_{ji}(x) = D(\varphi_j\circ\varphi_k^{-1})(\varphi_k(y))\cdot A_{ki}(x) \tag{1}$$

for every $y\in U_j\cap U_k$ and for a.e. $x\in U_i$.

Fix $j_0\in I$. Since M is connected, for every $j\in I$ we can choose a sequence $\{U_{j_s}\}_{s=1}^m$ of sets and a sequence $\{y_s\}_{s=1}^m$ of points such that $y_s\in U_{j_{s-1}}\cap U_{j_s}$ for $s=1,\dots,m$, where $j_m=j$. Let $\widetilde{\varphi}_j : U_j\to\mathbb{R}^k$ be defined by

$$\begin{aligned}\widetilde{\varphi}_j(x) &:= B_j\ \varphi_j(x)\\ &= D(\varphi_{j_0}\circ\varphi_{j_1}^{-1})(\varphi_{j_1}(y_1))\cdot\ldots\cdot D(\varphi_{j_{m-1}}\circ\varphi_{j_m}^{-1})(\varphi_{j_m}(y_m))\varphi_j(x).\end{aligned}$$

Clearly, $\{(U_i,\widetilde{\varphi}_i)\}_{i\in I}$ is a C^∞-atlas on M. Take $i,j\in I$. Then for a.e. $x\in U_i$ we have

$$\begin{aligned}&\frac{1}{n^\beta}\,D(\widetilde{\varphi}_j\circ f^n\circ\widetilde{\varphi}_i^{-1})(\widetilde{\varphi}_i(x))\\ &\quad= B_j\cdot\frac{1}{n^\beta}\,D(\varphi_j\circ f^n\circ\varphi_i^{-1})(\varphi_i(x))\cdot B_i^{-1}\\ &\quad\to D(\varphi_{j_0}\circ\varphi_{j_1}^{-1})(\varphi_{j_1}(y_1))\cdot\ldots\cdot D(\varphi_{j_{m-1}}\circ\varphi_{j_m}^{-1})(\varphi_{j_m}(y_m))\cdot A_{j_m i}(x)\cdot B_i^{-1}\\ &\quad= A_{j_0 i}(x)\cdot B_i^{-1},\end{aligned}$$

whenever $n\to\infty$ with $f^n x\in U_j$, by (1), and the proof is complete. ∎

By the above lemma, we can assume that for every $i \in I$ there exists $A_i : U_i \to \mathbb{R}^k$ such that

$$\frac{1}{n^\beta} D(\varphi_j \circ f^n \circ \varphi_i^{-1})(\varphi_i(x)) \to A_i(x)$$

whenever $n \to \infty$ with $f^n x \in U_j$.

Lemma 2. *For every $i, j, k \in I$ and for any natural n we have*

$$A_i(x) = D(\varphi_k \circ f^n \circ \varphi_j^{-1})(\varphi_j(y)) \cdot A_i(x)$$

for any $y \in U_j \cap f^{-n} U_k$ and for a.e. $x \in U_i$.

Proof. Take $y \in U_j \cap f^{-n} U_k$. Since for a.e. $x \in U_i$ the orbit $\{f^n x\}_{n=0}^\infty$ is dense in M (by the ergodicity of f), we can choose an increasing sequence $\{n_l\}_{l=1}^\infty$ of natural numbers such that

$$f^{n_l} x \in U_j \cap f^{-n} U_k \quad \text{and} \quad f^{n_l} x \to y.$$

Since

$$\begin{aligned} &\frac{1}{n_l^\beta} D(\varphi_k \circ f^{n+n_l} \circ \varphi_i^{-1})(\varphi_i(x)) \\ &\qquad = D(\varphi_k \circ f^n \circ \varphi_j^{-1})(\varphi_j(f^{n_l} x)) \cdot \frac{1}{n_l^\beta} D(\varphi_j \circ f^{n_l} \circ \varphi_i^{-1})(\varphi_i(x)), \end{aligned}$$

letting $l \to \infty$ we obtain the assertion. ■

Theorem 3. *Suppose that a C^∞-diffeomorphism $f : (M, \mu) \to (M, \mu)$ is ergodic and has β-polynomial growth of the derivative. Then there exists a C^∞-flow ψ^t on M such that*

- *$f \circ \psi^t = \psi^t \circ f$ for any real t,*
- *ψ^t has no fixed point.*

Proof. Fix $i \in I$. By Lemma 2, there exists $x \in U_i$ such that

$$A_i(x) \neq 0 \quad \text{and} \quad A_i(x) = D(\varphi_k \circ f^n \circ \varphi_j^{-1})(\varphi_j(y)) \cdot A_i(x)$$

whenever $y \in U_j \cap f^{-n} U_k$. Let $\overline{a} \in \mathbb{R}^k$ be a non-zero column of $A_i(x)$. Then

$$\overline{a} = D(\varphi_k \circ f^n \circ \varphi_j^{-1})(\varphi_j(y))\overline{a} \tag{2}$$

whenever $y \in U_j \cap f^{-n} U_k$. Consider the C^∞-vector field $X : M \to TM$ defined by

$$X(x) := D(\varphi_j^{-1})(\varphi_j(x))\overline{a}$$

whenever $x \in U_j$. Clearly, $X(x)$ does not depend on the choice of the chart (U_j, φ_j), by (2). Moreover, $X(x) \neq 0$ for all $x \in M$. Let ψ^t stand for the associated flow on M, i.e.

$$\frac{d}{dt}\psi^t(x) = X(\psi^t x).$$

Suppose that $x \in U_j$ and choose $\varepsilon > 0$ such that $\psi^t x \in U_j$ for any $t \in (-\varepsilon, \varepsilon)$. Then

$$\begin{aligned}\frac{d}{dt}\varphi_j \circ \psi^t(x) &= D(\varphi_j)(\psi^t(x)) \cdot \frac{d}{dt}\psi^t(x) \\ &= D(\varphi_j)(\psi^t(x)) \cdot X(\psi^t(x)) \\ &= D(\varphi_j)(\psi^t(x)) \cdot D(\varphi_j^{-1})(\varphi_j(\psi^t x))\overline{a} = \overline{a}\end{aligned}$$

whenever $t \in (-\varepsilon, \varepsilon)$. Now suppose that $x \in U_j \cap f^{-1}U_k$ and choose $0 < \varepsilon' \le \varepsilon$ such that $f \circ \psi^t(x), \psi^t \circ f(x) \in U_k$ for any $t \in (-\varepsilon', \varepsilon')$. Then

$$\begin{aligned}&\frac{d}{dt}(\varphi_k \circ f \circ \psi^t(x) - \varphi_k \circ \psi^t \circ f(x)) \\ &\qquad = D(\varphi_k \circ f \circ \varphi_j^{-1})(\varphi_j(\psi^t x)) \cdot \frac{d}{dt}\varphi_j \circ \psi^t(x) - \frac{d}{dt}\varphi_k \circ \psi^t(fx) \\ &\qquad = D(\varphi_k \circ f \circ \varphi_j^{-1})(\varphi_j(\psi^t x))\overline{a} - \overline{a} = 0\end{aligned}$$

for all $t \in (-\varepsilon', \varepsilon')$, by (2). Consequently, $f \circ \psi^t(x) = \psi^t \circ f(x)$ for all $t \in (-\varepsilon', \varepsilon')$. Since M is compact, we conclude that $f \circ \psi^t = \psi^t \circ f$ for all real t. ■

COROLLARY 4. *Let M be a connected compact C^∞-manifold. Suppose that there exists an ergodic positive C^∞-measure-preserving C^∞-diffeomorphism with polynomial growth of the derivative on M. Then the Euler characteristic $\chi(M)$ equals zero.* ■

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure-preserving automorphism of standard Borel space. We will denote by $\mathcal{I}_T(\mu)$ the σ-algebra of $\mathcal{B}$-measurable T-invariant sets. Each measurable function $f : X \to \mathbb{R}$ determines a cocycle over the automorphism T given by

$$f^{(n)}(x) = \begin{cases} f(x) + f(Tx) + \ldots + f(T^{n-1}x) & \text{for } n > 0, \\ 0 & \text{for } n = 0, \\ -(f(T^n x) + f(T^{n+1}x) + \ldots + f(T^{-1}x)) & \text{for } n < 0. \end{cases}$$

Denote by $T_f : (X \times \mathbb{R}, \mu \otimes \lambda_{\mathbb{R}}) \to (X \times \mathbb{R}, \mu \otimes \lambda_{\mathbb{R}})$ the skew product

$$T_f(x, y) = (Tx, y + f(x)).$$

Then $T_f^n(x, y) = (Tx, y + f^{(n)}(x))$ for any integer n.

REMARK 1. Suppose that a C^∞-diffeomorphism $f : M \to M$ preserves a positive probability C^∞-measure μ on M. Assume that a C^∞-diffeomorphism $g : M \to M$ commutes with f. It is easy to check that if the σ-algebra $\mathcal{I}_f(\mu)$ is finite, then g preserves μ as well.

3. 2-dimensional case. In this section we study the case where M is a surface. Let $f : M \to M$ be a C^∞-diffeomorphism and let μ be an f-invariant positive probability C^∞-measure on M. Suppose that f is ergodic and has polynomial growth of the derivative. By Corollary 4, the Euler characteristic of M must be zero. Therefore M is C^∞-diffeomorphic either to the torus or to the Klein bottle. Here is the main result of this paper.

THEOREM 5. *Let $f : M \to M$ be a C^∞-diffeomorphism of a connected compact C^∞-surface. Suppose that f has polynomial growth of the derivative and possesses an invariant ergodic positive probability C^∞-measure on M. Then f is C^∞-conjugate to a skew product of the form*

$$\mathbb{T}^2 \ni (x_1, x_2) \mapsto (x_1 + \alpha, x_2 + \beta(x_1)) \in \mathbb{T}^2.$$

Before passing to the proof of the theorem let us consider the case where M is the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. By $\lambda_{\mathbb{T}^2}$ we will denote the Lebesgue measure on $\mathbb{T}^2$. We will identify functions on $\mathbb{T}^2$ with $\mathbb{Z}^2$-periodic (i.e. 1-periodic in each coordinate) functions on $\mathbb{R}^2$. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a smooth diffeomorphism. We will identify f with its lift, i.e. with a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(x_1 + 1, x_2) = f(x_1, x_2) + (a_{11}, a_{21}),$$
$$f(x_1, x_2 + 1) = f(x_1, x_2) + (a_{12}, a_{22})$$

for every $(x_1, x_2) \in \mathbb{R}^2$, where $[a_{ij}]_{i,j=1,2} \in \mathrm{GL}_2(\mathbb{Z})$. Then the induced automorphism of the 1-homology group

$$f_{*1} : H_1(\mathbb{T}^2, \mathbb{Z}) \to H_1(\mathbb{T}^2, \mathbb{Z})$$

is determined by the matrix $[a_{ij}]_{i,j=1,2}$. Given $\alpha \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\beta : \mathbb{T} \to \mathbb{T}$ let $T_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2$ stand for the skew product

$$T_{\alpha,\beta}(x_1, x_2) = (x_1 + \alpha, x_2 + \beta(x_1)).$$

LEMMA 6. *Let $f : (\mathbb{T}^2, \mu) \to (\mathbb{T}^2, \mu)$ be a C^∞-diffeomorphism such that*

- *f has polynomial growth of the derivative*,
- *the σ-algebra $\mathcal{I}_f(\mu)$ is finite*,
- *there exists on $\mathbb{T}^2$ a C^∞-flow ψ^t which commutes with f and has no fixed point.*

Then f is C^∞-conjugate to a skew product $T_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2$, where $\alpha \in \mathbb{T}$ is irrational and $\beta : \mathbb{T} \to \mathbb{T}$ is a C^∞-mapping with non-zero topological degree.

Proof. The proof starts with the observation that we only need to prove the lemma in the case where $\mu = \lambda_{\mathbb{T}^2}$. Indeed, by Theorem 1 in [11], there exists a C^∞-diffeomorphism $\varrho : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\varrho^*\mu = \lambda_{\mathbb{T}^2}$. Then the diffeomorphism $\varrho \circ f \circ \varrho^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$ preserves the Lebesgue measure and satisfies the assumption of the lemma.

Therefore we can suppose that $\mu = \lambda_{\mathbb{T}^2}$. By Remark 1, the flow ψ^t preserves the Lebesgue measure as well. It follows that ψ^t is a Hamiltonian flow on $\mathbb{T}^2$ with no fixed point, i.e. there exists a C^∞-function $H : \mathbb{R}^2 \to \mathbb{R}$ such that DH is $\mathbb{Z}^2$-periodic, non-zero at each point and

$$\frac{d}{dt}\psi^t(x) = \begin{bmatrix} H_{x_2}(\psi^t(x)) \\ -H_{x_1}(\psi^t(x)) \end{bmatrix}.$$

Put

$$d := \begin{cases} \int_{\mathbb{T}^2} H_{x_1}(x)\,dx / \int_{\mathbb{T}^2} H_{x_2}(x)\,dx & \text{if } \int_{\mathbb{T}^2} H_{x_2}(x)\,dx \neq 0, \\ \infty & \text{otherwise.} \end{cases}$$

First suppose that d is irrational. Then ψ^t is C^∞-conjugate to a special flow constructed over the rotation by an irrational number a and under a positive C^∞-function $b : \mathbb{T} \to \mathbb{R}$ with $\int_{\mathbb{T}} b(x)\,dx = 1$ (see for instance [1, Ch. 16]), i.e. there exists an area-preserving C^∞-diffeomorphism $\varrho : \mathbb{R}^2 \to \mathbb{R}^2$ and a matrix $N \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$\psi^t \circ \varrho = \varrho \circ \sigma^t,$$

where $\sigma^t(x_1, x_2) = (x_1, x_2 + ct)$, $c > 0$, and

$$\varrho(x_1 + m_1 + m_2 a, x_2 - b^{(m_1)}(x_1)) = \varrho(x_1, x_2) + (m_1, m_2)N$$

for all $(m_1, m_2) \in \mathbb{Z}^2$. Let $T_{a,-b} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ stand for the skew product $T_{a,-b}(x_1, x_2) = (x_1 + a, x_2 - b(x_1))$. Consider the quotient space $M_{a,b} = \mathbb{T} \times \mathbb{R}/\!\sim$, where the relation $\sim$ is defined by $(x_1, x_2) \sim (y_1, y_2)$ iff $(x_1, x_2) = T^k_{a,-b}(y_1, y_2)$ for an integer k. Then the quotient flow $\sigma^t_{a,b}$ of the action σ^t by the relation $\sim$ is the special flow constructed over the rotation by a and under the function b. Moreover, $\varrho : M_{a,b} \to \mathbb{T}^2$ conjugates the flows $\sigma^t_{a,b}$ and ψ^t. Let $\widehat{f} : M_{a,b} \to M_{a,b}$ be given by $\widehat{f} = \varrho^{-1} \circ f \circ \varrho$. As $\widehat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ commutes with the flow σ^t we have

$$\widehat{f}(x_1, x_2) = \widehat{f} \circ \sigma^{x_2/c}(x_1, 0) = \sigma^{x_2/c} \circ \widehat{f}(x_1, 0) = (\widehat{f}_1(x_1, 0), x_2 + \widehat{f}_2(x_1, 0)).$$

Since $\widehat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ preserves area, $\frac{\partial}{\partial x_1}\widehat{f}_1(x_1, 0) = \det D\widehat{f} = \varepsilon = \pm 1$. Consequently,

$$\widehat{f}(x_1, x_2) = (\varepsilon x_1 + \alpha, x_2 + \beta(x_1)),$$

where $\beta(x) = \widehat{f}_2(x, 0)$. Since $\widehat{f}$ is a diffeomorphism of $M_{a,b}$, there exist $m_1, m_2 \in \mathbb{Z}$ such that

$$\begin{aligned} &(\varepsilon x_1 + \varepsilon + \alpha, x_2 + \beta(x_1 + 1)) \\ &\qquad = \widehat{f}(x_1 + 1, x_2) = T^{m_2}_{a,-b}\widehat{f}(x_1, x_2) + (m_1, 0) \\ &\qquad = (\varepsilon x_1 + \alpha + m_1 + m_2 a, x_2 + \beta(x_1) - b^{(m_2)}(\varepsilon x_1 + \alpha)). \end{aligned}$$

It follows that $m_1 = \varepsilon$, $m_2 = 0$, hence that $\beta : \mathbb{T} \to \mathbb{R}$. Moreover, there exist

$n_1, n_2 \in \mathbb{Z}$ such that

$$\begin{aligned}(\varepsilon x_1 + \varepsilon a + \alpha, x_2 - b(x_1) + \beta(x_1 + a))\\ &= \widehat{f} \circ T_{a,-b}(x_1, x_2) = T^{n_2}_{a,-b}\widehat{f}(x_1, x_2) + (n_1, 0)\\ &= (\varepsilon x_1 + \alpha + n_1 + n_2 a, x_2 + \beta(x_1) - b^{(n_2)}(\varepsilon x_1 + \alpha)).\end{aligned}$$

It follows that $n_1 = 0$, $n_2 = \varepsilon$, hence that $\beta(x) - b^{(\varepsilon)}(\varepsilon x + \alpha) = -b(x) + \beta(x + a)$. Suppose that $\varepsilon = -1$. Then

$$-2 = \int_{\mathbb{T}} (-b(-x + \alpha - a) - b(x))\, dx = \int_{\mathbb{T}} (\beta(x) - \beta(x + a))\, dx = 0.$$

Therefore

$$\widehat{f}(x_1, x_2) = (x_1 + \alpha, x_2 + \beta(x_1))$$

and the skew products $\widehat{f}$ and $T_{a,-b}$ commute. By Lemma 9 (see Appendix), $f : \mathbb{T}^2 \to \mathbb{T}^2$ is tangentially non-transient, which contradicts the fact that f has polynomial growth of the derivative. Consequently, d must be either rational or infinite. Then

$$Df(x)\begin{bmatrix} H_{x_2}(x) \\ -H_{x_1}(x) \end{bmatrix} = \frac{d}{dt} f \circ \psi^t(x)\Big|_{t=0} = \frac{d}{dt}\psi^t \circ f(x)\Big|_{t=0} = \begin{bmatrix} H_{x_2}(fx) \\ -H_{x_1}(fx) \end{bmatrix}.$$

It follows that

$$DH(fx) \cdot Df(x) = \varepsilon DH(x),$$

where $\varepsilon = \det Df = \pm 1$, and finally that

$$H \circ f(x) = \varepsilon H(x) + \alpha.$$

Since DH is $\mathbb{Z}^2$-periodic, we can represent H as

$$H(x_1, x_2) = \widetilde{H}(x_1, x_2) + d_1 x_1 + d_2 x_2,$$

where $d_i = \int_{\mathbb{T}^2} H_{x_i}(x)\, dx$, $i = 1, 2$ and $\widetilde{H} : \mathbb{T}^2 \to \mathbb{R}$. Without loss of generality we can assume that d_1, d_2 are relatively prime integer numbers and at least one of them is non-zero, because DH is non-zero at each point. Now notice that $\varepsilon = 1$. Indeed, suppose, contrary to our claim, that $\varepsilon = -1$. Let $\xi : \mathbb{T}^2 \to \mathbb{C}$ be given by $\xi(x_1, x_2) = \exp 2\pi i H(x_1, x_2)$. Then $\xi \circ f^2 = \xi$. Since the σ-algebra $\mathcal{I}_f$ is finite, it is easy to show that the σ-algebra $\mathcal{I}_{f^2}$ is finite as well. It follows that ξ and finally H is constant, which is impossible. In the same manner we can show that α is irrational. Now applying Theorem 13 of [6] we conclude that f is C^∞-conjugate to a skew product $T_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2$.

Finally, suppose, contrary to our claim, that the topological degree $d(\beta)$ of $\beta : \mathbb{T} \to \mathbb{T}$ equals zero. Then

$$DT^{q_n}_{\alpha,\beta} = \begin{bmatrix} 1 & 0 \\ D\beta^{(q_n)} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

uniformly (see [8, p. 189]), where $\{q_n\}$ is the sequence of the denominators of α, which contradicts the fact that f has polynomial growth of the derivative. ■

Let us turn to the case where M is the Klein bottle. Let $G \subset \operatorname{Aut}(\mathbb{R}^2)$ denote the subgroup of all group automorphisms of the plane generated by the following two isometries:

$$S(x_1, x_2) = (x_1 + 1, -x_2), \qquad T(x_1, x_2) = (x_1, x_2 + 1).$$

Then the compact surface $\mathbb{K} = \mathbb{R}^2/G$ is a model of the Klein bottle. Since $S^n \circ T^m = T^{(-1)^n m} \circ S^n$, each element of G can be represented as $T^m \circ S^n$, where $m, n \in \mathbb{Z}$. Moreover, G is isomorphic to the semidirect product $\mathbb{Z} \times_{\mathrm{ev}} \mathbb{Z}$, where $\mathrm{ev} : \mathbb{Z} \to \operatorname{Aut}(\mathbb{Z})$ is given by $\mathrm{ev}(n)m = (-1)^n m$, i.e. the multiplication is given by

$$(n_1, m_1) \circ (n_2, m_2) = (n_1 + n_2, m_1 + (-1)^{n_1} m_2).$$

The group isomorphism is established by the map

$$\mathbb{Z} \times_{\mathrm{ev}} \mathbb{Z} \ni (n, m) \mapsto T^m \circ S^n \in G.$$

Suppose that $f : \mathbb{K} \to \mathbb{K}$ is a C^∞-diffeomorphism. Let $\bar{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f. Then there exists a group automorphism $\Phi : G \to G$ such that

$$\bar{f}((n, m)x) = \Phi(n, m)\bar{f}(x)$$

for any $(n, m) \in G$. Put

$$(a_{11}, a_{12}) := \Phi(1, 0), \qquad (a_{21}, a_{22}) := \Phi(0, 1).$$

As $(1, 1) \circ (1, 1) = (2, 0) = (1, 0) \circ (1, 0)$, we have

$$\begin{aligned}&(2(a_{11} + a_{21}), (1 + (-1)^{a_{11}+a_{21}})(a_{22} + (-1)^{a_{21}} a_{12}))\\ &\qquad = \Phi((1, 1) \circ (1, 1)) = \Phi((1, 0) \circ (1, 0)) = (2a_{11}, (1 + (-1)^{a_{11}})a_{12}).\end{aligned}$$

It follows that $a_{21} = 0$ and $(1 + (-1)^{a_{11}})a_{22} = 0$. Consider the subgroup $G_0 \subset G$ of all elements of the form $(2n, m)$, $m, n \in \mathbb{Z}$. Then $G_0 \cong \mathbb{Z}^2$, because $(2n_1, m_1) \circ (2n_2, m_2) = (2n_1 + 2n_2, m_1 + m_2)$. Moreover,

$$\begin{aligned}\Phi(2n, m) &= \Phi(0, 1)^m \circ \Phi(1, 0)^{2n} = (0, a_{22})^m \circ (a_{11}, a_{12})^{2n}\\ &= (0, ma_{22}) \circ (2na_{11}, n(1 + (-1)^{a_{11}})a_{12})\\ &= (2na_{11}, ma_{22} + n(1 + (-1)^{a_{11}})a_{12}) \in G_0.\end{aligned}$$

It follows that $\Phi : G_0 \to G_0$ is a group automorphism, hence that there exists $B \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$\Phi(2n, m) = (2b_{11}n + 2b_{12}m, b_{21}n + b_{22}m).$$

However,

$$\begin{aligned}(2b_{11}, b_{21}) &= \Phi(2, 0) = (2a_{11}, (1 + (-1)^{a_{11}})a_{12}),\\ (2b_{12}, b_{22}) &= \Phi(0, 1) = (0, a_{22}).\end{aligned}$$

It follows that $b_{11} = a_{11}$, $b_{12} = 0$, $b_{21} = (1 + (-1)^{a_{11}})a_{12}$ and $b_{22} = a_{22}$, and finally that $a_{11}, a_{22} \in \{-1, 1\}$ and $b_{21} = 0$. Consider the C^∞-diffeomorphism $\widehat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\widehat{f}(x_1, x_2) := \big(\tfrac{1}{2}\bar{f}_1(2x_1, x_2), \bar{f}_2(2x_1, x_2)\big). \tag{3}$$

Then

$$\begin{aligned}\widehat{f}(x_1 + n, x_2 + m) &= \big(\tfrac{1}{2}\bar{f}_1((2n, m)(2x_1, x_2)), \bar{f}_2((2n, m)(2x_1, x_2))\big)\\ &= \big(\tfrac{1}{2}\bar{f}_1(2x_1, x_2) + nb_{11}, \bar{f}_2(2x_1, x_2) + mb_{22}\big)\\ &= \widehat{f}(x_1, x_2) + (nb_{11}, mb_{22}).\end{aligned}$$

Therefore $\widehat{f}$ can be treated as a C^∞-diffeomorphism of the torus.

COROLLARY 7. *For every smooth diffeomorphism $f : \mathbb{K} \to \mathbb{K}$ the matrix of the induced automorphism $\widehat{f}_{*1} : H_1(\mathbb{T}^2, \mathbb{Z}) \to H_1(\mathbb{T}^2, \mathbb{Z})$ of $\widehat{f} : \mathbb{T}^2 \to \mathbb{T}^2$ is diagonal.*

Denote by $\varrho : \mathbb{T}^2 \to \mathbb{K}$ the twofold cover of the Klein bottle given by

$$\varrho(x_1, x_2) := (2x_1, x_2).$$

Then

$$f \circ \varrho = \varrho \circ \widehat{f}.$$

Suppose that μ is an f-invariant positive probability C^∞-measure on $\mathbb{K}$. Then μ is equivalent to the Lebesgue measure $\lambda_{\mathbb{K}}$ on $\mathbb{K}$. Set $p := d\mu/d\lambda_{\mathbb{K}}$. Let $\widehat{p} : \mathbb{T}^2 \to \mathbb{R}$ be given by $\widehat{p} = p \circ \varrho$. Then the positive C^∞-measure $d\widehat{\mu} = \widehat{p}\, d\lambda_{\mathbb{T}^2}$ is $\widehat{f}$-invariant.

LEMMA 8. *If the diffeomorphism $f : (\mathbb{K}, \mathcal{B}_{\mathbb{K}}, \mu) \to (\mathbb{K}, \mathcal{B}_{\mathbb{K}}, \mu)$ is ergodic, then the σ-algebra of invariant sets of $\widehat{f} : (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \widehat{\mu}) \to (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \widehat{\mu})$ is finite.*

Proof. It is easy to check that the group of all measure-preserving automorphisms $g : (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \widehat{\mu}) \to (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \widehat{\mu})$ such that $g \circ \widehat{f} = \widehat{f} \circ g$ and $g^{-1}\varrho^{-1}\mathcal{B}_{\mathbb{K}} = \varrho^{-1}\mathcal{B}_{\mathbb{K}}$ equals $\{\mathrm{Id}_{\mathbb{T}^2}, I\}$, where $I(x_1, x_2) = (x_1 + 1/2, -x_2)$. By Lemma 1.8.1 of [9], $\widehat{f} : (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \widehat{\mu}) \to (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \widehat{\mu})$ is a $\mathbb{Z}_2$-extension of $f : (\mathbb{K}, \mathcal{B}_{\mathbb{K}}, \mu) \to (\mathbb{K}, \mathcal{B}_{\mathbb{K}}, \mu)$, i.e. $\widehat{f}$ is measure theoretically isomorphic to a skew product $f_\xi : (\mathbb{K} \times \mathbb{Z}_2, \mu \otimes (\delta_0 + \delta_1)/2) \to (\mathbb{K} \times \mathbb{Z}_2, \mu \otimes (\delta_0 + \delta_1)/2)$ given by

$$f_\xi(x, y) = (f(x), y + \xi(x)),$$

where $\xi : \mathbb{K} \to \mathbb{Z}_2$ is a measurable function. It is now easy to check that the σ-algebra $\mathcal{I}_{f_\xi}$ is generated by at most two sets. ∎

Proof of Theorem 5. By Lemma 6, it is sufficient to prove that there is no ergodic positive C^∞-measure-preserving C^∞-diffeomorphism of the Klein bottle with polynomial growth of the derivative on $\mathbb{K}$. Suppose, contrary to our claim, that $f : \mathbb{K} \to \mathbb{K}$ is such a diffeomorphism. Consider the associated C^∞-diffeomorphism $\widehat{f} : \mathbb{T}^2 \to \mathbb{T}^2$ given by (3). It is easy to see that $\widehat{f}$ has polynomial growth of the derivative as well. Let $\widehat{\mu}$ denote the $\widehat{f}$-invariant positive C^∞-measure on $\mathbb{T}^2$ given by $d\widehat{\mu} = ((d\mu/d\lambda_{\mathbb{K}}) \circ \varrho) d\lambda_{\mathbb{T}^2}$. By Lemma 8, the σ-algebra of invariant sets of $\widehat{f} : (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \widehat{\mu}) \to (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \widehat{\mu})$ is finite.

By Theorem 3, there exists an area-preserving C^∞-flow ψ^t on $\mathbb{K}$ such that $f \circ \psi^t = \psi^t \circ f$ for any real t and ψ^t has no fixed point. Denote by $\overline{\psi}^t$ a flow which is a lift of ψ^t to $\mathbb{R}^2$. Since ψ^t and f commute, there exists $(n, m) \in G$ such that

$$\overline{\psi}^t \circ \overline{f} = (n, m)\overline{f} \circ \overline{\psi}^t$$

for all real t. Letting $t = 0$, we see that $\overline{f} = (n, m)\overline{f}$, and finally that $(n, m) = (0, 0)$, because G acts freely on $\mathbb{R}^2$. Denote by $\widehat{\psi}^t$ the C^∞-flow on $\mathbb{T}^2$ defined by

$$\widehat{\psi}^t(x_1, x_2) := \left(\tfrac{1}{2}\overline{\psi}_1^t(2x_1, x_2), \overline{\psi}_2^t(2x_1, x_2)\right).$$

Then the flow $\widehat{\psi}^t$ has no fixed point and commutes with $\widehat{f}$. Therefore $\widehat{f}$ satisfies the assumption of Lemma 6. Consequently, $\widehat{f}$ is C^∞-conjugate to a skew product $T_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2$, where $\alpha \in \mathbb{T}$ is irrational and $\beta : \mathbb{T} \to \mathbb{T}$ is a C^∞-function with non-zero topological degree. This contradicts the fact that the matrix of the induced automorphism $\widehat{f}_{*1}$ is diagonal (see Corollary 7), which proves the theorem. ∎

A. Tangentially non-transient diffeomorphisms. Let $f : M \to M$ be a C^∞-diffeomorphism of a compact C^∞-manifold M.

Definition 2. We say that f is *tangentially non-transient* if there exists a Riemannian C^∞-structure on M such that

$$\liminf_{n\to\infty} \|Df^n(x)\| < \infty \quad \text{for a.e. } x \in M.$$

In fact, the above notion does not depend on the choice of the Riemannian structure, because all Riemannian structures are equivalent. Clearly, the notion of the tangential non-transience is invariant under C^∞-conjugation. Moreover, there is no diffeomorphism which is simultaneously tangentially non-transient and with polynomial growth of the derivative. In this section we present a class of tangentially non-transient diffeomorphisms of the torus.

Given $\alpha \in \mathbb{T}$ and $\beta : \mathbb{T} \to \mathbb{R}$ we will denote by $T_{\alpha,\beta} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ the skew product $T_{\alpha,\beta}(x_1, x_2) = (x_1 + \alpha, x_2 + \beta(x_1))$. Let $a \in \mathbb{T}$ be an irrational number and let $b : \mathbb{T} \to \mathbb{R}$ be a positive C^∞-function with $\int_{\mathbb{T}} b(x)\, dx = 1$. Let σ^t denote the flow on $\mathbb{T} \times \mathbb{R}$ given by $\sigma^t(x_1, x_2) = (x_1, x_2 + t)$. Consider the quotient space $M_{a,b} = \mathbb{T} \times \mathbb{R}/\!\sim$, where the relation $\sim$ is defined by $(x_1, x_2) \sim (y_1, y_2)$ iff $(x_1, x_2) = T^k_{a,-b}(y_1, y_2)$ for an integer k. Then the quotient flow $\sigma^t_{a,b}$ of the action σ^t by the relation $\sim$ is the special flow constructed over the rotation by a and under the function b. By Lemma 2 of [3] and Theorem 1 of [11], there exists a C^∞-diffeomorphism $\varrho : M_{a,b} \to \mathbb{T}^2$ such that the flow $\psi^t := \varrho \circ \sigma^t_{a,b} \circ \varrho^{-1}$ is a Hamiltonian flow on $\mathbb{T}^2$ with no fixed points, i.e. there exists a C^∞-function $H : \mathbb{R}^2 \to \mathbb{R}$ such that DH is $\mathbb{Z}^2$-periodic, non-zero at each point and

$$\frac{d}{dt}\psi^t(\overline{x}) = \begin{bmatrix} H_{x_2}(\psi^t(\overline{x})) \\ -H_{x_1}(\psi^t(\overline{x})) \end{bmatrix}.$$

We will identify ϱ with a diffeomorphism $\varrho : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\varrho(x_1+1, x_2) = \varrho(x_1, x_2) + (N_{11}, N_{12}),$$
$$\varrho(x_1+a, x_2 - b(x_1)) = \varrho(x_1, x_2) + (N_{21}, N_{22}),$$

for any $(x_1, x_2) \in \mathbb{R}^2$, where $N \in \mathrm{GL}_2(\mathbb{Z})$. Then

$$D\varrho(x_1+1, x_2) = D\varrho(x_1, x_2), \tag{4}$$

$$D\varrho(T^n_{a,-b}(x_1, x_2)) \begin{bmatrix} 1 & 0 \\ -Db^{(n)}(x_1) & 1 \end{bmatrix} = D\varrho(x_1, x_2), \tag{5}$$

for any integer n.

Let $T_{\alpha,\beta} : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ be a skew product commuting with $T_{a,-b}$, where $\beta : \mathbb{T} \to \mathbb{R}$ is of class C^∞. Then $T_{\alpha,\beta}$ can be treated as a C^∞-diffeomorphism of $M_{a,b}$. Denote by $f : \mathbb{T}^2 \to \mathbb{T}^2$ the area-preserving C^∞-diffeomorphism given by $f := \varrho \circ T_{\alpha,\beta} \circ \varrho^{-1}$.

LEMMA 9. *The diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is tangentially non-transient.*

To prove this lemma we will need some information on recurrent cocycles over $\mathbb{Z}^2$-actions. Let

$$\mathbb{Z}^2 \ni \overline{n} \mapsto T^{\overline{n}} \in \mathrm{Aut}(X, \mathcal{B}, \mu)$$

be a measure-preserving ergodic free $\mathbb{Z}^2$-action on a standard Borel space such that the automorphism $T^{(1,0)}$ is ergodic as well. Let $\Phi : \mathbb{Z}^2 \times X \to \mathbb{R}^2$ be an L^1-cocycle over the $\mathbb{Z}^2$-action T, i.e.

$$\Phi^{(\overline{n}_1+\overline{n}_2)}(x) = \Phi^{(\overline{n}_1)}(x) + \Phi^{(\overline{n}_2)}(T^{\overline{n}_1}x)$$

for all $\overline{n}_1, \overline{n}_2 \in \mathbb{Z}$ and $\|\Phi^{(\overline{n})}\| \in L^1(X, \mu)$ for all $\overline{n} \in \mathbb{Z}$. We will denote by T_Φ the $\mathbb{Z}^2$-action on $(X \times \mathbb{R}^2, \mu \otimes \lambda_{\mathbb{R}^2})$ given by the skew product

$$T^{\overline{n}}_\Phi(x, y) = (T^{\overline{n}}(x), y + \Phi^{(\overline{n})}(x)).$$

For the background on the theory of cocycles we refer to [12].

LEMMA 10. *Suppose that Φ is a recurrent cocycle such that the cocycle $\Phi^{(1,0)}_1$ over the automorphism $T^{(1,0)}$ is transient. Then for a.e. $x \in X$ and all $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $\overline{n} \in \mathbb{Z}^2$ such that $\|\Phi^{(\overline{n})}(x)\| \le \varepsilon$ and $n_2 > N$.*

Proof. Denote by $\preceq\subset\mathbb{Z}^2\times\mathbb{Z}^2$ the lexicographic order on $\mathbb{Z}^2$, i.e.

$$\overline{m}\preceq\overline{n}\ \Leftrightarrow\ m_1<n_1\vee(m_1=n_1\wedge m_2\le n_2).$$

Fix $\varepsilon>0$. First we show that for a.e. $x\in X$ we have $\|\Phi^{(\overline{n})}(x)\|\le\varepsilon$ for infinitely many $\overline{n}\succ 0$. Put

$$V:=X\times B(0,\varepsilon/2),\qquad D:=V\cap\bigcap_{\overline{m}\succ 0}T_{\Phi}^{-\overline{m}}V^c.$$

Since $T_{\Phi}^{\overline{m}}D\cap T_{\Phi}^{\overline{n}}D=\emptyset$ for $\overline{m}\ne\overline{n}$, we obtain $\mu\otimes\lambda_{\mathbb{R}^2}(D)=0$, because T_{Φ} is conservative. Consequently, for a.e. $x\in X$ there exists $\overline{n}\succ 0$ such that $\|\Phi^{(\overline{n})}(x)\|\le\varepsilon$, by the Fubini theorem. Therefore the set

$$F_\varepsilon=\bigcup_{k=1}^{\infty}\bigcup_{\overline{m}\in\mathbb{Z}^2}\bigcap_{\overline{n}\succ 0}\{x\in X:\|\Phi^{(\overline{n})}(T^{\overline{m}}x)\|>\varepsilon/2^k\}$$

has zero μ-measure. Clearly, if $x\in X\setminus F_\varepsilon$, then there exists a strictly increasing sequence $\{\overline{m}_i\}_{i\in\mathbb{N}}$ in $\{\overline{m}\in\mathbb{Z}^2:\overline{m}\succ 0\}$ such that $\|\Phi^{(\overline{m}_i)}(x)\|\le\varepsilon$. Now suppose that the set

$$B_\varepsilon=\{x\in X:\exists_{N\in\mathbb{N}}\ \forall_{\overline{n}\succ 0}\ (\|\Phi^{(\overline{n})}(x)\|\le\varepsilon\Rightarrow n_2\le N)\}$$

has positive μ-measure. Since $\Phi_1^{(1,0)}$ is a transient cocycle over the ergodic automorphism $T^{(1,0)}$, the set $C\subset X$ of all $x\in X$ such that

$$\frac{1}{n}\Phi_1^{(n,0)}(T^{\overline{m}}x)\to\int_X\Phi_1^{(1,0)}\,d\mu\ne 0$$

for every $\overline{m}\in\mathbb{Z}^2$ has full μ-measure. Suppose that $x\in B_\varepsilon\cap C\cap X\setminus F_\varepsilon$. Then there exists a strictly increasing sequence $\{\overline{m}^i\}_{i\in\mathbb{N}}$ in $\{\overline{m}\in\mathbb{Z}^2:\overline{m}\succ 0\}$ such that $\|\Phi^{(\overline{m}^i)}(x)\|\le\varepsilon$. As $x\in B_\varepsilon$, there exists a natural number N such that $m_2^i\le N$. Without loss of generality we can assume that $m_2^i=N$ for all $i\in\mathbb{N}$. Then $m_1^i\to\infty$ and

$$\|\Phi_1^{(m_1^i,0)}(T^{(0,N)}x)\|\le\|\Phi^{(\overline{m}^i)}(x)\|+\|\Phi^{(0,N)}(x)\|\le\varepsilon+\|\Phi^{(0,N)}(x)\|$$

is bounded, which contradicts the fact that $x\in C$. Consequently, $\mu(B_\varepsilon)=0$ for every $\varepsilon>0$. Finally, if $x\in X\setminus\bigcup_{k=1}^{\infty}B_{1/k}$, then for all $\varepsilon>0$ and $N\in\mathbb{N}$ there exists $\overline{n}\in\mathbb{Z}^2$ such that $\|\Phi^{(\overline{n})}(x)\|\le\varepsilon$ and $n_2>N$, which completes the proof. ∎

Proof of Lemma 9. Since

$$Df^n(\overline{x})=D\varrho(T^n_{\alpha,\beta}\circ\varrho^{-1}(\overline{x}))\begin{bmatrix}1&0\\ D\beta^{(n)}(\varrho_1^{-1}(\overline{x}))&1\end{bmatrix}D\varrho^{-1}(\overline{x}),$$

it suffices to show that

$$\liminf_{n\to\infty}\left\|D\varrho(T^n_{\alpha,\beta}(x_1,x_2))\begin{bmatrix}1&0\\ D\beta^{(n)}(x_1)&1\end{bmatrix}\right\|<\infty$$

for a.e. (x_1,x_2) in the set $M' = \{(x_1,x_2) : x_1 \in \mathbb{R},\ 0 \le x_2 \le b(x_1)\}$. Put $C := \|b\|_{\sup}$. From (4), for every $\gamma > 0$ there exists a positive constant K_γ such that

$$\|D\varrho(x_1,x_2)\| \le K_\gamma \quad \text{whenever} \quad (x_1,x_2) \in \mathbb{R}\times[-\gamma, C+\gamma].$$

Consider the $\mathbb{Z}^2$-action on $(\mathbb{T},\lambda_{\mathbb{T}})$ given by

$$T^{(n_1,n_2)}(x) = x + n_1 a + n_2\alpha$$

and the cocycle $\Phi : \mathbb{Z}^2 \times \mathbb{T} \to \mathbb{R}^2$ over the action T given by

$$\Phi^{(n_1,n_2)}(x) = (-b^{(n_1)}(x+n_2\alpha) + \beta^{(n_2)}(x), -Db^{(n_1)}(x+n_2\alpha) + D\beta^{(n_2)}(x)).$$

Case 1. Suppose that the $\mathbb{Z}^2$-action T is free. Since

$$\operatorname{rank}\begin{bmatrix} \int_{\mathbb{T}} \Phi^{(1,0)}(x)\,dx \\ \int_{\mathbb{T}} \Phi^{(0,1)}(x)\,dx \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -1 & 0 \\ \int_{\mathbb{T}} \beta(x)\,dx & 0 \end{bmatrix} = 1,$$

the cocycle $\Phi : \mathbb{Z}^2 \times \mathbb{T} \to \mathbb{R}^2$ is recurrent, by Corollary of [2]. By Lemma 10, the set $A \subset \mathbb{T}$ of all $x \in \mathbb{T}$ for which there exists a sequence $\{\overline{n}^i\}_{i\in\mathbb{N}}$ in $\mathbb{Z}^2$ such that $n_2^i \to \infty$ and $\|\Phi^{(\overline{n}^i)}(x)\| \le 1$ has full Lebesgue measure. Suppose that $(x_1,x_2) \in M'$ and $x_1 \in A$. From (5) we have

$$\begin{aligned}&\left\|D\varrho(T_{\alpha,\beta}^{n_2^i}(x_1,x_2))\begin{bmatrix} 1 & 0 \\ D\beta^{(n_2^i)}(x_1) & 1 \end{bmatrix}\right\| \\ &\qquad = \left\|D\varrho(T^{\overline{n}^i}x_1, x_2 + \Phi_1^{(\overline{n}^i)}(x_1)))\begin{bmatrix} 1 & 0 \\ \Phi_2^{(\overline{n}^i)}(x_1) & 1 \end{bmatrix}\right\| \le 3K_1,\end{aligned}$$

because $(T^{\overline{n}^i}x_1, x_2 + \Phi_1^{(\overline{n}^i)}(x_1))) \in \mathbb{R}\times[-1, C+1]$ for any natural i, which implies the tangential non-transience of f.

Case 2. Suppose that the $\mathbb{Z}^2$-action T is not free. Then there exist $k_1, k_2, k \in \mathbb{Z}$ such that $k_1 a + k_2\alpha = k$ and $k_2 \ne 0$. As $T_{\alpha,\beta}^{k_2} \circ T_{a,-b}^{k_1} = T_{a,-b}^{k_1} \circ T_{\alpha,\beta}^{k_2}$ we have

$$-b^{(k_1)}(x) + \beta^{(k_2)}(x + k_1 a) = \beta^{(k_2)}(x) - b^{(k_1)}(x + k_2\alpha).$$

Consequently,

$$b^{(-k_1)}(x + k_1 a) + \beta^{(k_2)}(x + k_1 a) = \beta^{(k_2)}(x) + b^{(-k_1)}(x)$$

and $\beta^{(k_2)}(x) = -b^{(-k_1)}(x) + c$, because a is irrational. Then

$$\begin{aligned} T_{\alpha,\beta}^{k_1}(x_1,x_2) &= (x_1 + k_1\alpha, x_2 + \beta^{(k_1)}(x_1)) \\ &= (x_1 - k_2 a, x_2 - b^{(k_2)}(x_1) + c) \sim (x_1, x_2 + c). \end{aligned}$$

Since the tangential non-transience of f^{k_1} implies that of f, we can narrow our consideration down to the case where $\alpha = 0$ and β is a constant function. If $\beta = 0$, then $f = \mathrm{Id}_{\mathbb{T}}$. Assume that $\beta \ne 0$. Denote by $\{q_n\}_{n\in\mathbb{N}}$

the sequence of the denominators of the irrational number $a \in \mathbb{T}$. Set $m_n := \operatorname{sgn}(\beta)([q_n/\beta]+1)$, where $\operatorname{sgn}(\beta) = \beta/|\beta|$. Then $m_n \to +\infty$ and $0 < \beta m_n - \operatorname{sgn}(\beta) q_n \le |\beta|$. Suppose that $(x_1, x_2) \in M'$. From (5) we have

$$\begin{aligned}
&\left\| D\varrho(T^{m_n}_{\alpha,\beta}(x_1,x_2)) \begin{bmatrix} 1 & 0 \\ D\beta^{(m_n)}(x_1) & 1 \end{bmatrix} \right\| \\
&= \|D\varrho(x_1, x_2 + m_n\beta)\| \\
&= \left\| D\varrho(x_1 + \operatorname{sgn}(\beta) q_n a, x_2 + m_n\beta - b^{(\operatorname{sgn}(\beta)q_n)}(x_1)) \begin{bmatrix} 1 & 0 \\ -Db^{(\operatorname{sgn}(\beta)q_n)}(x_1) & 1 \end{bmatrix} \right\|.
\end{aligned}$$

Since $b^{(q_n)} - q_n$ and $Db^{(q_n)}$ tend uniformly to zero (see [8, p. 189]),

$$-|\beta| < m_n\beta - b^{(\operatorname{sgn}(\beta)q_n)}(x_1) \le 2|\beta|, \qquad |Db^{(\operatorname{sgn}(\beta)q_n)}(x_1)| \le 1,$$

for any $x_1 \in \mathbb{R}$ and for all n large enough. It follows that

$$(x_1 + \operatorname{sgn}(\beta) q_n a, x_2 + m_n\beta - b^{(\operatorname{sgn}(\beta)q_n)}(x_1)) \in \mathbb{R} \times [-2|\beta|, C + 2|\beta|],$$

and finally that

$$\|D\varrho(x_1, x_2 + m_n\beta)\| \le 3K_{2|\beta|}$$

for all $(x_1, x_2) \in M'$ and for all n large enough, which completes the proof. ∎

REFERENCES

[1] I. P. Cornfeld, S. W. Fomin and Ya. G. Sinai, *Ergodic Theory*, Springer, Berlin, 1982.

[2] J. Depauw, *Un critère de récurrence pour les cocycles d'actions de $\mathbb{Z}^d$*, C. R. Acad. Sci. Paris Sér. I Math. 318 (1994), 483–486.

[3] B. Fayad, A. Katok and A. Windsor, *Mixed spectrum reparameterizations of linear flows on $\mathbb{T}^2$*, Moscow Math. J. 1 (2001), 521–537.

[4] K. Frączek, *Linear growth of the derivative for measure-preserving diffeomorphism*, Colloq. Math. 84/85 Part 1 (2000), 147–157.

[5] —, *Measure-preserving diffeomorphisms of the torus*, Ergodic Theory Dynam. Systems 21 (2001), 1759–1781.

[6] —, *Polynomial growth of the derivative for diffeomorphisms on tori*, Discr. Cont. Dynam. Sys., to appear, preprint math.DS/0205044.

[7] J. Franks, *Anosov diffeomorphisms*, in: Global Analysis, Proc. Sympos. Pure Math. 14, Amer. Math. Soc., 1970, 61–93.

[8] M. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Mat. IHES 49 (1979), 5–234.

[9] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory Dynam. Systems 7 (1987), 531–557.

[10] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, Cambridge, 1995.

[11] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. 120 (1965), 286–294.

[12] K. Schmidt, *Cocycles of Ergodic Transformation Groups*, Macmillan Lectures in Math. 1, Delhi, 1977.

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