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STARTING AND ENDING COMPONENTS OF THE AUSLANDER–REITEN QUIVERS OF A CLASS OF SPECIAL BISERIAL ALGEBRAS

BY

ZYGMUNT POGORZAŁY and MIROSŁAWA SUFRANEK (Toruń)

Dedicated to Professor Otto Kerner
on the occasion of his sixtieth birthday

Abstract. The class of n-fundamental algebras is introduced. It is a subclass of string algebras. For n-fundamental algebras we study the problem of when the Auslander–Reiten quiver contains, at the beginning or at the end, a component which is not generalized standard.

Introduction. Let K be a fixed algebraically closed field. We shall consider only finite-dimensional, associative K-algebras with a unit element. All algebras will be assumed to be basic and connected. For a fixed finite-dimensional K-algebra A, we shall denote by $\mathrm{mod}(A)$ the category of right finite-dimensional A-modules. For every finite-dimensional K-algebra A we can study its Auslander–Reiten quiver Γ_A [1, 3]. Even if A is of tame representation type, it is difficult to describe the whole quiver Γ_A. Consequently, one usually studies the properties of the connected components of Γ_A.

A. Skowroński introduced in [17] a useful notion of a generalized standard component. A standard trick in representation theory is to indicate a generalized standard component $\mathcal{C}$ of Γ_A; if it has nice properties then one can derive some interesting information about the algebra A.

Our objective is different. We consider the following question. The Auslander–Reiten quivers of a wide class of triangular algebras have some components at the beginning and some components at the end. Is it possible that at least one of them is not generalized standard? We shall indicate a class of algebras for which this phenomenon can occur. This class is a subclass of special biserial algebras.

Biserial rings were introduced by K. Fuller [9]. Later A. Skowroński and J. Waschbüsch observed that any representation-finite biserial K-algebra is special biserial [18]. Further B. Wald and J. Waschbüsch proved that any

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special biserial algebra is of tame representation type [19]. The same result was obtained by P. Dowbor and A. Skowroński in [6] by an application of Galois covering techniques. Finally, W. Crawley-Boevey proved in [5] that every finite-dimensional biserial K-algebra is of tame representation type. Nevertheless our knowledge of Auslander–Reiten quivers of biserial algebras is still poor, even in the case of representation-infinite special biserial algebras. The main aim of this paper is to look for nongeneralized standard components at the beginning or end of their quivers.

The paper is organized as follows. Section 1 contains all needed definitions and facts from representation theory.

Section 2 is devoted to one-point extensions. It also contains some fundamental information about vector space categories and their subspace categories.

Section 3 contains a description of the Auslander–Reiten quivers of a narrow class of special biserial algebras.

The class of fundamental algebras is introduced in Section 4. The structure of their Auslander–Reiten quivers is also studied.

In Section 5 the class of n-fundamental algebras is introduced. Moreover, 2-fundamental algebras are studied. This section contains Theorem 5.7, which is our first main result.

Section 6 is devoted to n-fundamental algebras for arbitrary $n \geq 2$. Theorem 6.8 gives a sufficient condition for the Auslander–Reiten quiver of an n-fundamental algebra to have a starting or an ending component which is not generalized standard.

We shall use freely all information on the Auslander–Reiten sequences and irreducible morphisms which can be found in [1, 2, 3]. Moreover, we shall apply the description of morphisms between indecomposable modules from [11]. Furthermore, we shall view our algebras as factor algebras KQ/I of the path algebras KQ of some quivers Q modulo admissible two-sided ideals I. Then to each vertex x of a quiver Q we can attach a right simple KQ/I-module S_x, a right projective KQ/I-module P_x and a right injective KQ/I-module E_x.

1. Preparatory facts

1.1. Recall that a finite-dimensional K-algebra A is said to be *tame* provided that for every dimension d there exist finitely many $K[X]$-A-bimodules Q_i, $1 \leq i \leq n_d$, which are free left $K[X]$-modules of finite rank and satisfy the following condition: all but finitely many isoclasses of indecomposable right A-modules of dimension d are isoclasses of A-modules of the form $K[X]/(X-\lambda) \otimes_{K[X]} Q_i$ for some $\lambda \in K$ and some $1 \leq i \leq n_d$ (see [7]).

Let $\mu_A(d)$ denote the smallest number of bimodules Q_i satisfying the above conditions. Then the algebra A is said to be of *polynomial growth* if there is a positive integer m such that $\mu_A(d) \le d^m$ (see [16]).

1.2. Let A be a finite-dimensional K-algebra. Following Gabriel [10] we can associate to A a bound quiver (Q_A, I_A) in such a way that $A \cong KQ_A/I_A$, where KQ_A is the path algebra of the quiver Q_A, and I_A is a two-sided ideal in KQ_A contained in the square of the two-sided ideal generated by the arrows. The algebra A is called *triangular* if Q_A has no oriented cycles.

1.3. An algebra A is said to be *special biserial* (see [18]) if there exists a bound quiver (Q_A, I_A) with $A \cong KQ_A/I_A$ such that:

(1) Every vertex of Q_A is the source of at most two arrows.
(2) Every vertex of Q_A is the target of at most two arrows.
(3) For every arrow α in Q_A there exists at most one arrow β (resp. γ) such that $\alpha\beta \notin I_A$ (resp. $\gamma\alpha \notin I_A$).

Throughout the paper we shall always consider special biserial algebras of the form KQ_A/I_A with (Q_A, I_A) satisfying the above conditions.

1.4. Let (Q, I) be a bound quiver. Recall that a *walk in the quiver* Q is a formal composition of arrows and their formal inverses. We shall also consider trivial walks e_x attached to vertices x of Q. A *walk* w *in the bound quiver* (Q, I) is a walk in Q such that no subpath v in w or its formal inverse belongs to I.

We are interested in closed walks, i.e. ones with start vertices coinciding with end vertices. A closed walk w in a bound quiver (Q, I) will be called *small* if it is not of the form v^n for any integer $n \ge 2$, and for any positive integer m the walk w^m does not contain $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$, and it is not of the form $w^m = w_1 u w_2$, where u is a path (resp. its formal inverse) such that either u (resp. u^{-1}) lies in I, or $u - z$ (resp. $u^{-1} - z$) belongs to I for some path z in Q.

A pair of two different small closed walks w_1, w_2 is said to be *inadmissible* if:

(i) w_1, w_2 have the same start vertex,
(ii) for every prime p and any decompositions $p = \sum_{j=1}^{t}(i_j + l_j)$, $i_j, l_j \ge 1$, the closed walks $w_1^{i_1} w_2^{l_1} w_1^{i_2} w_2^{l_2} \cdots w_1^{i_t} w_2^{l_t}$ are small and pairwise different.

1.5. LEMMA. *Let* $A = KQ_A/I_A$ *be a special biserial* K*-algebra. If there is an inadmissible pair of walks* w_1, w_2 *in a bound quiver* (Q_A, I_A) *then the algebra* A *is not of polynomial growth.*

Proof. Repeat the arguments from the proof of Lemma 1 in [16].

1.6. Let $A = KQ_A/I_A$ be a special biserial algebra which is a *string algebra*, that is, I_A is generated only by paths. Then there is a full classification of indecomposable finite-dimensional right A-modules (see [6, 19]). For every such module X we have two possibilities. The first is that X is induced by a walk w satisfying: $w \neq w_1\alpha\alpha^{-1}w_2$, $w \neq w_1\beta^{-1}\beta w_2$ and w does not contain a subwalk of the form u or u^{-1} with $u \in I_A$. In this case we shall denote X by $X(w)$. The other possibility is that there is a small closed walk v, an integer $n \geq 1$ and an element $\lambda \in K^*$ such that X is uniquely determined (up to isomorphism) by these data. In this case we write $X \cong X(v, n, \lambda)$.

Under the above notation we have the following algorithm for computing Auslander–Reiten sequences, found by Skowroński and Waschbüsch in [18]. If $X \cong X(w)$ for some walk w in Q_A then we construct a walk w_R in the following way. If

$$w = \alpha_{1,s_1} \cdots \alpha_{1,1}\alpha_{2,1}^{-1} \cdots \alpha_{2,s_2}^{-1} \cdots \alpha_{r-1,s_{r-1}} \cdots \alpha_{r-1,1}\alpha_{r,1}^{-1} \cdots \alpha_{r,s_r}^{-1},$$

where each $\alpha_{j,t}$ is an arrow in Q_A and $\alpha_{1,s_1} \cdots \alpha_{1,1}$ or $\alpha_{r,1}^{-1} \cdots \alpha_{r,s_r}^{-1}$ may be trivial, then

$$w_R = \alpha_{1,s_1} \cdots \alpha_{1,1}\alpha_{2,1}^{-1} \cdots \alpha_{2,s_2}^{-1} \cdots \alpha_{r-1,s_{r-1}} \cdots \alpha_{r-1,1} \cdot \alpha_{r,1}^{-1} \cdots \alpha_{r,s_r}^{-1}\alpha_{r,s_r+1}^{-1}\alpha_{r+1,s_{r+1}} \cdots \alpha_{r+1,1},$$

where $\alpha_{r+1,s_{r+1}} \cdots \alpha_{r+1,1} \notin I_A$ is a maximal path, provided that such a walk w_R exists. If there is no walk $\alpha_{r,s_r+1}^{-1}\alpha_{r+1,s_{r+1}} \cdots \alpha_{r+1,1}$ then $w_R = \alpha_{1,s_1} \cdots \alpha_{1,1}\alpha_{2,1}^{-1} \cdots \alpha_{2,s_2}^{-1} \cdots \alpha_{r-1,s_{r-1}} \cdots \alpha_{r-1,2}$. Similarly we can construct a walk w_L using the same rules on the other end of the walk w. Then we can compose our constructions and obtain a walk w_{RL}. Finally, if $X(w)$ is noninjective then we have the following Auslander–Reiten sequence in $\mathrm{mod}(A)$:

$$0 \to X(w) \to X(w_R) \oplus X(w_L) \to X(w_{RL}) \to 0.$$

Furthermore, if $X \cong X(v, n, \lambda)$ then it is known from [19] that the Auslander–Reiten sequence ending at X is of the form

$$0 \to X(v, n, \lambda) \to X(v, n-1, \lambda) \oplus X(v, n+1, \lambda) \to X(v, n, \lambda) \to 0,$$

where $X(v, 0, \lambda)$ is always the zero module.

Following Auslander and Reiten (see [2, 3]) we attach to any K-algebra A its Auslander–Reiten quiver Γ_A. The vertices of Γ_A are the isoclasses $[M]$ of indecomposable finite-dimensional right A-modules M. The number of arrows from $[M]$ to $[N]$ is $\dim_K \mathrm{Irr}(M, N)/\mathrm{Irr}^2(M, N)$, where $\mathrm{Irr}(\mathrm{mod}(A))$ is the two-sided ideal in $\mathrm{mod}(A)$ generated by the irreducible morphisms. We shall not distinguish between indecomposable A-modules and their isoclasses.

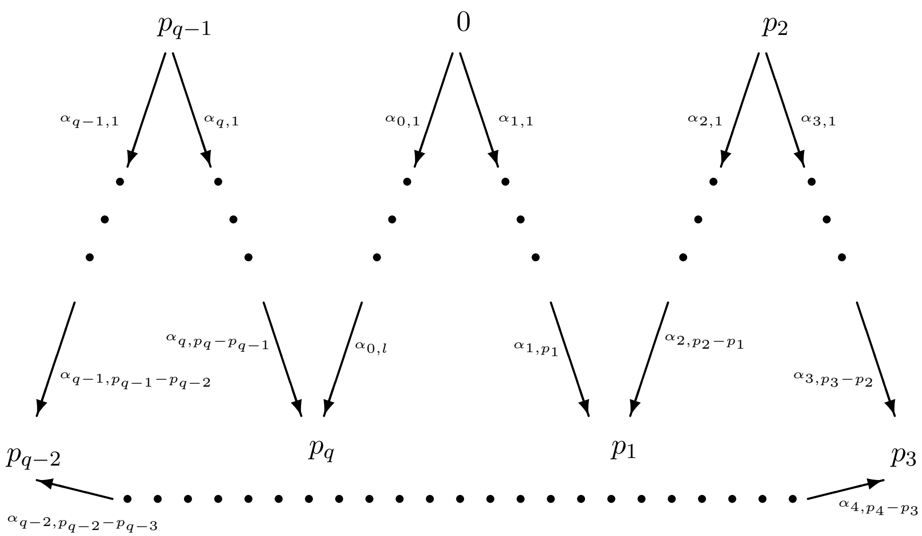
A *component* in Γ_A will always mean a connected component.

Following Ringel (see [14]) two components $\mathcal{C}_1$, $\mathcal{C}_2$ in Γ_A are said to be *orthogonal* if $\mathrm{Hom}_A(M,N) = 0 = \mathrm{Hom}_A(N,M)$ for any $M \in \mathcal{C}_1$ and $N \in \mathcal{C}_2$. A family $\{\mathcal{C}_j\}_{j\in J}$ of pairwise orthogonal components in Γ_A *separates* a component $\mathcal{C}$ from a component $\mathcal{C}'$ provided that:

(1) $\Gamma_A = \mathcal{C} \sqcup \bigsqcup_{j\in J} \mathcal{C}_j \sqcup \mathcal{C}'$.
(2) $\mathrm{Hom}_A(\mathcal{C}', \mathcal{C}) = \mathrm{Hom}_A(\mathcal{C}', \bigsqcup_{j\in J} \mathcal{C}_j) = \mathrm{Hom}_A(\bigsqcup_{j\in J} \mathcal{C}_j, \mathcal{C}) = 0$.
(3) For any nonzero morphism $f : M \to N$ with $M \in \mathcal{C}$, $N \in \mathcal{C}'$ and for any $j \in J$ there exists a finite-dimensional module X_j in the additive category formed by the modules from $\mathcal{C}_j$ and there are homomorphisms $f_1 : M \to X_j$ and $f_2 : X_j \to N$ such that $f = f_2 f_1$.

1.7. Throughout the paper $A = KQ_A/I_A$ will denote a string algebra which is triangular. We define a triangular string algebra A to be $\widetilde{\mathbb{A}}_n$-*separated* provided that for any two subquivers Q', Q'' in Q_A of type $\widetilde{\mathbb{A}}_n$ such that $KQ' \cap I_A = 0 = KQ'' \cap I_A$ we have $Q'_0 \cap Q''_0 = \emptyset$, where Q'_0, Q''_0 denote the sets of vertices of Q', Q'', respectively.

1.8. Let $\underline{p} = (p_1, \ldots, p_q)$ denote a strictly increasing sequence of positive integers. Let $l \geq 1$ be an integer. Consider a quiver $Q_{(\underline{p},l)}$ of the form



The path algebra $KQ_{(\underline{p},l)} = A_{(\underline{p},l)}$ is a tame hereditary algebra. It is well known (see [14]) that its Auslander–Reiten quiver is a disjoint union

$$\Gamma_{A_{(\underline{p},l)}} = \mathcal{P}(A_{(\underline{p},l)}) \sqcup \mathcal{C}_0(A_{(\underline{p},l)}) \sqcup \mathcal{C}_\infty(A_{(\underline{p},l)}) \sqcup \bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A_{(\underline{p},l)}) \sqcup \mathcal{I}(A_{(\underline{p},l)})$$

of components, where $\mathcal{P}(A_{(\underline{p},l)})$ is the preprojective component and $\mathcal{I}(A_{(\underline{p},l)})$

is the preinjective component. Moreover, the family $\mathcal{C}_0(A_{(\underline{p},l)}) \sqcup \mathcal{C}_\infty(A_{(\underline{p},l)}) \sqcup \bigsqcup_{\lambda \in K^*} \mathcal{C}_\lambda(A_{(\underline{p},l)})$ of pairwise orthogonal components separates $\mathcal{P}(A_{(\underline{p},l)})$ from $\mathcal{I}(A_{(\underline{p},l)})$. Furthermore, for every $\lambda \in K^*$, $\mathcal{C}_\lambda(A_{(\underline{p},l)})$ is a tube of rank 1 in the sense of [8]. The component $\mathcal{C}_0(A_{(\underline{p},l)})$ is a tube of rank $p_1 + p_3 - p_2 + p_5 - p_4 + \ldots + p_q - p_{q-1}$, and $\mathcal{C}_\infty(A_{(\underline{p},l)})$ is a tube of rank $l + p_2 - p_1 + p_4 - p_3 + \ldots + p_{q-2} - p_{q-1}$. Finally, the following $(*)$-*condition* is satisfied:

$(*_1)$ if S_i is a simple $A_{(\underline{p},l)}$-module which is neither projective nor injective and the vertex i belongs to a clockwise oriented path then $S_i \in \mathcal{C}_0(A_{(\underline{p},l)})$,

$(*_2)$ if S_i is a simple $A_{(\underline{p},l)}$-module which is neither projective nor injective and i belongs to a counter-clockwise oriented path then $S_i \in \mathcal{C}_\infty(A_{(\underline{p},l)})$,

$(*_3)$ if $M \cong M(w)$ is an $A_{(\underline{p},l)}$-module and w is a maximal path which is counter-clockwise oriented then $M(w) \in \mathcal{C}_0(A_{(\underline{p},l)})$,

$(*_4)$ if $M \cong M(w)$ is an $A_{(\underline{p},l)}$-module and w is a maximal path which is clockwise oriented then $M(w) \in \mathcal{C}_\infty(A_{(\underline{p},l)})$.

1.9. Let B be an algebra. Following Skowroński [17] we shall say that a component $\mathcal{C}$ of Γ_B is *generalized standard* if $\operatorname{rad}^\infty(X,Y) = 0$ for any indecomposable right B-modules X, Y whose isoclasses belong to $\mathcal{C}$, where $\operatorname{rad}^\infty(\operatorname{mod}(B))$ denotes the intersection of all natural powers of the Jacobson radical $\operatorname{rad}(\operatorname{mod}(B))$ of the category $\operatorname{mod}(B)$.

A connected component $\mathcal{C}$ in Γ_B is defined to be *starting* (resp. *ending*) if there is no nonzero morphism $f : X \to Y$ between indecomposable modules X, Y such that $Y \in \mathcal{C}$ and $X \notin \mathcal{C}$ (resp. $X \in \mathcal{C}$ and $Y \notin \mathcal{C}$). An example of a starting component is the preprojective component $\mathcal{P}(A_{(\underline{p},l)})$. It is also obvious that the preinjective component $\mathcal{I}(A_{(\underline{p},l)})$ is an ending component.

2. One-point extensions

2.1. Let B be a finite-dimensional triangular K-algebra. Consider the algebra

$$C = \begin{pmatrix} K & {}_K M_B \\ 0 & B \end{pmatrix},$$

where ${}_K M_B$ is a finite-dimensional K-B-bimodule. It is clear that C is a finite-dimensional triangular K-algebra. Moreover, we can treat finite-dimensional, right C-modules as triples (V, X_B, f), where V is a finite-dimensional K-linear space, X_B is a finite-dimensional right B-module and $f : V \to \operatorname{Hom}_B({}_K M_B, X)$ is a K-linear morphism. The algebra C is said to be a *one-point extension* of B by ${}_K M_B$ (see [13, 15]).

2.2. We can associate a vector space category $\mathcal{X}_{M_B}$ (see [15]) to the bimodule ${}_KM_B$; the indecomposable objects of $\mathcal{X}_{M_B}$ are the indecomposable finite-dimensional right B-modules X with $\mathrm{Hom}_B({}_KM_B, X) \neq 0$, and morphisms are of the form $\mathrm{Hom}_B({}_KM_B, f)$ for $f \in \mathrm{Hom}_B(X, Y)$. The structure of a left K-linear space on ${}_KM_B$ yields the structure of a right K-linear space on $\mathrm{Hom}_B({}_KM_B, X)$ for any $X \in \mathcal{X}_{M_B}$, and the functor $|-| : \mathcal{X}_{M_B} \to \mathrm{mod}(K)$ is of the form $|-| = \mathrm{Hom}_B({}_KM_B, -)$. We know from [13] that there exists a functor $\eta : \mathcal{U}(\mathcal{X}_{M_B}) \to \mathrm{mod}(C)$ which is full and faithful and establishes an equivalence between the subspace category $\mathcal{U}(\mathcal{X}_{M_B})$ and the full subcategory of $\mathrm{mod}(C)$ consisting of the modules without direct summands of the form $(0, X, 0)$. Moreover, there is an equivalence of categories $(\mathrm{mod}(C))/[\mathrm{mod}(B)] \cong \mathcal{U}(\mathcal{X}_{M_B})$ (see [15]).

A vector space category of the form $\mathcal{X}_{M_B}$ is said to be *linear* if

$$\dim_K \mathrm{Hom}_B({}_KM_B, X) = 1$$

for every indecomposable object $X \in \mathcal{X}_{M_B}$ and the partially ordered set attached to $\mathcal{X}_{M_B}$ is linearly ordered.

The next two lemmas were proved by Nazarova and Roiter in [12].

2.3. Lemma. *Let $\mathcal{X}_{M_B}$ be a linear vector space category. Then the triples of the form (K, X, f), where X is an indecomposable object from $\mathcal{X}_{M_B}$ and $f : K \to \mathrm{Hom}_B({}_KM_B, X)$ is the identity morphism, and $(K, 0, 0)$ form a full list of nonisomorphic indecomposable objects of the subspace category $\mathcal{U}(\mathcal{X}_{M_B})$.*

2.4. Lemma. *Let $\mathcal{X}_{M_B}$ be a vector space category which is equivalent to an additive category $\mathrm{add}(KS)$, where S is a disjoint union of two linearly ordered sets S_1, S_2. Then the triples of the form (K, X, id), (K, Y, id), $(K, X \oplus Y, \Delta)$, $(K, 0, 0)$, where X is an indecomposable object of $\mathcal{X}_{M_B}$ contained in S_1, Y is an indecomposable object of $\mathcal{X}_{M_B}$ contained in S_2, and $\Delta : K \to \mathrm{Hom}_B({}_KM_B, X \oplus Y) = K^2$ is given by $\Delta(k) = (k, k)$, form a full list of nonisomorphic indecomposable objects of $\mathcal{U}(\mathcal{X}_{M_B})$.*

3. One-point extensions of $A_{(\underline{p},l)}$

3.1. Now we shall consider the algebra

$$A = \begin{pmatrix} K & M_{A_{(\underline{p},l)}} \\ 0 & A_{(\underline{p},l)} \end{pmatrix},$$

where $M_{A_{(\underline{p},l)}} \cong M(w)$ is a simple regular $A_{(\underline{p},l)}$-module in the sense of [14]. Then $M_{A_{(\underline{p},l)}}$ is either a simple $A_{(\underline{p},l)}$-module which is neither projective nor injective, or a simple regular $A_{(\underline{p},l)}$-module which is not simple. In both cases $M_{A_{(\underline{p},l)}} \in \mathcal{C}_0(A_{(\underline{p},l)})$ or $M_{A_{(\underline{p},l)}} \in \mathcal{C}_\infty(A_{(\underline{p},l)})$. In these notations we have

3.2. LEMMA. (1) *The vector space category* $\mathcal{X}_{M_{A_{(\underline{p},l)}}}$ *is linear.*

(2) $\Gamma_A = \mathcal{P}(A_{(\underline{p},l)}) \sqcup \mathcal{C}_0(A) \sqcup \mathcal{C}_\infty(A) \sqcup \bigsqcup_{\lambda \in K^*} \mathcal{C}_\lambda(A_{(\underline{p},l)}) \sqcup \mathcal{I}(A)$ *and:*

(2i) *If* $M_{A_{(\underline{p},l)}} \in \mathcal{C}_0(A_{(\underline{p},l)})$ *then* $\mathcal{C}_\infty(A) = \mathcal{C}_\infty(A_{(\underline{p},l)})$.

(2ii) *If* $M_{A_{(\underline{p},l)}} \in \mathcal{C}_\infty(A_{(\underline{p},l)})$ *then* $\mathcal{C}_0(A) = \mathcal{C}_0(A_{(\underline{p},l)})$.

(2iii) *Every indecomposable projective* A*-module which is not an* $A_{(\underline{p},l)}$*-module belongs to the component which contains* $M_{A_{(\underline{p},l)}}$.

(2iv) $\mathcal{C}_0(A) \sqcup \mathcal{C}_\infty(A) \sqcup \bigsqcup_{\lambda \in K^*} \mathcal{C}_\lambda(A_{(\underline{p},l)})$ *separates* $\mathcal{P}(A_{(\underline{p},l)})$ *from* $\mathcal{I}(A)$.

(2v) $\mathcal{I}(A)$ *contains all indecomposable injective* A*-modules and is an ending component.*

Proof. See [14].

3.3. Let

$$B = \begin{pmatrix} A_{(\underline{p},l)} & 0 \\ {}_K M_{A_{(\underline{p},l)}} & K \end{pmatrix},$$

where $M_{A_{(\underline{p},l)}}$ is either a simple $A_{(\underline{p},l)}$-module which is neither projective nor injective, or a simple regular $A_{(\underline{p},l)}$-module which is not simple. Then the algebra B is called a *one-point coextension* of the algebra $A_{(\underline{p},l)}$ by the K-$A_{(\underline{p},l)}$-bimodule ${}_K M_{A_{(\underline{p},l)}}$. Under the above notations we have

LEMMA. $\Gamma_B = \mathcal{P}(B) \sqcup \mathcal{C}_0(B) \sqcup \mathcal{C}_\infty(B) \sqcup \bigsqcup_{\lambda \in K^*} \mathcal{C}_\lambda(A_{(\underline{p},l)}) \sqcup \mathcal{I}(A_{(\underline{p},l)})$ *and:*

(i) *If* $M_{A_{(\underline{p},l)}} \in \mathcal{C}_0(A_{(\underline{p},l)})$ *then* $\mathcal{C}_\infty(B) = \mathcal{C}_\infty(A_{(\underline{p},l)})$.

(ii) *If* $M_{A_{(\underline{p},l)}} \in \mathcal{C}_\infty(A_{(\underline{p},l)})$ *then* $\mathcal{C}_0(B) = \mathcal{C}_0(A_{(\underline{p},l)})$.

(iii) *Every injective indecomposable* B*-module which is not an* $A_{(\underline{p},l)}$*-module belongs to the component which contains* $M_{A_{(\underline{p},l)}}$.

(iv) $\mathcal{C}_0(B) \sqcup \mathcal{C}_\infty(B) \sqcup \bigsqcup_{\lambda \in K^*} \mathcal{C}_\lambda(A_{(\underline{p},l)})$ *separates* $\mathcal{P}(B)$ *from* $\mathcal{I}(A_{(\underline{p},l)})$.

(v) $\mathcal{P}(B)$ *contains all indecomposable projective* B*-modules and is a starting component.*

Proof. See [14].

4. Fundamental algebras

4.1. A triangular string algebra A is defined to be *fundamental* if $A \cong KQ_A/I_A$ is connected and in the bound quiver (Q_A, I_A) there exists exactly one full subquiver Q' of type $\widetilde{\mathbb{A}}_n$ such that $KQ' \cap I_A = 0$ and the quiver obtained from Q_A by removing all arrows belonging to Q' and identifying all vertices in Q' with vertex 0 is a tree.

For a K-algebra B, a right finite-dimensional B-module M is said to be *uniserial* if the lattice of its submodules is a chain.

4.2. LEMMA. *If A is a fundamental K-algebra then there exists a sequence $\underline{p}$, an integer $l \geq 1$ and a sequence $A_0, A_1, \ldots, A_r$ of fundamental algebras such that*:

(1) $A_0 \cong A_{(\underline{p},l)}$.
(2) *For each $i = 1, \ldots, r$ the algebra A_i is a one-point extension or a one-point coextension of A_{i-1} by a uniserial module.*
(3) $A_r \cong A$.

Proof. Let Q' be as in the definition of a fundamental algebra. Then there exists a sequence $\underline{p}$ and an integer $l \geq 1$ such that $Q' = Q_{(\underline{p},l)}$. We put $A_0 = A_{(\underline{p},l)}$. Let $\overline{Q}$ denote the quiver obtained from Q_A by removing all arrows in Q' and identifying all vertices in Q' with vertex 0. Since $\overline{Q}$ is a tree, there is a vertex $x \neq 0$ which is either the source of exactly one arrow and the target of none, or the target of exactly one arrow and the source of none. If $\overline{Q}$ has $r+1$ vertices then we put $A_r = A$. Let Q_{r-1} be the quiver obtained from Q_A by removing the vertex x and the only arrow α whose source or target is x. Let I_{r-1} be the two-sided ideal in KQ_{r-1} generated by the paths in I_A which do not contain α. We put $A_{r-1} = KQ_{r-1}/I_{r-1}$. Then A_{r-1} is fundamental by construction.

Suppose that the removed arrow α has source x. Let P_x be an indecomposable projective right A-module which is not an A_{r-1}-module. It is clear that $\operatorname{rad}(P_x)$ is a uniserial right A-module which is an A_{r-1}-module. Thus clearly

$$A \cong \begin{pmatrix} K & \operatorname{rad}(P_x) \\ 0 & A_{r-1} \end{pmatrix}$$

and $\operatorname{rad}(P_x)$ is a uniserial A_{r-1}-module.

If α has target x then consider an indecomposable injective right A-module E_x which is not an A_{r-1}-module. Again it is clear that $E_x/\operatorname{soc}(E_x)$ is a uniserial A-module which is an A_{r-1}-module. Thus

$$A \cong \begin{pmatrix} A_{r-1} & 0 \\ E_x/\operatorname{soc}(E_x) & K \end{pmatrix}$$

and $E_x/\operatorname{soc}(E_x)$ is a uniserial right A_{r-1}-module.

Consequently, $A_r = A$ is a one-point extension or coextension of A_{r-1} by a uniserial A_{r-1}-module.

Repeating the above arguments we construct algebras $A_{r-2}, \ldots, A_1$ such that the fundamental algebras $A_0, A_1, \ldots, A_r$ satisfy (1)–(3).

4.3. LEMMA. *If $A = KQ_A/I_A$ is a fundamental K-algebra then for any vertex x in Q_A there exists at most one walk w in Q_A of minimal length which starts at x and ends at a vertex of Q'.*

Proof. Since the quiver $\overline{Q}$ obtained from Q_A by removing all arrows in Q' and identifying all vertices in Q' with vertex 0 is a tree, the assertion is obvious.

Define a $\pm$-*arrow* of a quiver Q to be an arrow of Q or its formal inverse.

4.4. PROPOSITION. *Let A be a fundamental K-algebra. Then*

$$\Gamma_A = \mathcal{P}(A) \sqcup \mathcal{C}_0(A) \sqcup \bigsqcup_{\lambda \in K^*} \mathcal{C}_\lambda(A) \sqcup \mathcal{C}_\infty(A) \sqcup \mathcal{I}(A)$$

and the following conditions are satisfied:

(1) *If $X \in \mathcal{P}(A)$ then $X \cong X(w)$ for some walk w in (Q_A, I_A); conversely, $\mathcal{P}(A)$ contains all $X(w)$ for the walks w satisfying one of the following conditions*:

(1i) *w is a walk in $Q_{(\underline{p},l)}$ with $X(w) \in \mathcal{P}(A_{(\underline{p},l)})$.*

(1ii) *$w = w''\overline{w}w'$ for some walks w'', w' which do not contain any $\pm$-arrow from $Q_{(\underline{p},l)}$, and $w' = \alpha^{-1}w'_1$, where α is an arrow with source in $Q_{(\underline{p},l)}$ different from $0, p_2, p_4, \ldots, p_{q-1}$, and $\overline{w}$ is a walk in $Q_{(\underline{p},l)}$ such that $X(\overline{w}) \in \mathcal{P}(A_{(\underline{p},l)})$.*

(1iii) *w does not contain any $\pm$-arrow in $Q_{(\underline{p},l)}$ and there exists a walk w' (maybe trivial) which does not contain any $\pm$-arrow in $Q_{(\underline{p},l)}$ such that $w' = \alpha^{-1}w''$, where α is an arrow with source in $Q_{(\underline{p},l)}$ different from $0, p_2, p_4, \ldots, p_{q-1}$ and the other frame vertex of w' is the ending point of w. Moreover, if the source of α is different from $p_1, p_3, \ldots, p_q$ then $w' = w_1\beta^{-1}$ for some arrow β whose target coincides with the ending point of w.*

(2) *If $X_0 \in \mathcal{C}_0(A)$ then $X_0 \cong X_0(w)$ for some walk w in (Q_A, I_A); conversely, $\mathcal{C}_0(A)$ contains all $X_0(w)$ for the walks w satisfying one of the following conditions*:

(2i) *w is a walk in $Q_{(\underline{p},l)}$ such that $X_0(w) \in \mathcal{C}_0(A_{(\underline{p},l)})$.*

(2ii) *$w = w''\overline{w}w'$ for some walks w'', w' which do not contain any $\pm$-arrow from $Q_{(\underline{p},l)}$, and either the end of w' is a vertex in $Q_{(\underline{p},l)}$ which belongs to a maximal counter-clockwise oriented path in $Q_{(\underline{p},l)}$ and is neither the starting nor the ending point of this path, or the end of w' is one of $0, p_1, p_2, \ldots, p_q$. Furthermore, $\overline{w}$ is contained in $Q_{(\underline{p},l)}$ and $X_0(\overline{w}) \in \mathcal{C}_0(A_{(\underline{p},l)})$.*

(2iii) *w consists of $\pm$-arrows which do not belong to $Q_{(\underline{p},l)}$ and there is a walk w' (maybe trivial) which does not contain any $\pm$-arrow from $Q_{(\underline{p},l)}$ such that either $w' = \alpha^{-1}w''$, or $w' = \alpha w''$. If $w' = \alpha^{-1}w''$ then α is an arrow whose source is a vertex of some maximal counter-clockwise oriented path in $Q_{(\underline{p},l)}$ and is neither* *the starting nor the ending point of this path, and* $w' = w_1\beta$ *for some arrow* β *whose source coincides with the ending point of* w*. If* $w' = \alpha w''$ *then* α *is an arrow whose target belongs to a maximal counter-clockwise oriented path in* $Q_{(\underline{p},l)}$ *and is neither the starting nor the ending point of this path. Moreover,* $w' = w_1\beta^{-1}$ *for some arrow* β *whose target coincides with the ending point of* w*.*

(3) *If* $X_\infty \in \mathcal{C}_\infty(A)$ *then* $X_\infty \cong X_\infty(w)$ *for some walk* w *in* (Q_A, I_A)*; conversely,* $\mathcal{C}_\infty(A)$ *contains all* $X_\infty(w)$ *for the walks* w *satisfying one of the following conditions*:

(3i) w *is a walk in* $Q_{(\underline{p},l)}$ *such that* $X_\infty(w) \in \mathcal{C}_\infty(A_{(\underline{p},l)})$*.*

(3ii) $w = w''\overline{w}w'$ *for some walks* w''*,* w' *which do not contain any* $\pm$*-arrow from* $Q_{(\underline{p},l)}$*, and either the ending point of* w' *belongs to a maximal clockwise oriented path in* $Q_{(\underline{p},l)}$ *and is neither the starting nor the ending point of this path, or the ending point of* w' *is one of* $0, p_1, p_2, \dots, p_q$*. Furthermore,* $\overline{w}$ *is contained in* $Q_{(\underline{p},l)}$ *and* $X_\infty(\overline{w}) \in \mathcal{C}_\infty(A_{(\underline{p},l)})$*.*

(3iii) w *contains no* $\pm$*-arrow from* $Q_{(\underline{p},l)}$ *and there is a walk* w' *(maybe trivial) which does not contain any* $\pm$*-arrow from* $Q_{(\underline{p},l)}$ *such that either* $w' = \alpha^{-1}w''$ *or* $w' = \alpha w''$*. If* $w' = \alpha^{-1}w''$ *then* α *is an arrow whose source is a vertex of a maximal clockwise oriented path in* $Q_{(\underline{p},l)}$ *and is neither the starting nor the ending point of this path. Moreover,* $w' = w_1\beta$ *for some arrow* β *whose source coincides with the ending point of* w*. If* $w' = \alpha w''$ *then* α *is an arrow whose target belongs to a maximal clockwise oriented path in* $Q_{(\underline{p},l)}$ *and is neither the starting nor the ending point of this path. Moreover,* $w' = w_1\beta^{-1}$ *for some arrow* β *whose target coincides with the ending point of* w*.*

(4) *Any* $X_\lambda \in \mathcal{C}_\lambda(A)$*,* $\lambda \in K^*$*, is an* $A_{(\underline{p},l)}$*-module which belongs to* $\mathcal{C}_\lambda(A_{(\underline{p},l)})$*.*

(5) *If* $Y \in \mathcal{I}(A)$ *then* $Y \cong Y(w)$ *for some walk* w *in* (Q_A, I_A)*; conversely,* $\mathcal{I}(A)$ *contains all* $Y(w)$ *for the walks* w *satisfying one of the following conditions*:

(5i) w *is a walk in* $Q_{(\underline{p},l)}$ *such that* $Y(w) \in \mathcal{I}(A_{(\underline{p},l)})$*.*

(5ii) $w = w''\overline{w}w'$ *for some walks* w''*,* w' *which do not contain any* $\pm$*-arrow from* $Q_{(\underline{p},l)}$*, and* $w' = \alpha w'_1$*, where* α *is an arrow whose target belongs to* $Q_{(\underline{p},l)}$ *and is different from* $p_1, p_3, \dots, p_q$*. Furthermore,* $\overline{w}$ *is contained in* $Q_{(\underline{p},l)}$ *and* $Y(\overline{w}) \in \mathcal{I}(A_{(\underline{p},l)})$*.*

(5iii) *w does not contain any $\pm$-arrow from $Q_{(\underline{p},l)}$ and there is a walk w' (maybe trivial) which does not contain any $\pm$-arrow from $Q_{(\underline{p},l)}$ such that $w' = \alpha w''$, where α is an arrow whose target belongs to $Q_{(\underline{p},l)}$ and is different from $p_1, p_3, \dots, p_q$ and the other frame vertex of w' is the ending point of w. Moreover, if the target of α is different from $0, p_2, p_4, \dots, p_{q-1}$ then $w' = w_1\beta$ for some arrow β whose source coincides with the ending point of w.*

Proof. Let $A_0, A_1, \dots, A_r$ be given by Lemma 4.2. We shall prove the assertion by induction on r.

If $r = 1$ then the assertion is clear by Lemmas 3.2, 3.3.

Assume that the assertion is true for all fundamental algebras A with $r \leq r_0$. Let A' be a fundamental algebra such that there is a sequence of fundamental algebras $A_0, A_1, \dots, A_{r_0}, A_{r_0+1}$ which satisfies the relevant conditions. Assume that $A_{r_0+1} = A'$ is a one-point extension of $A_{r_0} = A$. Every bound quiver $(Q_{A'}, I_{A'})$ of A' is obtained from a bound quiver (Q_A, I_A) of A by adding to Q_A one vertex $0'$ and one arrow κ with source $0'$ and target $x \in Q_A$. Furthermore, $I_{A'}$ is a two-sided ideal which contains I_A and possibly new paths starting with κ.

Consider $\operatorname{rad}(P_{0'})$, which is a uniserial A-module. There is a nonzero path $\varepsilon_n \cdots \varepsilon_1$ in (Q_A, I_A) starting at x such that $\operatorname{rad}(P_{0'}) \cong M(\varepsilon_n \cdots \varepsilon_1)$, because A is a string algebra. It is clear that A' is a one-point extension of A by the module $M(\varepsilon_n \cdots \varepsilon_1) = M$.

If no walk in (Q_A, I_A) starts at x and ends at a vertex of $Q_{(\underline{p},l)}$ then the vector space category $\mathcal{X}_M$ contains only finitely many indecomposable A-modules $Z_1, \dots, Z_m$ with $Z_j \cong S_x$ or $Z_j \cong Z_j(w_j)$, where $w_j = w'\varepsilon_i \cdots \varepsilon_1$, $i = 1, \dots, n$, and either w' is trivial, or $w' = w''\tau^{-1}$, or else $w_j = w'\tau^{-1}$, where τ is an arrow in Q_A whose target is x. Then $\mathcal{X}_M$ is linear by [12].

If there is a walk w in (Q_A, I_A) which starts at x and ends at a vertex from $Q_{(\underline{p},l)}$ then by Lemma 4.3 there exists exactly one such walk w of minimal length. If $w = w'\delta\varepsilon_n \cdots \varepsilon_1$ then $\operatorname{Hom}_A(M, M(w)) = 0$ and $\mathcal{X}_M$ consists of finitely many indecomposable A-modules of the above form. Hence $\mathcal{X}_M$ is linear. If $w = w'\delta^{-1}\varepsilon_i \cdots \varepsilon_1$, $i = 1, \dots, n$, or $w = w'\delta^{-1}$, where δ is an arrow in Q_A whose target is x, then $\operatorname{Hom}_A(M, M(w)) \cong K$. Let $y \in Q_{(\underline{p},l)}$ be the end of w. Then there are walks $\overline{w}$ in $Q_{(\underline{p},l)}$ which start at y such that $\operatorname{Hom}_A(M, M(\overline{w}w)) \cong K$. Now consider the case $w = \eta\widetilde{w''}\delta^{-1}\varepsilon_i \cdots \varepsilon_1$, $i = 1, \dots, n$, or $w = \eta\widetilde{w''}\delta^{-1}$. Then y is the target of the arrow η. Thus $y \neq p_1, p_3, \dots, p_q$. If $y \in \{0, p_2, p_4, \dots, p_{q-1}\}$ then by the inductive assumption for every walk $\overline{w}$ in $Q_{(\underline{p},l)}$ starting at y we have either $M(\overline{w}) \cong Y(\overline{w}) \in \mathcal{I}(A)$, or $M(\overline{w}) \cong X_0(\overline{w}) \in \mathcal{C}_0(A)$, or else $M(\overline{w}) \cong X_\infty(\overline{w}) \in \mathcal{C}_\infty(A)$. Since A is fundamental, we have either $\mathcal{X}_M \subset \mathcal{C}_0(A) \sqcup \mathcal{I}(A)$ or $\mathcal{X}_M \subset \mathcal{C}_\infty(A) \sqcup \mathcal{I}(A)$.

Consider the case $\mathcal{X}_M \subset \mathcal{C}_0(A) \sqcup \mathcal{I}(A)$. First suppose that $y \in \{0, p_2, p_4, \dots, p_{q-1}\}$. Then $X_0(\overline{w}w) \in \mathcal{C}_0(A) \cap \mathcal{X}_M$, where $\overline{w} = \overline{w}'\alpha_{i,p_i-p_{i-1}} \cdots \alpha_{i,1}$ if $y = p_i$, and $p_i - p_{i-1} = l$ if $y = 0$. In this case all $X_0(\overline{w}w)$ are of the form

$$X_0(\alpha_{i,p_i-p_{i-1}} \cdots \alpha_{i,1}w), \quad X_0(\alpha^{-1}_{i-1,p_{i-1}-p_{i-2}}\alpha_{i,p_i-p_{i-1}} \cdots \alpha_{i,1}w), \ \dots,$$

and it is easy to see that

$$\mathrm{Hom}_{\mathcal{X}_M}(X_0(\overline{w}w), X_0(\overline{w}_1w)) \cong \begin{cases} K & \text{if } l(\overline{w}) \le l(\overline{w}_1), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $X_0(w''\overline{w}w) \in \mathcal{C}_0(A) \cap \mathcal{X}_M$ provided that $X_0(\overline{w}w) \in \mathcal{C}_0(A) \cap \mathcal{X}_M$ and $\overline{w}$ is as above. Then

$$\mathrm{Hom}_{\mathcal{X}_M}(X_0(w''\overline{w}w), X_0(w''_1\overline{w}_1w)) \cong \begin{cases} K & \text{if } l(\overline{w}) < l(\overline{w}_1), \\ K & \text{if } l(\overline{w}) = l(\overline{w}_1) \text{ and} \\ & \mathrm{Hom}_A(X(w''\overline{w}), X(w''_1\overline{w}_1)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $Y(\overline{w}w) \in \mathcal{I}(A) \cap \mathcal{X}_M$ provided that

$$\overline{w} = \alpha_{i,1},\ \alpha_{i,2}\alpha_{i,1},\ \dots,\ \alpha_{i,p_i-p_{i-1}} \cdots \alpha_{i,1}, \\ \alpha^{-1}_{i-1,1} \cdots \alpha^{-1}_{i-1,p_{i-1}-p_{i-2}}\alpha_{i,p_i-p_{i-1}} \cdots \alpha_{i,1},\ \dots$$

Then it is easy to see that

$$\mathrm{Hom}_{\mathcal{X}_M}(Y(\overline{w}w), Y(\overline{w}_1w)) \cong \begin{cases} K & \text{if } l(\overline{w}) \le l(\overline{w}_1), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $Y(w''\overline{w}w) \in \mathcal{I}(A) \cap \mathcal{X}_M$ provided that $Y(\overline{w}w) \in \mathcal{I}(A) \cap \mathcal{X}_M$ and $\overline{w}$ is as above. Then

$$\mathrm{Hom}_{\mathcal{X}_M}(Y(w''\overline{w}w), Y(w''_1\overline{w}_1w)) \cong \begin{cases} K & \text{if } l(\overline{w}) < l(\overline{w}_1), \\ K & \text{if } l(\overline{w}) = l(\overline{w}_1) \text{ and} \\ & \mathrm{Hom}_A(Y(w''\overline{w}), Y(w''_1\overline{w}_1)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, all indecomposable modules $Z(w')$ in $\mathcal{X}_M$ for the walks w' of the form $w' = \widetilde{w}\tau^{-1}\varepsilon_a \cdots \varepsilon_1$ with $a > i$ such that w' is disjoint from $Q_{(\underline{p},l)}$ form a linear vector space category by [12], and

$$\mathrm{Hom}_{\mathcal{X}_M}(Z(w'), X_0(w''\overline{w}w)) \cong K \cong \mathrm{Hom}_{\mathcal{X}_M}(Z(w'), Y(w''_1\overline{w}_1w)), \\ \mathrm{Hom}_{\mathcal{X}_M}(X_0(w''\overline{w}w), Z(w')) = 0 = \mathrm{Hom}_{\mathcal{X}_M}(Y(w''_1\overline{w}_1w), Z(w')).$$

Likewise, the indecomposable modules $Z(w')$ in $\mathcal{X}_M$ for the walks w' of the form $w' = \widetilde{w}\tau^{-1}\varepsilon_b \cdots \varepsilon_1$ with $b < i$ such that w' is disjoint from $Q_{(\underline{p},l)}$ form a linear vector space category and

$$\mathrm{Hom}_{\mathcal{X}_M}(X_0(w''\overline{w}w), Z(w')) \cong K \cong \mathrm{Hom}_{\mathcal{X}_M}(Y(w''_1\overline{w}_1w), Z(w')).$$

Finally, the indecomposable modules $Z(w_1)$ in $\mathcal{X}_M$ for $w_1 = w'_1\delta^{-1}\varepsilon_i \cdots \varepsilon_1$ disjoint from $Q_{(\underline{p},l)}$ also form a linear vector space category. Next, for every walk w_1 it is easy to see that either

$$\mathrm{Hom}_{\mathcal{X}_M}(Z(w_1), X_0(w''\overline{w}w)) \cong K \cong \mathrm{Hom}_{\mathcal{X}_M}(Z(w_1), Y(w_1''\overline{w}_1w))$$

and

$$\mathrm{Hom}_{\mathcal{X}_M}(X_0(w''\overline{w}w), Z(w_1)) = 0 = \mathrm{Hom}_{\mathcal{X}_M}(Y(w_1''\overline{w}_1w), Z(w_1)),$$

or vice versa.

In view of the above remarks the vector space category $\mathcal{X}_M$ is linear.

Now consider the case $y \neq 0, p_2, p_4, \ldots, p_{q-1}$. Then y is a vertex of a maximal counter-clockwise oriented path in $Q_{(p,l)}$, and y is neither the starting nor the ending point of this path. To simplify notation assume that y is the target of the arrow $\alpha_{0,i}$, $i \neq l$. Then $X_0(\overline{w}w) \in \mathcal{C}_0(A) \cap \mathcal{X}_M$ provided that $\overline{w}$ is one of the following walks:

$$e_y, \alpha_{0,i}^{-1},\ \alpha_{0,i-1}^{-1}\alpha_{0,i}^{-1},\ \ldots, \alpha_{0,2}^{-1}\cdots\alpha_{0,i}^{-1},\ \alpha_{1,p_1}\cdots\alpha_{1,1}\alpha_{0,1}^{-1}\cdots\alpha_{0,i}^{-1},\ \ldots$$

It is easy to see that

$$\mathrm{Hom}_{\mathcal{X}_M}(X_0(\overline{w}w), X_0(\overline{w}_1w)) \cong \begin{cases} K & \text{if } l(\overline{w}) \leq l(\overline{w}_1), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $X_0(w''\overline{w}w) \in \mathcal{C}_0(A) \cap \mathcal{X}_M$ if $\overline{w}$ is as above. Then

$$\mathrm{Hom}_{\mathcal{X}_M}(X_0(w''\overline{w}w), X_0(w_1''\overline{w}_1w)) \cong \begin{cases} K & \text{if } l(\overline{w}) < l(\overline{w}_1), \\ K & \text{if } l(\overline{w}) = l(\overline{w}_1) \text{ and} \\ & \mathrm{Hom}_A(X_0(w''\overline{w}), X_0(w_1''\overline{w}_1)) \cong K, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $Y(\overline{w}w) \in \mathcal{X}_M \cap \mathcal{I}(A)$ provided that $\overline{w}$ is one of the following walks:

$$\alpha_{0,1}^{-1}\cdots\alpha_{0,i}^{-1},\ \alpha_{1,1}\alpha_{0,1}^{-1}\cdots\alpha_{0,i}^{-1}\cdot\alpha_{1,2}\alpha_{1,1}\alpha_{0,1}^{-1}\cdots\alpha_{0,i}^{-1},\ \ldots$$

Then it is easy to see that

$$\mathrm{Hom}_{\mathcal{X}_M}(Y(\overline{w}w), Y(\overline{w}_1w)) \cong \begin{cases} K & \text{if } l(\overline{w}_1) \leq l(\overline{w}), \\ 0 & \text{if } l(\overline{w}_1) > l(\overline{w}). \end{cases}$$

Furthermore, $Y(w''\overline{w}w) \in \mathcal{X}_M \cap \mathcal{I}(A)$ if $\overline{w}$ is as above. Then

$$\mathrm{Hom}_{\mathcal{X}_M}(Y(w''\overline{w}w), Y(w_1''\overline{w}_1w)) \cong \begin{cases} K & \text{if } l(\overline{w}_1) < l(\overline{w}), \\ K & \text{if } l(\overline{w}_1) = l(\overline{w}) \text{ and} \\ & \mathrm{Hom}_A(Y(w''\overline{w}), Y(\overline{w}_1w)) \cong K, \\ 0 & \text{otherwise.} \end{cases}$$

Additionally, for every $X_0(\overline{w}w) \in \mathcal{X}_M \cap \mathcal{C}_0(A)$ and every $Y(\overline{w}_1w) \in \mathcal{X}_M \cap \mathcal{I}(A)$ we obviously have

$$\mathrm{Hom}_{\mathcal{X}_M}(X_0(\overline{w}w), Y(\overline{w}_1w)) \cong K, \quad \mathrm{Hom}_{\mathcal{X}_M}(Y(\overline{w}_1w), X_0(\overline{w}w)) = 0.$$

Thus for every $X_0(w''\overline{w}w) \in \mathcal{X}_M \cap \mathcal{C}_0(A)$ and every $Y(w_1''\overline{w}_1 w) \in \mathcal{X}_M \cap \mathcal{I}(A)$ we have

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{X}_M}(X_0(w''\overline{w}w), Y(w_1''\overline{w}_1 w)) \cong K,\\ &\mathrm{Hom}_{\mathcal{X}_M}(Y(w_1''\overline{w}_1 w), X_0(w''\overline{w}w)) = 0.\end{aligned}$$

Next, the indecomposable modules $Z(w')$ in $\mathcal{X}_M$ for $w' = \widetilde{w}\tau^{-1}\varepsilon_a \cdots \varepsilon_1$ with $a > i$ such that w' is disjoint from $Q_{(\underline{p},l)}$ form a linear vector space category by [12], and

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{X}_M}(Z(w'), X_0(w''\overline{w}w)) \cong K \cong \mathrm{Hom}_{\mathcal{X}_M}(Z(w'), Y(w_1''\overline{w}_1 w)),\\ &\mathrm{Hom}_{\mathcal{X}_M}(X_0(w''\overline{w}w), Z(w')) = 0 = \mathrm{Hom}_{\mathcal{X}_M}(Y(w_1''\overline{w}_1 w), Z(w')).\end{aligned}$$

The indecomposable modules $Z(w')$ for $w' = \widetilde{w}\tau^{-1}\varepsilon_b \cdots \varepsilon_1$ with $b < i$ such that w' is disjoint from $Q_{(\underline{p},l)}$ also form a linear vector space category, and

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{X}_M}(Z(w'), X_0(w''\overline{w}w)) = 0 = \mathrm{Hom}_{\mathcal{X}_M}(Z(w'), Y(w_1''\overline{w}_1 w)),\\ &\mathrm{Hom}_{\mathcal{X}_M}(X_0(w''\overline{w}w), Z(w')) \cong K \cong \mathrm{Hom}_{\mathcal{X}_M}(Y(w_1''\overline{w}_1 w), Z(w')).\end{aligned}$$

Finally, the indecomposable modules $Z(w_1)$ for $w_1 = w_1'\delta^{-1}\varepsilon_i \cdots \varepsilon_1$ or $w_1 = \varepsilon_i \cdots \varepsilon_1$ such that w_1 is disjoint from $Q_{(\underline{p},l)}$ also form a linear vector space category. Furthermore, either

$$\mathrm{Hom}_{\mathcal{X}_M}(Z(w_1), X_0(w''\overline{w}w)) \cong K \cong \mathrm{Hom}_{\mathcal{X}_M}(Z(w_1), Y(w_1''\overline{w}_1 w))$$

and

$$\mathrm{Hom}_{\mathcal{X}_M}(X_0(w''\overline{w}w), Z(w_1)) = 0 = \mathrm{Hom}_{\mathcal{X}_M}(Y(w_1''\overline{w}_1 w), Z(w_1)),$$

or vice versa.

Consequently, the vector space category $\mathcal{X}_M$ is linear.

The case when $\mathcal{X}_M \subset \mathcal{C}_\infty(A) \cup \mathcal{I}(A)$ is similar.

If $w = \eta^{-1}\widetilde{w}_1\delta^{-1}\varepsilon_i \cdots \varepsilon_1$, $i = 1, \ldots, n$, or $w = \eta^{-1}\widetilde{w}_1\delta^{-1}$, then a similar analysis shows that $\mathcal{X}_M$ is linear; here the indecomposable modules $Z(w''\overline{w}w)$ belong to $\mathcal{P}(A) \sqcup \mathcal{C}_0(A)$ or to $\mathcal{P}(A) \sqcup \mathcal{C}_\infty(A)$.

Now Lemma 2.3 shows that the indecomposable A'-modules which are not A-modules can be identified with the following objects of the subspace category $\mathcal{U}(\mathcal{X}_M)$: $(K, 0, 0)$, (K, Z, id) for all indecomposable $Z \in \mathcal{X}_M$. Since every indecomposable $Z \in \mathcal{X}_M$ is of the form $Z \cong Z(w)$ for some walk w in (Q_A, I_A) which starts at x, $(K, 0, 0)$ is in fact the simple A'-module $S_{0'}$. However every object $(K, Z(w), \mathrm{id})$ is in fact an A'-module of the form $Z(w\kappa)$. Therefore we obtain condition (4) for the algebra A'.

If there is no walk connecting x to $Q_{(\underline{p},l)}$ in (Q_A, I_A) then $\mathcal{X}_M$ is a finite vector space category whose indecomposable objects are $Z(w)$ for the walks w of the form $e_x, w'\tau^{-1}, w_1'\sigma^{-1}\varepsilon_i \cdots \varepsilon_1$, $i \in \{1, \ldots, n\}$, disjoint from $Q_{(\underline{p},l)}$. Then the indecomposable A'-modules which are not A-modules are of the form $Z(w\kappa)$ for the above w. Moreover, if there is an irreducible morphism

$f : Z(w) \to Z_1$ in $\mathrm{mod}(A)$ then either there are irreducible morphisms $Z(w) \xrightarrow{g} Z(w\kappa) \xrightarrow{h} Z_1$ in $\mathrm{mod}(A')$ or f is also irreducible in $\mathrm{mod}(A')$. Hence $Z(w\kappa)$ and $Z(w)$ belong to the same component and so passing from Γ_A to $\Gamma_{A'}$ we have no gluing of components. Therefore $\Gamma_{A'} = \mathcal{P}(A') \sqcup \mathcal{C}_0(A') \sqcup \bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A') \sqcup \mathcal{C}_\infty(A') \sqcup \mathcal{I}(A')$ and conditions (1)–(5) are satisfied.

If there is a walk in (Q_A, I_A) which connects x to $Q_{(\underline{p},l)}$ then let w be such a walk of minimal length. By the first part of the proof the vector space category $\mathcal{X}_M$ is linear, and the indecomposable A-modules of the form $Z(w''\overline{w}w')$ which belong to $\mathcal{X}_M$ are contained in $\mathcal{C}_0(A)\sqcup\mathcal{I}(A)$, $\mathcal{C}_\infty(A)\sqcup\mathcal{I}(A)$, $\mathcal{P}(A) \sqcup \mathcal{C}_0(A)$, $\mathcal{P}(A) \sqcup \mathcal{C}_\infty(A)$.

If $\mathcal{X}_M \subset \mathcal{C}_0(A) \sqcup \mathcal{I}(A)$ then $\mathcal{P}(A') = \mathcal{P}(A)$, $\mathcal{C}_\infty(A') = \mathcal{C}_\infty(A)$ and $\mathcal{C}_\lambda(A') = \mathcal{C}_\lambda(A)$, $\lambda \in K^*$. Hence conditions (1), (3), (4) hold for $\Gamma_{A'}$.

In order to check (2), (5), notice that the new walks in $(Q_{A'}, I_{A'})$ are of the form $w\kappa$. Thus if $f : Z(w) \to Z_1$ is an irreducible morphism in $\mathrm{mod}(A)$ then either it is irreducible in $\mathrm{mod}(A')$, or there are irreducible morphisms $Z(w) \xrightarrow{g} Z(w\kappa) \xrightarrow{h} Z_1$. Hence passing from Γ_A to $\Gamma_{A'}$ we do not glue any different components, and so $\Gamma_{A'} = \mathcal{P}(A')\sqcup\mathcal{C}_0(A')\sqcup\bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A')\sqcup\mathcal{C}_\infty(A')\sqcup \mathcal{I}(A')$. Furthermore, it is obvious that if (2) (resp. (5)) is satisfied for w then it is also satisfied for $w\kappa$.

The other cases can be checked similarly. We omit the details.

Consequently, A_{r_0+1} is a one-point extension of A_{r_0}, and conditions (1)–(5) hold for A_{r_0+1}.

The case when A_{r_0+1} is a one-point coextension of A_{r_0} is similar.

4.5. Proposition. *Let A be a fundamental algebra. Then $\mathcal{C}_0(A) \sqcup \bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A) \sqcup \mathcal{C}_\infty(A)$ separates $\mathcal{P}(A)$ from $\mathcal{I}(A)$ in Γ_A.*

Proof. We keep the notation of the previous proof and again argue by induction on r. If $r = 1$ then the assertion holds by Lemmas 3.2 and 3.3.

Assume that the assertion is true for a fixed r_0. Let A be such that the above r for A is $r_0 + 1$. Set $A_{r_0} = A'$ for $A \cong A_{r_0+1}$. By the inductive assumption, the required condition holds for A'.

Suppose that A is a one-point extension of A' by a uniserial A'-module M. Then a bound quiver (Q_A, I_A) is obtained from $(Q_{A'}, I_{A'})$ by adding one vertex $0'$ and one arrow κ with source $0'$ and target $x \in Q_{A'}$. Furthermore, the two-sided ideal I_A contains $I_{A'}$ and possibly some new paths starting with κ.

Consider the uniserial A'-module $M \cong \mathrm{rad}(P_{0'})$. There exists a nonzero path $\varepsilon_n \cdots \varepsilon_1$ in $(Q_{A'}, I_{A'})$ starting at x such that $M \cong M(\varepsilon_n \cdots \varepsilon_1)$.

First we check that $\mathcal{C}_0(A) \sqcup \bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A) \sqcup \mathcal{C}_\infty(A)$ is a family of pairwise orthogonal components. Notice that if Z, U are indecomposable A'-modules which belong to different components then

$$\mathrm{Hom}_A(Z, U) = 0 = \mathrm{Hom}_A(U, Z)$$

by the inductive assumption. Let $X_0(w\kappa) \in \mathcal{C}_0(A)$ and $X_\lambda \in \mathcal{C}_\lambda$, $\lambda \in K^*$. Then by Proposition 4.4, X_λ is an A'-module. If $f : X_0(w\kappa) \to X_\lambda$ is a homomorphism of A-modules then $f\iota$ is a homomorphism of A'-modules, where $\iota : X_0(w) \to X_0(w\kappa)$ is the inclusion. But ι is an irreducible morphism by the Skowroński–Waschbüsch algorithm. Hence $f\iota = 0$ by the inductive assumption, and so $f = 0$. If $g : X_\lambda \to X_0(w\kappa)$ is a homomorphism of A-modules then $g = \iota g_1$ for some homomorphism $g_1 : X_\lambda \to X_0(w)$. Since g_1 is a homomorphism of A'-modules, $g_1 = 0$ by the inductive assumption. Thus $g = 0$. Consequently, $\mathcal{C}_0(A)$ is orthogonal to all components $\mathcal{C}_\lambda(A)$, $\lambda \in K^*$.

One shows similarly that $\mathcal{C}_\infty(A)$ is orthogonal to all $\mathcal{C}_\lambda(A)$, $\lambda \in K^*$. Moreover, the inductive assumption and Proposition 4.4 imply that the $\mathcal{C}_\lambda(A)$, $\lambda \in K^*$, are pairwise orthogonal.

Let $X_0(w) \in \mathcal{C}_0(A)$ and $X_\infty(w_1) \in \mathcal{C}_\infty(A)$. Consider the case $w = w'\kappa$ and $w_1 \neq w_1'\kappa$. Then $X_0(w'\kappa)$ is not an A'-module and $X_\infty(w_1)$ is an A'-module. Suppose that $f : X_0(w'\kappa) \to X_\infty(w_1)$ is a homomorphism of A-modules. Then $f\iota$ is a homomorphism of A'-modules, where $\iota : X_0(w') \to X_0(w'\kappa)$ is the inclusion. By the inductive assumption, $f\iota = 0$. Hence $f = 0$. If $g : X_\infty(w_1) \to X_0(w'\kappa)$ is a homomorphism of A-modules then $g = \iota g_1$ for some homomorphism $g_1 : X_\infty(w_1) \to X_0(w')$ of A'-modules. But the inductive assumption yields $g_1 = 0$. Thus $g = 0$.

If $w_1 = w_1'\kappa$ and $w \neq w'\kappa$ then similar arguments show that

$$\mathrm{Hom}_A(X_0(w), X_\infty(w_1)) = 0 = \mathrm{Hom}_A(X_\infty(w_1), X_0(w)).$$

Now suppose that $w_1 = w_1'\kappa$ and $w = w'\kappa$. Let $f : X_0(w'\kappa) \to X_\infty(w_1'\kappa)$ be a homomorphism of A-modules. Then it is clear that $X_0(w') \not\subset \ker(f)$ provided that $f \neq 0$. Thus $f\iota \neq 0$ for the irreducible monomorphism $\iota : X_0(w') \to X_0(w'\kappa)$. But by the above considerations $f\iota = 0$, because $X_0(w')$ is an A'-module from $\mathcal{C}_0(A)$. Thus $f = 0$. One shows similarly that $\mathrm{Hom}_A(X_\infty(w_1'\kappa), X_0(w'\kappa)) = 0$. Consequently, $\mathcal{C}_0(A)$ and $\mathcal{C}_\infty(A)$ are orthogonal, which finishes the proof that $\mathcal{C}_0(A) \sqcup \bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A) \sqcup \mathcal{C}_\infty(A)$ is a family of pairwise orthogonal components in Γ_A.

Let $X(w) \in \mathcal{P}(A)$ and $X_\lambda \in \mathcal{C}_\lambda(A)$, $\lambda \in K^*$. If $w \neq w'\kappa$ then $X(w)$ is an A'-module and so $\mathrm{Hom}_A(X_\lambda, X(w)) = 0$ by Proposition 4.4 and the inductive assumption. If $w = w'\kappa$ then the Skowroński–Waschbüsch algorithm yields an irreducible monomorphism $\iota : X(w') \to X(w'\kappa)$ such that any homomorphism $g : X_\lambda \to X(w'\kappa)$ is of the form $\iota g_1 = g$ for some homomorphism $g_1 : X_\lambda \to X(w')$ of A'-modules. But $g_1 = 0$ by the inductive assumption. Hence $g = 0$.

Let $X(w) \in \mathcal{P}(A)$ and $X_0(w_1) \in \mathcal{C}_0(A)$. If $w \neq w'\kappa$ and $w_1 \neq w_1'\kappa$ then both modules are A'-modules and $\mathrm{Hom}_A(X_0(w_1), X(w)) = 0$ by the inductive assumption. If $w = w'\kappa$ and $w_1 \neq w_1'\kappa$ then any homomorphism $f : X_0(w_1) \to X(w)$ of A-modules is of the form $f = \iota f_1$, where $f_1 : X_0(w_1) \to X(w')$ is a homomorphism of A'-modules and $\iota : X(w') \to X(w'\kappa)$ is irreducible. But $f_1 = 0$ by the inductive assumption, and so $f = 0$. If $w \neq w'\kappa$ and $w_1 = w_1'\kappa$ then for a nonzero homomorphism $f : X_0(w_1'\kappa) \to X(w)$ of A-modules we have $f\varrho \neq 0$, where $\varrho : X_0(w_1') \to X_0(w_1'\kappa)$ is irreducible. But $f\varrho = 0$ by the inductive assumption. Thus $f = 0$. If $w = w'\kappa$ and $w_1 = w_1'\kappa$ then for any nonzero homomorphism $f : X_0(w_1'\kappa) \to X(w'\kappa)$ we have $f\varrho \neq 0$, where $\varrho : X_0(w_1') \to X_0(w_1'\kappa)$ is irreducible. But from the above considerations we deduce that $f\varrho = 0$, which shows that $f = 0$. Consequently, $\mathrm{Hom}_A(\mathcal{C}_0(A) \sqcup \bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A), \mathcal{P}(A)) = 0$.

A similar analysis shows that $\mathrm{Hom}_A(\mathcal{C}_\infty(A), \mathcal{P}(A)) = 0$, which implies that $\mathrm{Hom}_A(\mathcal{C}_0 \sqcup \bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A) \sqcup \mathcal{C}_\infty(A), \mathcal{P}(A)) = 0$.

Dually one shows that $\mathrm{Hom}_A(\mathcal{I}(A), \mathcal{C}_0(A) \sqcup \bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A) \sqcup \mathcal{C}_\infty(A)) = 0$.

Now consider $X(w) \in \mathcal{P}(A)$ and $Y(w_1) \in \mathcal{I}(A)$. Let $f : X(w) \to Y(w_1)$ be a nonzero homomorphism of A-modules. If $w \neq w'\kappa$ and $w_1 \neq w_1'\kappa$ then $X(w), Y(w_1)$ are A'-modules and by the inductive assumption there are A'-modules $X_i \in \mathrm{add}(\mathcal{C}_i(A))$, $i \in K \cup \{\infty\}$, and homomorphisms $h_i : X \to X_i$, $g_i : X_i \to Y$, $i \in K \cup \{\infty\}$, such that $f = g_i h_i$. Thus the required condition is satisfied.

If $w = w'\kappa$ and $w_1 \neq w_1'\kappa$ then $w' = w''\varepsilon_i \cdots \varepsilon_1\kappa$ for $i \in \{1, \ldots, n\}$ and $w'' = w'''\delta^{-1}$ or w'' is trivial, or else $w' = w''\delta^{-1}$, or w' is trivial.

If $w = \kappa$ then there is no nonzero homomorphism from $X(w)$ to any A'-module. If $w = \varepsilon_i \cdots \varepsilon_1\kappa$ then there is no nonzero homomorphism from $X(w)$ to any A'-module. Therefore $w = w'''\delta^{-1}\varepsilon_i \cdots \varepsilon_1\kappa$ or $w = w''\delta^{-1}\kappa$. But the Skowroński–Waschbüsch algorithm yields an irreducible homomorphism $\iota_1 : X(w) \to X(w''')$ or $\iota_2 : X(w) \to X(w')$. Since $Y(w_1)$ is an A'-module, we have $\ker(\iota_1) \subset \ker(f)$ and $\ker(\iota_2) \subset \ker(f)$ for any nonzero homomorphism $f : X(w) \to Y(w_1)$. Thus $f = f_1\iota_j$, $j = 1, 2$, for some homomorphism $f_1 : X(w') \to Y(w_1)$ or $f_1 : X(w''') \to Y(w_1)$ of A'-modules. Since the required condition holds for f_1, it also holds for f.

If $w_1 = w_1'\kappa$ and $w \neq w'\kappa$ then any homomorphism $f : X(w) \to Y(w_1)$ is of the form ϱf_1, where $\varrho : Y(w_1') \to Y(w_1'\kappa)$ is irreducible and $f_1 : X(w) \to Y(w_1')$ is a homomorphism of A'-modules. Since the required condition holds for f_1, it also holds for f.

If $w = w'\kappa$ and $w_1 = w_1'\kappa$ then we consider an irreducible homomorphism $\varrho : Y(w_1') \to Y(w_1'\kappa)$. If $f = \varrho f_1$ for some $f_1 : X(w) \to Y(w_1')$ then we deduce from the above analysis that f_1 satisfies the required condition, and hence so does f. If $f \neq \varrho f_1$ then either $f\iota \neq 0$ for some irreducible $\iota : X(w') \to X(w'\kappa)$ or $Y(w_1) \cong S_{0'}$. First consider the case $f\iota \neq 0$. Then the above considerations show that the required condition holds for $f\iota$. In particular $f\iota$ factorizes through a module $X_\lambda \in \operatorname{add}(\mathcal{C}_\lambda(A))$ for some $\lambda \in K^*$. Thus $\operatorname{im}(f\iota)$ is an $A_{(\underline{p},l)}$-module. But $x \in \operatorname{supp}(\operatorname{im}(f\iota))$. Hence $x \in Q_{(\underline{p},l)}$. Then by Proposition 4.4(1), either $X(w')$ is an $A_{(\underline{p},l)}$-module, or $X(w') = X(w''\overline{w})$ with $X(\overline{w}) \in \mathcal{P}(A_{(\underline{p},l)})$, or else w' does not contain any $\pm$-arrow from $Q_{(\underline{p},l)}$. In the last case we have no factorization of $f\iota$ through any module $X_\lambda \in \operatorname{add}(\mathcal{C}_\lambda(A))$, for $\lambda \in K^*$. Thus the last case is impossible. If $X(w')$ is an $A_{(\underline{p},l)}$-module from $\mathcal{P}(A_{(\underline{p},l)})$ then $X(w)$ is not an indecomposable A-module by Lemma 3.3. If $X(w') \cong X(w''\overline{w})$ with $X(\overline{w}) \in \mathcal{P}(A_{(\underline{p},l)})$ then $X(w''\overline{w}\kappa)$ is not an indecomposable A-module from $\mathcal{P}(A)$ by Proposition 4.4(1).

If $Y(w_1) \cong S_{0'}$ then f factorizes through the indecomposable injective A-module E_x and a similar analysis shows that $X(w)$ cannot be an indecomposable A-module from $\mathcal{P}(A)$. Therefore $f = \varrho f_1$ and the required condition holds for f_1, and so for f.

Consequently, $\mathcal{C}_0(A) \sqcup \bigsqcup_{\lambda\in K^*} \mathcal{C}_\lambda(A) \sqcup \mathcal{C}_\infty(A)$ separates $\mathcal{P}(A)$ from $\mathcal{I}(A)$ if $A \cong A_{r_0+1}$ is a one-point extension of A_{r_0}. If $A \cong A_{r_0+1}$ is a one-point coextension of A_{r_0} we proceed dually. Thus, the proof is finished.

5. 2-fundamental algebras

5.1. Let n be a fixed positive integer. A triangular string $\widetilde{\mathbb{A}}_m$-separated algebra A is defined to be *n-fundamental* if $A \cong KQ_A/I_A$ is connected and the following conditions are satisfied:

(1) There exist exactly n full subquivers $Q'_1, \ldots, Q'_n$ of type $\widetilde{\mathbb{A}}_m$ in (Q_A, I_A) which are pairwise disjoint and such that $KQ'_j \cap I_A = 0$ and the quiver $\overline{Q}_A$, obtained from Q_A by removing the arrows from Q'_j, $j = 1, \ldots, n$, and identifying the vertices of Q'_j with vertex 0_j, $j = 1, \ldots, n$, is a tree.

(2) For any vertex 0_j in $\overline{Q}_A$ there exists either a maximal path v in $\overline{Q}_A$ starting at 0_j such that $v \notin I_A$, or a maximal path u in $\overline{Q}_A$ ending at 0_j such that $u \notin I_A$. If v (treated as a path in Q_A) starts at some vertex x in Q'_j which is the ending point of two maximal paths v_1, v_2 in Q'_j then $vv_1 \notin I_A$ or $vv_2 \notin I_A$. If u (treated as a path in Q_A) ends at some vertex y in Q'_j which is the starting point of two maximal paths u_1, u_2 in Q'_j then $u_1u \notin I_A$ or $u_2u \notin I_A$.

It is clear that any 1-fundamental algebra is fundamental.

In this section we shall study Auslander–Reiten quivers of 2-fundamental algebras.

A 2-fundamental algebra A is defined to be *minimal* if the quiver $\overline{Q}_A$ is of type $\mathbb{A}_m$.

5.2. Lemma. *Let A be a minimal 2-fundamental algebra. If $\overline{Q}_A$ is a path w such that $w \notin I_A$ then Γ_A has only one starting component $\mathcal{P}(A)$ and only one ending component $\mathcal{I}(A)$; both are generalized standard.*

Proof. Since A is 2-fundamental, there are exactly two disjoint subquivers $Q_{(\underline{p},l)}$ and $Q'_{(\underline{p}',l')}$ in (Q_A, I_A). Moreover, $\overline{Q}_A$ is a path $w = \beta_m \cdots \beta_1$ by assumption. Assume that the source of β_1 belongs to $Q_{(\underline{p},l)}$ and the target of β_m belongs to $Q'_{(\underline{p}',l')}$.

We start by studying the component $\mathcal{P}(A)$ which contains the simple projective A-modules $S_{i'}$ for $i' = p'_1, p'_3, \ldots, p'_{q'}$. By the Skowroński–Waschbüsch algorithm, $\mathcal{P}(A) = \mathcal{P}(A'_{(\underline{p}',l')})$ for $A'_{(\underline{p}',l')} = KQ'_{(\underline{p}',l')}$. Thus $\mathcal{P}(A)$ consists of the indecomposable $A'_{(\underline{p}',l')}$-modules $X(\overline{w})$ such that $X(\overline{w}) \in \mathcal{P}(A'_{(\underline{p}',l')})$. If there is a nonzero homomorphism $f : Y \to X(\overline{w})$, where $X(\overline{w}) \in \mathcal{P}(A)$ and Y is an indecomposable A-module which is not an $A'_{(\underline{p}',l')}$-module, then $Y \cong X(v\overline{w}')$ for some walk $\overline{w}'$ in $Q'_{(\underline{p}',l')}$ and some nontrivial walk $v = v'\beta_m^{-1}$. But then $fg \neq 0$ for the inclusion $g : X(\overline{w}') \to X(v\overline{w}')$. Therefore $X(\overline{w}') \in \mathcal{P}(A)$ by the above considerations. Thus $X(\beta_m^{-1}\overline{w}') \in \mathcal{P}(A)$, which is impossible, because $\mathcal{P}(A) = \mathcal{P}(A'_{(\underline{p}',l')})$. Hence $\mathcal{P}(A)$ is a starting component in Γ_A, and it is generalized standard.

Dual arguments show that the component $\mathcal{I}(A)$ which contains the simple injective A-modules S_i for $i = 0, p_2, p_4, \ldots, p_{q-1}$ is ending and generalized standard. We omit the details.

The fact that $\mathcal{P}(A)$ is the only starting component and $\mathcal{I}(A)$ the only ending component in Γ_A is a direct consequence of Propositions 4.4 and 4.5.

5.3. Lemma. *Let A be a minimal 2-fundamental algebra. If $\overline{Q}_A$ contains a path $v \in I_A$ then Γ_A has a starting component $\mathcal{P}(A)$ which is generalized standard and an ending component $\mathcal{I}(A)$ which is generalized standard.*

Proof. If $\overline{Q}_A$ is a path then the arguments from the proof of Lemma 5.2 yield the desired components.

Suppose $\overline{Q}_A$ is not a path. Let $v = \beta_m \cdots \beta_1$ be a path in $\overline{Q}_A$ such that $v \in I_A$. Again, there are exactly two disjoint subquivers $Q_{(\underline{p},l)}$ and $Q'_{(\underline{p}',l')}$ in (Q_A, I_A). Let $\overline{Q}_A$ be a walk of the form $w_2\beta_m \cdots \beta_1 w_1$, where the starting point of w_1 belongs to $Q_{(\underline{p},l)}$ and the ending point of w_2 belongs to $Q'_{(\underline{p}',l')}$. Consider the full subquiver Q' in Q_A which contains the arrows of $Q_{(\underline{p},l)}$ and of $\beta_{m-1} \cdots \beta_1 w_1$. Let $I' = I_A \cap KQ'$. Then the algebra $A' = KQ'/I'$ is obviously fundamental. Denote by Q'' the full subquiver in Q_A which contains the arrows of $Q'_{(\underline{p}',l')}$ and of $w_2\beta_m \cdots \beta_2$. We put $I'' = I_A \cap KQ''$. Then $A'' = KQ''/I''$ is fundamental. Moreover, $\Gamma_A = \Gamma_{A'} \cup \Gamma_{A''}$, where $\Gamma_{A'} \cap \Gamma_{A''} = \Gamma_B$ for $B = KQ_B/I_B$ given by the full subquiver Q_B of Q_A which contains the arrows $\beta_2, \ldots, \beta_{m-1}$, and the two-sided ideal $I_B = KQ_B \cap I_A$. Now Propositions 4.4 and 4.5 show that $\Gamma_{A'}$ contains the starting component $\mathcal{P}(A')$ and the ending component $\mathcal{I}(A')$. Similarly, $\Gamma_{A''}$ contains the starting component $\mathcal{P}(A'')$ and the ending component $\mathcal{I}(A'')$. It is clear that $\mathcal{P}(A')$, $\mathcal{P}(A'')$ are starting components in Γ_A provided that $\mathcal{P}(A') \cap \mathcal{P}(A'') = \emptyset$, and $\mathcal{P}(A') \cup \mathcal{P}(A'')$ is a starting component in Γ_A otherwise. Moreover, $\mathcal{I}(A')$, $\mathcal{I}(A'')$ are ending components in Γ_A provided that $\mathcal{I}(A') \cap \mathcal{I}(A'') = \emptyset$, and $\mathcal{I}(A') \cup \mathcal{I}(A'')$ is an ending component in Γ_A otherwise.

Now we show that if $\mathcal{P}(A') \cap \mathcal{P}(A'') \neq \emptyset$ then $\mathcal{P}(A') \cup \mathcal{P}(A'')$ is generalized standard. Let $X_1(v_1), X_2(v_2) \in \mathcal{P}(A') \cup \mathcal{P}(A'')$. Suppose that there is a nonzero homomorphism $f \in \operatorname{rad}^\infty(X_1(v_1), X_2(v_2))$. If $X_1(v_1), X_2(v_2) \in \mathcal{P}(A')$ and both are $A_{(\underline{p},l)}$-modules, then $\operatorname{rad}^\infty(X_1(v_1), X_2(v_2)) = 0$, because $\mathcal{P}(A')$ contains $\mathcal{P}(A_{(\underline{p},l)})$ and $\mathcal{P}(A_{(\underline{p},l)})$ is a generalized standard component of $\Gamma_{A_{(\underline{p},l)}}$. If $X_1(v_1)$ is an $A_{(\underline{p},l)}$-module and $X_2(v_2)$ is not, then there exists a nonzero homomorphism $f : X_1(v_1) \to X_2(v_2)$ provided that v_2 is of the form $v_2 = \overline{w}_2\alpha^{-1}w_2'$ by Proposition 4.4(1), where α is an arrow whose source belongs to $Q_{(\underline{p},l)}$ and is different from $0, p_2, p_4, \ldots, p_{q-1}$ and $\overline{w}_2$ is a walk in $Q_{(\underline{p},l)}$ such that $X(\overline{w}_2) \in \mathcal{P}(A_{(\underline{p},l)})$. But in this case there is a nonzero homomorphism $g : X_2(v_2) \to X(\overline{w}_2)$ and $gf \neq 0$. Thus $gf \in \operatorname{rad}^\infty(X_1(v_1), X(\overline{w}_2))$, which is impossible.

If $X_2(v_2)$ is an $A_{(\underline{p},l)}$-module and $X_1(v_1)$ is not, then Proposition 4.4(1) implies that $v_1 = \overline{w}_1\alpha^{-1}w_1'$, where α is an arrow with source in $Q_{(\underline{p},l)}$ different from $0, p_2, p_4, \ldots, p_{q-1}$, and $\overline{w}_1$ is a walk in $Q_{(\underline{p},l)}$ such that $X(\overline{w}_1) \in \mathcal{P}(A_{(\underline{p},l)})$. Then $\overline{w}_1 = \overline{w}_1'\delta$, where δ is an arrow in $Q_{(\underline{p},l)}$ whose source coincides with that of α. Hence we have a monomorphism $h : X(\overline{w}_1') \to X_1(v_1)$ and $X(\overline{w}_1') \in \mathcal{P}(A_{(\underline{p},l)})$. If $f \neq 0$ then $fh \in \operatorname{rad}^\infty(X(\overline{w}_1'), X_2(v_2))$, which is impossible. Thus $fh = 0$. But in this case $\operatorname{im}(h) \subset \ker(f)$. Thus f factorizes through $X(\alpha^{-1}w_1') \notin \mathcal{P}(A')$, which is impossible, because $\mathcal{P}(A')$ is a starting component in $\Gamma_{A'}$.

If $X_1(v_1)$ and $X_2(v_2)$ are not $A_{(\underline{p},l)}$-modules then by Proposition 4.4(1), $v_1 = \overline{w}_1\alpha^{-1}w_1'$ and $v_2 = \overline{w}_2\alpha^{-1}w_2'$. Moreover, we have an epimorphism $g : X_2(v_2) \to X(\overline{w}_2)$. We infer that $gf = 0$. Hence $\operatorname{im}(f) \subset X(w_2')$. But there is a monomorphism $h : X(w_1') \to X_1(v_1)$ and $fh \neq 0$ by the last inclusion. Thus $fh \in \operatorname{rad}^\infty(X(w_1'), X_2(v_2))$. Furthermore, $gfh \neq 0$ is in $\operatorname{rad}^\infty(X(w_1'), X(w_2'))$. Since it is easily seen that no homomorphism from $X(w_1')$ to $X(w_2')$ can factorize through an A-module which is not a $K\overline{Q}_A/\overline{I}_A$-module for $\overline{I}_A = K\overline{Q}_A \cap I_A$, we have $gfh = 0$, and so f is zero.

Similar considerations in the cases $X_1(v_1), X_2(v_2) \in \mathcal{P}(A'')$, or $X_1(v_1) \in \mathcal{P}(A')$, $X_2(v_2) \in \mathcal{P}(A'')$, or else $X_1(v_1) \in \mathcal{P}(A'')$, $X_2(v_2) \in \mathcal{P}(A')$ show that

$\mathcal{P}(A') \cup \mathcal{P}(A'')$ is a generalized standard component. In particular, when $\mathcal{P}(A') \cap \mathcal{P}(A'') = \emptyset$, both $\mathcal{P}(A')$, $\mathcal{P}(A'')$ are generalized standard.

Dual arguments show that $\mathcal{I}(A') \cup \mathcal{I}(A'')$ is generalized standard when $\mathcal{I}(A') \cap \mathcal{I}(A'') \neq \emptyset$, and both $\mathcal{I}(A')$, $\mathcal{I}(A'')$ are generalized standard if $\mathcal{I}(A') \cap \mathcal{I}(A'') = \emptyset$. This finishes the proof of the lemma.

5.4. PROPOSITION. *Let A be a minimal 2-fundamental algebra. If $\overline{Q}_A$ is a quiver of type $\mathbb{A}_n$ which is not a path and there is no subpath v in $\overline{Q}_A$ such that $v \in I_A$ then Γ_A has at least one starting component $\mathcal{P}(A)$ and at least one ending component $\mathcal{I}(A)$. Moreover, the total number of starting and ending components in Γ_A is not greater than 3.*

Proof. The assumptions imply that there exists a vertex r in $\overline{Q}_A$ which is either the source of no arrow in Q_A, or the target of no arrow in Q_A. If r is not the target of any arrow in Q_A then there are fundamental subalgebras A_1, A_2 in A such that

$$A = \begin{pmatrix} K & \operatorname{rad}(P_r) \\ 0 & A_1 \times A_2 \end{pmatrix}$$

and $\operatorname{rad}(P_r) \cong M \oplus N$ for some uniserial A_1-module M and some uniserial A_2-module N. Since there is no subpath $v \in I_A$ in $\overline{Q}_A$, the A_1-module M is either projective, or simple, or else simple regular, and similarly for the A_2-module N. Moreover, Q_A is obtained from Q_{A_1} and Q_{A_2} by adding the vertex r and two arrows: an arrow ε with source r and target $x \in Q_{A_1}$, and an arrow τ with source r and target $y \in Q_{A_2}$. Then the proof of Proposition 4.4 shows that the vector space categories $\mathcal{X}_M$, $\mathcal{X}_N$ are linear. Since $\mathcal{X}_M$ consists of A_1-modules and $\mathcal{X}_N$ consists of A_2-modules, $\mathcal{X}_{M\oplus N} = \mathcal{X}_M \sqcup \mathcal{X}_N$ is a vector space category which satisfies the assumptions of Lemma 2.4. Furthermore, by Proposition 4.4,

$$\Gamma_{A_1} = \mathcal{P}(A_1) \sqcup \mathcal{C}_0(A_1) \sqcup \bigsqcup_{\lambda \in K^*} \mathcal{C}_\lambda(A_1) \sqcup \mathcal{C}_\infty(A_1) \sqcup \mathcal{I}(A_1),$$

$$\Gamma_{A_2} = \mathcal{P}(A_2) \sqcup \mathcal{C}_0(A_2) \sqcup \bigsqcup_{\lambda \in K^*} \mathcal{C}_\lambda(A_2) \sqcup \mathcal{C}_\infty(A_2) \sqcup \mathcal{I}(A_2).$$

By the proof of Proposition 4.4, $\mathcal{X}_M$ is contained either in $\mathcal{P}(A_1) \sqcup \mathcal{C}_0(A_1)$, or in $\mathcal{P}(A_1) \sqcup \mathcal{C}_\infty(A_1)$, or in $\mathcal{C}_0(A_1) \sqcup \mathcal{I}(A_1)$, or else in $\mathcal{C}_\infty(A_1) \sqcup \mathcal{I}(A_1)$, and similarly for $\mathcal{X}_N$.

If $\mathcal{X}_M \subset \mathcal{P}(A_1) \sqcup \mathcal{C}_0(A_1)$ and $\mathcal{X}_N \subset \mathcal{P}(A_2) \sqcup \mathcal{C}_0(A_2)$ then Proposition 4.4 shows that any $X \in \mathcal{X}_M$ is of the form $X \cong X(w)$ for some walk w in (Q_{A_1}, I_{A_1}) which starts at x. Moreover, if $X(w) \in \mathcal{P}(A_1)$ then either w is a walk in $\overline{Q}_A$ or $w = \overline{w}w'$, where w' is a walk in $\overline{Q}_A$ and $\overline{w}$ is a walk without $\pm$-arrows which belong to $\overline{Q}_A$ and $w' = \alpha^{-1}w''$ for some arrow α. If $X(w) \in \mathcal{C}_0(A_1)$ then either w is a walk in $\overline{Q}_A$ or $w = \overline{w}w'$ for some walk w' in $\overline{Q}_A$ and some walk $\overline{w}$ without $\pm$-arrows from $\overline{Q}_A$.

Similarly, any $Y \in \mathcal{X}_N$ is of the form $Y \cong Y(u)$ for some walk u in (Q_{A_2}, I_{A_2}) which starts at y. Furthermore, u is either contained in $\overline{Q}_A$ or $u = \overline{u}u'$, where u' is a walk contained in $\overline{Q}_A$ which does not contain any $\pm$-arrow from $\overline{Q}_A$.

By Lemma 2.4 the indecomposable A-modules which are not $A_1 \times A_2$-modules are in 1-1 correspondence with the following objects of the subspace category $\mathcal{U}(\mathcal{X}_{M\oplus N})$: $(K,0,0)$, (K,X,id) for all indecomposable $X \in \mathcal{X}_M$, (K,Y,id) for all indecomposable $Y \in \mathcal{X}_N$ and $(K, X \oplus Y, \Delta)$ for all indecomposable $X \in \mathcal{X}_M$, $Y \in \mathcal{X}_N$. But $(K,0,0)$ corresponds to the simple A-module S_r; $(K, X(w), \mathrm{id})$ corresponds to $X(w\varepsilon)$; $(K, Y(u), \mathrm{id})$ corresponds to $Y(u\tau)$; and $(K, X(w) \oplus Y(u), \Delta)$ corresponds to $XY(w\varepsilon\tau^{-1}u^{-1})$.

Notice that in passing from $\Gamma_{A_1\times A_2} = \Gamma_{A_1} \sqcup \Gamma_{A_2}$ to Γ_A the components $\mathcal{C}_\lambda(A_1)$, $\mathcal{C}_\lambda(A_2)$, $\lambda \in K^*$, remain components in Γ_A. Similarly, for $\mathcal{C}_\infty(A_1)$, $\mathcal{C}_\infty(A_2)$, $\mathcal{I}(A_1)$, $\mathcal{I}(A_2)$ the Skowroński–Waschbüsch algorithm acts identically in $\mathrm{mod}(A)$ and in $\mathrm{mod}(A_1 \times A_2)$. Thus they too are components in Γ_A. We now show that $\mathcal{P}(A_1)$, $\mathcal{P}(A_2)$, $\mathcal{C}_0(A_1)$, $\mathcal{C}_0(A_2)$ are glued into a common component $\mathcal{P}(A)$ in Γ_A.

Since A is a minimal 2-fundamental algebra and $\overline{Q}_A$ is of type $\mathbb{A}_n$, is not a path and is relation-free, it follows that Q_{A_1} contains a subquiver $Q_{(\underline{p},l)}$ and a subquiver $\overline{Q}_{A_1} = Q_{A_1} \cap \overline{Q}_A$. The arrows of Q_{A_1} which belong to $Q_{(\underline{p},l)}$ will be denoted by α_{ij} (as in 1.8). By Proposition 4.4(1), $\overline{Q}_{A_1}$ is of the form

$$z \xrightarrow{\varrho_{1,1}} \cdots \xrightarrow{\varrho_{1,s_1}} \xleftarrow{\varrho_{2,s_2}} \cdots \xleftarrow{\varrho_{2,1}} \cdots \xrightarrow{\varrho_{t,1}} \cdots \xrightarrow{\varrho_{t,s_t}} \xleftarrow{\varepsilon_m} \cdots \xleftarrow{\varepsilon_1} x$$

with $t \ge 1$, $s_i \ge 0$, $m \ge 0$, where $m = 0$ denotes that x is the target of the arrow ϱ_{t,s_t}, and z is the only vertex of $\overline{Q}_{A_1}$ which belongs to $Q_{(\underline{p},l)}$. Similarly Q_{A_2} contains a subquiver $Q'_{(\underline{p}',l')}$ and a subquiver $\overline{Q}_{A_2} = Q_{A_2} \cap \overline{Q}_A$. The arrows in Q_{A_2} which belong to $Q'_{(\underline{p}',l')}$ will be denoted by $\alpha'_{i',j'}$. Proposition 4.4(1) implies that $\overline{Q}_{A_2}$ is of the form

$$y \xrightarrow{\tau_1} \cdots \xrightarrow{\tau_n} \xleftarrow{\eta_{c,a_c}} \cdots \xleftarrow{\eta_{c,1}} \cdots \xrightarrow{\eta_{2,1}} \cdots \xrightarrow{\eta_{2,a_2}} \xleftarrow{\eta_{1,a_1}} \cdots \xleftarrow{\eta_{1,1}} z'$$

with $c \ge 1$, $a_i \ge 0$, $n \ge 0$, where $n = 0$ denotes that y is the target of the arrow η_{c,a_c}, and z' is the only vertex of $\overline{Q}_{A_2}$ which belongs to $Q'_{(\underline{p}',l')}$.

In the above notation, $M \cong X(\varepsilon_m \cdots \varepsilon_1)$ and $N \cong Y(\tau_1^{-1} \cdots \tau_n^{-1})$. Then Proposition 4.4(1iii), (2iii) shows that $M \in \mathcal{P}(A_1)$ and $N \in \mathcal{P}(A_2)$. Furthermore, we have irreducible morphisms $M \to XY(\varepsilon_m \cdots \varepsilon_1\varepsilon\tau^{-1}\tau_1^{-1} \cdots \tau_n^{-1}) \cong P_r$ and $N \to P_r$ in $\mathrm{mod}(A)$. Then applying the Skowroński–Waschbüsch algorithm, we find that if $X(w) \in \mathcal{X}_M \cap \mathcal{P}(A_1)$ then we have an irreducible morphism $X(w) \to XY(w\varepsilon\tau^{-1}\tau_1^{-1} \cdots \tau_n^{-1})$ in $\mathrm{mod}(A)$. Similarly, if $Y(u) \in \mathcal{X}_N \cap \mathcal{P}(A_2)$ then there is an irreducible morphism $Y(u) \to XY(\varepsilon_m \cdots \varepsilon_1\varepsilon\tau^{-1}u^{-1})$ in $\mathrm{mod}(A)$. It is easy to see, applying the Skowroński–

Waschbüsch algorithm, that if $X(w) \in \mathcal{X}_M \cap \mathcal{P}(A_1)$ and $Y(u) \in \mathcal{X}_N \cap \mathcal{P}(A_2)$ then $XY(w\varepsilon\tau^{-1}u^{-1}) \in \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the component in Γ_A which contains the projective module P_r.

Now notice that we have the chain of irreducible morphisms

$$M \to X(\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1) \to \cdots \to X(\varrho_{1,2}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1) \to X$$

in $\mathrm{mod}(A)$, where $X \cong X(\varrho_{1,1}^{-1}\varrho_{1,2}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1)$ in case z is the end point of a maximal path in $Q_{(\underline{p},l)}$. Furthermore, again by the Skowroński–Waschbüsch algorithm, there is a chain of irreducible morphisms of the form $S_z \to \cdots \to X$ in $\mathrm{mod}(A)$ in case z is the end point of a maximal path in $Q_{(\underline{p},l)}$, and a chain $X(\alpha_{i,p_i-p_{i-1}}\cdots\alpha_{i,j}) \to \cdots \to X$ otherwise. Further in the respective cases we have the following chains of irreducible morphisms in $\mathrm{mod}(A)$:

$$X(\varrho_{1,1}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1\varepsilon) \to \cdots \to X(\varrho_{1,1}^{-1}) \to S_z,$$
$$X(\alpha_{i,p_i-p_{i-1}}\cdots\alpha_{i,j}\varrho_{1,1}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1\varepsilon) \to \cdots$$
$$\to X(\alpha_{i,p_i-p_{i-1}}\cdots\alpha_{i,j}\varrho_{1,1}^{-1}) \to X(\alpha_{i,p_i-p_{i-1}}\cdots\alpha_{i,j}).$$

In both cases the sources of the chains correspond to the objects $(K, X(w), \mathrm{id})$ for $X(w) \in \mathcal{X}_M \cap \mathcal{P}(A_1)$. Repeating the above arguments for any $X(w) \in \mathcal{X}_M \cap \mathcal{P}(A_1)$, we obtain $X(w\varepsilon) \in \mathcal{P}(A)$. Symmetrically one shows that $Y(\tau^{-1}u^{-1}) \in \mathcal{P}(A)$ for any $Y(u) \in \mathcal{X}_N \cap \mathcal{P}(A_2)$.

Further it is easy to see that for any $X(w) \in \mathcal{X}_M \cap \mathcal{P}(A_1)$ we have the following chain of irreducible morphisms in $\mathrm{mod}(A)$: $XY(w\varepsilon\tau^{-1}u^{-1}) \to \cdots \to XY(w\varepsilon\tau^{-1}) \to X(w\varepsilon)$, where $Y(u) \in \mathcal{X}_N \cap \mathcal{C}_0(A_2)$. Hence the objects $(K, X(w) \oplus Y(u), \Delta)$ belong to $\mathcal{P}(A)$, where $X(w) \in \mathcal{X}_M \cap \mathcal{P}(A_1)$ and $Y(u) \in \mathcal{X}_N \cap \mathcal{C}_0(A_2)$. Symmetrically, $(K, X(w) \oplus Y(u), \Delta) \in \mathcal{P}(A)$, where $X(w) \in \mathcal{X}_M \cap \mathcal{C}_0(A_1)$ and $Y(u) \in \mathcal{X}_N \cap \mathcal{P}(A_2)$. Since the above considerations show that for any $X(w) \in \mathcal{X}_M \cap \mathcal{C}_0(A_1)$ and $Y(u) \in \mathcal{X}_N \cap \mathcal{P}(A_2)$ we have $XY(w\varepsilon\tau^{-1}u^{-1}) \in \mathcal{P}(A)$, in particular $XY(\varepsilon\tau^{-1}\tau_1^{-1}\cdots\tau_n^{-1}) \in \mathcal{P}(A)$. But there is an irreducible morphism $S_x \to XY(\varepsilon\tau^{-1}\tau_1^{-1}\cdots\tau_n^{-1})$. Thus $S_x \in \mathcal{P}(A)$. Similarly, $S_y \in \mathcal{P}(A)$. Consequently, the objects of $\mathcal{X}_M \cap \mathcal{C}_0(A_1) \cup \mathcal{X}_N \cap \mathcal{C}_0(A_2)$ belong to $\mathcal{P}(A)$.

Furthermore, the Skowroński–Waschbüsch algorithm shows that for S_z nonprojective there exists the following chain of irreducible morphisms in $\mathrm{mod}(A)$:

$$X(\varrho_{1,1}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1) \to \cdots \to X(\varepsilon_2\varepsilon_1) \to X(\varepsilon_1) \to S_x$$

when z is not the end point of a maximal path in $Q_{(\underline{p},l)}$, and

$$X(\alpha_{i,1}^{-1}\cdots\alpha_{i,p_i-p_{i-1}}^{-1}\varrho_{1,1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1) \to \cdots$$
$$\to X(\varepsilon_2\varepsilon_1) \to X(\varepsilon_1) \to S_x$$

when z is the end point of the maximal path $\alpha_{i,p_i-p_{i-1}}\cdots\alpha_{i,1}$ in $Q_{(\underline{p},l)}$. Applying the algorithm again, we obtain the chains

$$S_z \to \cdots \to X(\varrho_{1,1}^{-1}\cdots\varrho_{1,s_1}^{-1}) \to \cdots \to X(\varrho_{1,1}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_2) \\ \to X(\varrho_{1,1}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1)$$

and

$$X(\alpha_{i,1}^{-1}\cdots\alpha_{i,p_i-p_{i-1}}^{-1}) \to \cdots \\ \to X(\alpha_{i,1}^{-1}\cdots\alpha_{i,p_i-p_{i-1}}^{-1}\varrho_{1,1}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1)$$

in the respective cases.

Moreover, we have the chains

$$X(\varrho_{1,1}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1\varepsilon) \to \cdots \to X(\varrho_{1,1}^{-1}) \to S_z$$

and

$$X(\alpha_{i,1}^{-1}\cdots\alpha_{i,p_i-p_{i-1}}^{-1}\varrho_{1,1}^{-1}\cdots\varrho_{1,s_1}^{-1}\cdots\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1}\varepsilon_m\cdots\varepsilon_1\varepsilon) \to \cdots \\ \to X(\alpha_{i,1}^{-1}\cdots\alpha_{i,p_i-p_{i-1}}^{-1}\varrho_{1,1}^{-1}) \to X(\alpha_{i,1}^{-1}\cdots\alpha_{i,p_i-p_{i-1}}^{-1})$$

in the same cases. Thus by the Skowroński–Waschbüsch algorithm, for any $X(w) \in \mathcal{X}_M \cap \mathcal{C}_0(A_1)$ the A-module $X(w\varepsilon)$ belongs to $\mathcal{P}(A)$. Symmetrically, for any $Y(u) \in \mathcal{X}_N \cap \mathcal{C}_0(A_2)$ the A-module $Y(\tau^{-1}u^{-1})$ belongs to $\mathcal{P}(A)$. Moreover, $XY(w\varepsilon\tau^{-1}) \in \mathcal{P}(A)$ for any $X(w) \in \mathcal{C}_0(A_1) \cap \mathcal{X}_M$, since we have an irreducible morphism $XY(w\varepsilon\tau^{-1}) \to X(w\varepsilon)$ in $\operatorname{mod}(A)$. Therefore, $XY(w\varepsilon\tau^{-1}u^{-1}) \in \mathcal{P}(A)$ for any $X(w) \in \mathcal{X}_M \cap C_0(A_1)$ and $Y(u) \in \mathcal{X}_N \cap \mathcal{C}_0(A_2)$.

If S_x is a simple projective A-module then the above arguments can be applied for $X(\varrho_{t,1}^{-1}\cdots\varrho_{t,s_t}^{-1})$ instead for S_x to conclude that for any $X(w) \in \mathcal{X}_M \cap \mathcal{C}_0(A_1)$ and $Y(u) \in \mathcal{X}_N \cap \mathcal{C}_0(A_2)$ the modules $X(w\varepsilon)$, $Y(\tau^{-1}u^{-1})$, $XY(w\varepsilon\tau^{-1}u^{-1})$ belong to $\mathcal{P}(A)$.

Consequently, $\mathcal{P}(A)$ is the only component of Γ_A which contains the A-modules from $\mathcal{P}(A_1) \cup \mathcal{P}(A_2) \cup \mathcal{C}_0(A_1) \cup \mathcal{C}_0(A_2)$ and the indecomposable A-modules which are not $A_1 \times A_2$-modules. Therefore

$$\Gamma_A = \mathcal{P}(A) \sqcup \mathcal{C}_\infty(A_1) \sqcup \mathcal{C}_\infty(A_2) \sqcup \bigsqcup_{\lambda\in K^*} (\mathcal{C}_\lambda(A_1) \sqcup \mathcal{C}_\lambda(A_2)) \sqcup \mathcal{I}(A_1) \sqcup \mathcal{I}(A_2).$$

The arguments from the proof of Proposition 4.5 imply that $\mathcal{C}_\infty(A_1) \sqcup \mathcal{C}_\infty(A_2) \sqcup \bigsqcup_{\lambda\in K^*}(\mathcal{C}_\lambda(A_1) \sqcup \mathcal{C}_\lambda(A_2))$ separates $\mathcal{P}(A)$ from $\mathcal{I}(A_1) \sqcup \mathcal{I}(A_2)$. By the arguments from the proof of Lemma 5.2, $\mathcal{P}(A)$ is a starting component in Γ_A, and $\mathcal{I}(A_1)$, $\mathcal{I}(A_2)$ are ending components in Γ_A. Moreover, $\mathcal{I}(A_1)$, $\mathcal{I}(A_2)$ are generalized standard.

If either $\mathcal{X}_M \subset \mathcal{P}(A_1) \sqcup \mathcal{C}_\infty(A_1)$ and $\mathcal{X}_N \subset \mathcal{P}(A_2) \sqcup \mathcal{C}_0(A_2)$, or $\mathcal{X}_M \subset \mathcal{P}(A_1) \sqcup \mathcal{C}_\infty(A_1)$ and $\mathcal{X}_N \subset \mathcal{P}(A_2) \sqcup \mathcal{C}_\infty(A_2)$, or else $\mathcal{X}_M \subset \mathcal{P}(A_1) \sqcup \mathcal{C}_0(A_1)$

and $\mathcal{X}_N \subset \mathcal{P}(A_2) \sqcup \mathcal{C}_\infty(A_2)$, then similar arguments yields respectively

$$\Gamma_A = \mathcal{P}(A) \sqcup \mathcal{C}_0(A_1) \sqcup \mathcal{C}_\infty(A_2) \sqcup \bigsqcup_{\lambda\in K^*} (\mathcal{C}_\lambda(A_1) \sqcup \mathcal{C}_\lambda(A_2)) \sqcup \mathcal{I}(A_1) \sqcup I(A_2)$$

or

$$\Gamma_A = \mathcal{P}(A) \sqcup \mathcal{C}_0(A_1) \sqcup \mathcal{C}_0(A_2) \sqcup \bigsqcup_{\lambda\in K^*} (\mathcal{C}_\lambda(A_1) \sqcup \mathcal{C}_\lambda(A_2)) \sqcup \mathcal{I}(A_1) \sqcup \mathcal{I}(A_2)$$

or else

$$\Gamma_A = \mathcal{P}(A) \sqcup \mathcal{C}_\infty(A_1) \sqcup \mathcal{C}_0(A_2) \sqcup \bigsqcup_{\lambda\in K^*} (\mathcal{C}_\lambda(A_1) \sqcup \mathcal{C}_\lambda(A_2)) \sqcup \mathcal{I}(A_1) \sqcup \mathcal{I}(A_2).$$

Moreover, in each case $\mathcal{P}(A)$ is a starting component and $\mathcal{I}(A_1)$, $\mathcal{I}(A_2)$ are generalized standard ending components.

Dual arguments show that if $\mathcal{X}_M \subset \mathcal{C}_{i_1}(A_1) \sqcup \mathcal{I}(A_1)$ and $\mathcal{X}_N \subset \mathcal{C}_{i_2}(A_2) \sqcup \mathcal{I}(A_2)$ then $\Gamma_A = \mathcal{P}(A_1) \sqcup \mathcal{P}(A_2) \sqcup \mathcal{C}_{j_1}(A_1) \sqcup \mathcal{C}_{j_2}(A_2) \sqcup \bigsqcup_{\lambda\in K^*}(\mathcal{C}_\lambda(A_1) \sqcup \mathcal{C}_\lambda(A_2)) \sqcup \mathcal{I}(A)$, where $i_1, j_1 \in \{0,\infty\}$ are different and $i_2, j_2 \in \{0,\infty\}$ are different. Moreover, $\mathcal{C}_{j_1}(A_1) \sqcup \mathcal{C}_{j_2}(A_2) \sqcup \bigsqcup_{\lambda\in K^*}(\mathcal{C}_\lambda(A_1) \sqcup \mathcal{C}_\lambda(A_2))$ separates $\mathcal{P}(A_1) \sqcup \mathcal{P}(A_2)$ from $\mathcal{I}(A)$. Further $\mathcal{P}(A_1)$, $\mathcal{P}(A_2)$ are generalized standard starting components in Γ_A, and $\mathcal{I}(A)$ is an ending component.

A similar analysis to that in the first part of the proof shows that if $\mathcal{X}_M \subset \mathcal{P}(A_1) \sqcup \mathcal{C}_{i_1}(A_1)$ and $\mathcal{X}_N \subset \mathcal{C}_{i_2}(A_2) \sqcup \mathcal{I}(A_2)$ then $\Gamma_A = \mathcal{P}(A_2) \sqcup \mathcal{C}(A) \sqcup \mathcal{C}_{j_1}(A_1) \sqcup \mathcal{C}_{j_2}(A_2) \sqcup \bigsqcup_{\lambda\in K^*}(\mathcal{C}_\lambda(A_1) \sqcup \mathcal{C}_\lambda(A_2)) \sqcup \mathcal{I}(A_1)$, where $i_1, j_1 \in \{0,\infty\}$ are different, $i_2, j_2 \in \{0,\infty\}$ are different, and $\mathcal{C}(A)$ is a component which contains $\mathcal{C}_{i_1}(A_1) \sqcup \mathcal{P}(A_1) \sqcup \mathcal{C}_{i_2}(A_2) \sqcup \mathcal{I}(A_2)$ and the indecomposable A-modules which are not $A_1 \times A_2$-modules. Furthermore, $\mathcal{P}(A_2)$ is a starting component, $\mathcal{I}(A_1)$ is an ending component, and both are generalized standard.

Consequently, the assertion is shown in the case when r is not the target of any arrow in Q_A. If r is not the source of any arrow in Q_A then A is a one-point coextension of $A_1 \times A_2$ and a similar analysis yields the assertion.

5.5. Corollary. *Let A be a minimal 2-fundamental algebra. If $\overline{Q}_A$ is a quiver of type $\mathbb{A}_n$ which is not a path and there is no path v in $\overline{Q}_A$ such that $v \in I_A$, and Γ_A contains exactly one starting component, and exactly one ending component, then these components are generalized standard.*

Proof. This is a direct consequence of the proof of Proposition 5.4.

5.6. Lemma. *Let A be a minimal 2-fundamental algebra for which the quiver $\overline{Q}_A$ is of type $\mathbb{A}_n$, is not a path, and there is no path v in $\overline{Q}_A$ with $v \in I_A$.*

(1) *If Γ_A has exactly one starting component $\mathcal{P}(A)$ and two ending components $\mathcal{I}_1(A)$, $\mathcal{I}_2(A)$ then $\mathcal{P}(A)$ is not generalized standard.*
(2) *If Γ_A has exactly one ending component $\mathcal{I}(A)$ and two starting components $\mathcal{P}_1(A)$, $\mathcal{P}_2(A)$ then $\mathcal{I}(A)$ is not generalized standard.*

Proof. For (1) notice that $\overline{Q}_A$ has the form

$$z_1 \xrightarrow{\varrho_{1,1}} \cdots \xrightarrow{\varrho_{1,s_1}} \xleftarrow{\varrho_{2,s_2}} \cdots \xleftarrow{\varrho_{2,1}} \xrightarrow{\varrho_{3,1}} \cdots \xrightarrow{\varrho_{3,s_3}} \cdots \xleftarrow{\varrho_{t,s_t}} \cdots \xleftarrow{\varrho_{t,1}} z_2,$$

$t \geq 2$, from the proof of Proposition 5.4, because by Proposition 4.4 this is the only case when we obtain exactly one starting component $\mathcal{P}(A)$ and exactly two ending components $\mathcal{I}_1(A)$, $\mathcal{I}_2(A)$. It is easy to see that in this case the indecomposable projective A-modules belong to $\mathcal{P}(A)$. Furthermore, Q_A is of the form $Q_A \cup Q_{(\underline{p},l)} \cup \overline{Q}_A \cup Q'_{(\underline{p}',l')}$, where $Q_{(\underline{p},l)} \cap \overline{Q}_A = \{z_1\}$ and $Q'_{(\underline{p}',l')} \cap \overline{Q}_A = \{z_2\}$. Consider the simple A-module S_{z_1} if z_1 is not the end point of a maximal path in $Q_{(\underline{p},l)}$, and the indecomposable A-module $X(\alpha_{i,1}^{-1} \cdots \alpha_{i,p_i-p_{i-1}}^{-1})$ if z_1 is the end point of the maximal path $\alpha_{i,p_i-p_{i-1}} \cdots \alpha_{i,1}$ in $Q_{(\underline{p},l)}$ such that $\varrho_{1,s_1} \cdots \varrho_{1,1}\alpha_{i,p_i-p_{i-1}} \cdots \alpha_{i,1} \notin I_A$. Then by the Skowroński–Waschbüsch algorithm, $\tau(S_{z_1}) \cong N(\alpha_{i,j}\varrho_{1,1}^{-1})$, $\tau^2(S_{z_1}) \cong N(\alpha_{i,j+1}\alpha_{i,j}\varrho_{1,1}^{-1}\varrho_{1,2}^{-1}), \ldots$ Therefore for any positive integer n we obtain $\tau^n(S_{z_1}) \cong M(w_n)$, $\tau^{n+1}(S_{z_1}) \cong M(w_{n+1})$ and the length $l(w_{n+1})$ is greater than $l(w_n)$. Thus S_{z_1} does not belong to the τ-orbit of any indecomposable projective A-module. Further it is easy to see that for any indecomposable $M \cong M(v)$ such that there exists a chain of irreducible morphisms $M(v) \to \cdots \to S_{z_1}$, the module $M(v)$ is not projective. Thus there is no nonzero morphism $f : P \to S_{z_1}$ such that $f \notin \operatorname{rad}^\infty(P, S_{z_1})$ and P is a projective A-module. But there exists a nonzero homomorphism $g : P_{z_1} \to S_{z_1}$. Hence $g \in \operatorname{rad}^\infty(P_{z_1}, S_{z_1})$. Consequently, the component $\mathcal{P}(A)$ is not generalized standard.

In the case of the module $X(\alpha_{i,1}^{-1} \cdots \alpha_{i,p_i-p_{i-1}}^{-1})$ similar arguments show that $\mathcal{P}(A)$ is not generalized standard, which finishes the proof of (1).

Dual arguments yield (2).

5.7. Theorem. *Let A be a minimal 2-fundamental algebra.*

(1) *Γ_A contains a starting component and an ending component.*
(2) *If $\overline{Q}_A$ is a path or contains a subpath v which belongs to I_A then the starting components and the ending components in Γ_A are generalized standard.*
(3) *If $\overline{Q}_A$ is not a path and does not contain a subpath which belongs to I_A then Γ_A contains at most three components which are starting or ending and the following conditions are satisfied*:
   (3a) *If Γ_A contains exactly two components which are starting or ending then both are generalized standard and one of them is starting*, *and the other ending.*
   (3b) *If Γ_A contains one starting component $\mathcal{P}(A)$ and two ending components $\mathcal{I}_1(A)$, $\mathcal{I}_2(A)$ then $\mathcal{P}(A)$ is not generalized standard*, *but $\mathcal{I}_1(A)$, $\mathcal{I}_2(A)$ are generalized standard.*

(3c) *If* Γ_A *contains two starting components* $\mathcal{P}_1(A)$, $\mathcal{P}_2(A)$ *and one ending component* $\mathcal{I}(A)$ *then* $\mathcal{I}(A)$ *is not generalized standard and* $\mathcal{P}_1(A)$, $\mathcal{P}_2(A)$ *are generalized standard.*

Proof. (1) follows from Lemmas 5.2, 5.3 and Proposition 5.4; (2) can be deduced from Lemmas 5.2 and 5.3; (3) is a consequence of Proposition 5.4, in particular, (3a) can be deduced from Corollary 5.5, and (3b), (3c) are consequences of Lemma 5.6.

6. Multifundamental algebras

6.1. Let A be an n-fundamental algebra for some positive integer n. If $n \geq 2$ then we just say that A is *multifundamental* whenever n is not essential.

We say that a multifundamental algebra A contains a *lower minimal* 2*-fundamental subalgebra* A' if:

(i) $A' = KQ_{A'}/I_{A'}$ is a minimal 2-fundamental algebra.

(ii) The bound quiver $(Q_{A'}, I_{A'})$ has the property: $\overline{Q}_{A'}$ is of the form $\rightarrow \;—\cdots\; — \leftarrow$ and contains no subpath which belongs to $I_{A'}$.

(iii) $Q_{A'}$ is a full subquiver of Q_A and $KQ_{A'} \cap I_A = I_{A'}$.

(iv) Let $Q_{(\underline{p},l)}$, $Q'_{(\underline{p}',l')}$ be two different subquivers of $Q_{A'}$ of type $\widetilde{\mathbb{A}}_m$. If $Q''_{(\underline{p}'',l'')}$ is a subquiver of Q_A of type $\widetilde{\mathbb{A}}_m$ which is different from $Q'_{(\underline{p}',l')}$, $Q_{(\underline{p},l)}$ then no walk of the form $u_1w_1u_2\gamma^{-1}w_2\beta^{-1}u_3$ in (Q_A, I_A) satisfies the following conditions:

(a) u_1 is either a walk in $Q_{(\underline{p},l)}$ whose end point is not the source of any arrow in $Q_{(\underline{p},l)}$, or a trivial walk attached to a vertex in $Q_{(\underline{p},l)}$ which is not the source of any arrow in $Q_{(\underline{p},l)}$.

(b) w_1 is a walk in $(Q_{A'}, I_{A'})$ which contains every arrow of $\overline{Q}_{A'}$ or its formal inverse.

(c) u_2 is a walk in $Q'_{(\underline{p}',l')}$ passing through a vertex which is not the source of any arrow in $Q'_{(\underline{p}',l')}$.

(d) γ is an arrow in Q_A with source in $Q'_{(\underline{p}',l')}$ and target not in $Q'_{(\underline{p}',l')}$.

(e) w_2 is a walk in (Q_A, I_A); if w_2 is trivial then possibly $\gamma = \beta$.

(f) β is an arrow with target in $Q''_{(\underline{p}'',l'')}$ and source not in $Q''_{(\underline{p}'',l'')}$.

(g) u_3 is either a walk in $Q''_{(\underline{p}'',l'')}$ whose start point is not the target of any arrow in $Q''_{(\underline{p}'',l'')}$, or a trivial walk attached to a vertex in $Q''_{(\underline{p}'',l'')}$ which is not the target of any arrow in $Q''_{(\underline{p}'',l'')}$.

Dually we say that a multifundamental algebra A contains an *upper minimal* 2-*fundamental subalgebra* A' if:

(i^0) $A' = KQ_{A'}/I_{A'}$ is a minimal 2-fundamental algebra.

(ii^0) The bound quiver $(Q_{A'}, I_{A'})$ has the property: $\overline{Q}_{A'}$ is of the form $\leftarrow \ \text{—} \cdots \text{—} \rightarrow$ and contains no subpath belonging to $I_{A'}$.

(iii^0) $Q_{A'}$ is a full subquiver of Q_A and $KQ_{A'} \cap I_A = I_{A'}$.

(iv^0) Let $Q_{(\underline{p},l)}$, $Q'_{(\underline{p}',l')}$ be two different subquivers of $Q_{A'}$ of type $\widetilde{\mathbb{A}}_m$. If $Q''_{(\underline{p}'',l'')}$ is a subquiver of Q_A of type $\widetilde{\mathbb{A}}_m$ which is different from $Q_{(\underline{p},l)}$, $Q'_{(\underline{p}',l')}$ then no walk $u_1w_1u_2\gamma w_2\beta u_3$ in (Q_A, I_A) satisfies the following conditions:

(a^0) u_1 is either a walk in $Q_{(\underline{p},l)}$ whose end point is not the target of any arrow in $Q_{(\underline{p},l)}$, or a trivial walk attached to a vertex in $Q_{(\underline{p},l)}$ which is not the target of any arrow in $Q_{(\underline{p},l)}$.

(b^0) w_1 is a walk in $(Q_{A'}, I_{A'})$ which contains every arrow of $\overline{Q}_{A'}$ or its formal inverse.

(c^0) u_2 is a walk in $Q'_{(\underline{p}',l')}$ passing through a vertex which is not the target of any arrow in $Q'_{(\underline{p}',l')}$.

(d^0) γ is an arrow in Q_A with target in $Q'_{(\underline{p}',l')}$ and source not in $Q'_{(\underline{p}',l')}$.

(e^0) w_2 is a walk in (Q_A, I_A); if w_2 is trivial then possibly $\gamma = \beta$.

(f^0) β is an arrow with source in $Q''_{(\underline{p}'',l'')}$ and target not in $Q''_{(\underline{p}'',l'')}$.

(g^0) u_3 is either a walk in $Q''_{(\underline{p}'',l'')}$ whose start point is not the source of any arrow in $Q''_{(\underline{p}'',l'')}$, or a trivial walk attached to a vertex of $Q''_{(\underline{p}'',l'')}$ which is not the source of any arrow in $Q''_{(\underline{p}'',l'')}$.

6.2. An n-fundamental algebra A is defined to be *minimal* provided that the quiver $\overline{Q}_A$ is a tree such that $\overline{Q}_{j_1,j_2}$ is of type $\mathbb{A}_m$ or empty for any two $j_1, j_2 \in \{1, \ldots, n\}$, where $\overline{Q}_{j_1,j_2}$ is the full subquiver of $\overline{Q}_A$ formed by the vertices $x \in Q_A$ such that if there is a walk in $\overline{Q}_A$ from 0_{j_1} to x which does not pass through any 0_j, $j \in \{1, \ldots, n\}$, then there is a walk in $\overline{Q}_A$ from x to 0_{j_2} which does not pass through any 0_j, $j \in \{1, \ldots, n\}$.

6.3. Lemma. *Let n be a fixed positive integer. For any n-fundamental algebra A there exists a sequence of n-fundamental algebras $A_0, A_1, \ldots, A_t$, $t \geq 0$, such that*:

(1) *A_0 is a minimal n-fundamental algebra.*

(2) *For each $i = 1, \ldots, t$ the algebra A_i is a one-point extension or coextension of A_{i-1} by a uniserial A_{i-1}-module.*

(3) *$A_t \cong A$.*

Proof. If $n = 1$ then the assertion holds by Lemma 4.2. Now assume that $n \geq 2$. Then $A \cong KQ_A/I_A$ and there are exactly n disjoint subquivers Q'_j of type $\widetilde{\mathbb{A}}_m$ in Q_A such that $KQ'_j \cap I_A = 0$. If A is a minimal n-fundamental algebra then $t = 0$ and $A_0 = A$.

If A is not minimal then there are $j_1, j_2 \in \{1, \dots, n\}$ such that $\overline{Q}_{j_1,j_2}$ is not of type $\mathbb{A}_m$. On the other hand, $\overline{Q}_{j_1,j_2}$ is a tree. Hence there is a vertex $x \in \overline{Q}_{j_1,j_2}$ such that $x \neq 0_{j_1}, 0_{j_2}$ and either x is not the target of any arrow in $\overline{Q}_{j_1,j_2}$ or x is not the source of any arrow in $\overline{Q}_{j_1,j_2}$. Since $x \neq 0_{j_1}, 0_{j_2}$, it is neither the target nor the source of any arrow in Q_A. Since $\overline{Q}_A$ is a tree, we can choose x in such a way that either it is not the target of any arrow in $\overline{Q}_A$ and the source of exactly one arrow α, or it is not the source of any arrow in $\overline{Q}_A$ and the target of exactly one arrow β. Then let Q' be the quiver obtained from Q_A by removing x and either α or β. Let $I' = KQ' \cap I_A$. It is clear that $A' = KQ'/I'$ is an n-fundamental algebra by its construction. If we have removed α then

$$A \cong \begin{pmatrix} K & \operatorname{rad}(P_x) \\ 0 & A' \end{pmatrix}$$

and $\operatorname{rad}(P_x)$ is a uniserial A'-module, while if we have removed β then

$$A \cong \begin{pmatrix} A' & 0 \\ E_x/\operatorname{soc}(E_x) & K \end{pmatrix}$$

and $E_x/\operatorname{soc}(E_x)$ is a uniserial A'-module.

Proceeding similarly with A', after finitely many steps we obtain a minimal n-fundamental algebra A_0.

6.4. Lemma. *Let A be a multifundamental algebra whose Auslander–Reten quiver Γ_A contains a starting component $\mathcal{P}(A)$ which is not generalized standard.*

(1) *If a multifundamental algebra A_1 is a one-point extension of A by a uniserial A-module then there exists a starting component $\mathcal{P}(A_1)$ in Γ_{A_1} which is not generalized standard.*
(2) *If a multifundamental algebra A_1 is a one-point coextension of A by a uniserial A-module then there exists a starting component $\mathcal{P}(A_1)$ in Γ_{A_1} which is not generalized standard.*

Proof. Let A_1 be a multifundamental algebra which is a one-point extension of A by a uniserial A-module M. Then Q_{A_1} is obtained from Q_A by adding one vertex z and one arrow ε from z to $x \in Q_A$. Moreover, $M \cong M(\varepsilon_m \cdots \varepsilon_1)$ for some path $\varepsilon_m \cdots \varepsilon_1$ starting at x. This path may be trivial.

If $\mathcal{X}_M \cap \mathcal{P}(A) = \emptyset$ then there is no walk w in (Q_A, I_A) starting at x and such that $X(w) \in \mathcal{P}(A)$. Then for any indecomposable A_1-module

$X(u\varepsilon)$ there is an irreducible morphism $X(u) \to X(u\varepsilon)$ by the Skowroński–Waschbüsch algorithm. If $X(u\varepsilon)$ belongs to the component in Γ_{A_1} which contains the A-modules from $\mathcal{P}(A)$ then so does $X(u)$. Thus $X(u) \in \mathcal{X}_M \cap \mathcal{P}(A)$ contrary to our assumption. Therefore $\mathcal{P}(A)$ is also a component in Γ_{A_1}, which will be denoted by $\mathcal{P}(A_1)$. A routine verification shows that $\mathcal{P}(A_1)$ is a starting component which is not generalized standard.

If $\mathcal{X}_M \cap \mathcal{P}(A) \neq \emptyset$ then there exists a walk w in (Q_A, I_A) starting at x and such that $X(w) \in \mathcal{P}(A)$. Every such walk has one of the following forms: $w = w_j\varepsilon_j \cdots \varepsilon_1$, where $w_j = \overline{w}_j\delta^{-1}$ or w_j is a trivial walk, $w = w_0 e_x$, where $w_0 = \overline{w}_0\delta^{-1}$ or w_0 is a trivial walk.

Furthermore, by the properties of string algebras, the indecomposable A_1-modules which are not A-modules are of the form $X(w_j\varepsilon_j \cdots \varepsilon_1\varepsilon)$, $X(w_0\varepsilon)$ or S_z. Then the Skowroński–Waschbüsch algorithm yields the chain $X(w_j\varepsilon_j \cdots \varepsilon_1) \to X(w_j\varepsilon_j \cdots \varepsilon_1\varepsilon) \to X(\overline{w}_j)$ of irreducible morphisms in $\mathrm{mod}(A_1)$. Hence there exists a component $\mathcal{P}(A_1)$ which contains the A-modules from $\mathcal{P}(A)$. A routine verification shows that $\mathcal{P}(A_1)$ is a starting component which is not generalized standard. Thus condition (1) is proved.

Condition (2) can be proved dually; we omit the details.

6.5. Lemma. *Let A be a multifundamental algebra whose Auslander–Reiten quiver Γ_A contains an ending component $\mathcal{I}(A)$ which is not generalized standard.*

(1) *If a multifundamental algebra A_1 is a one-point extension of A by a uniserial A-module then there exists an ending component $\mathcal{I}(A_1)$ in Γ_{A_1} which is not generalized standard.*
(2) *If a multifundamental algebra A_1 is a one-point coextension of A by a uniserial A-module then there exists an ending component $\mathcal{I}(A_1)$ in Γ_{A_1} which is not generalized standard.*

Proof. Apply dual arguments to those in the proof of Lemma 6.4.

6.6. Proposition. *Let A be a minimal multifundamental algebra which contains a lower minimal 2-fundamental subalgebra A'. Then there exists a starting component $\mathcal{P}(A)$ in Γ_A which is not generalized standard.*

Proof. By assumption A is n-fundamental for some $n \geq 2$. We argue by induction on n.

If $n = 2$ then $A' = A$. By the definition of a lower minimal 2-fundamental subalgebra and by Theorem 5.7, Γ_A contains exactly one starting component $\mathcal{P}(A)$ which is not generalized standard.

Assume that the assertion holds for any integer n with $2 \leq n \leq n_0$ and a starting component $\mathcal{P}(A)$ containing the A'-modules from $\mathcal{P}(A')$ is as required. Moreover, assume that if $X(w) \in \mathcal{P}(A)$ then there is a walk w'

such that $w'w = w_0\kappa u_1 w_1 u_2 \gamma^{-1} w_2$, where u_1 is a walk in Q'_1 and u_2 is a walk in Q'_2.

Now let A be a minimal (n_0+1)-fundamental algebra which contains a lower minimal 2-fundamental subalgebra A'. Then Q_A contains n_0+1 pairwise disjoint subquivers Q'_j, $j = 1, \ldots, n_0+1$, of type $\widetilde{\mathbb{A}}_m$ such that $KQ'_j \cap I_A = 0$. Two of them, say Q'_1, Q'_2, are subquivers of $Q_{A'}$. Since A is multifundamental, $\overline{Q}_A$ is a tree. Since A is minimal, there exists a vertex 0_{j_0} in $\overline{Q}_A$ which is different from 0_1, 0_2 and either is not the source of any arrow in $\overline{Q}_A$, or is not the target of any arrow in $\overline{Q}_A$.

Consider the case when 0_{j_0} is not the target of any arrow in $\overline{Q}_A$. Then 0_{j_0} is the source of an arrow β. Let A_1 be a subalgebra of A such that Q_{A_1} is obtained from Q_A by removing the vertices in Q'_{j_0} except the source x of β (treated as an arrow in Q_A), and by removing the arrows from Q'_{j_0}. We have $I_{A_1} = KQ_{A_1} \cap I_A$ and $A_1 = KQ_{A_1}/I_{A_1}$. It is obvious that A_1 contains A' as a lower minimal 2-fundamental subalgebra. We shall show that the component in Γ_A which contains the A'-modules from $\mathcal{P}(A')$ is a starting component. By the inductive assumption the component $\mathcal{P}(A_1)$ of Γ_{A_1} containing the A'-modules from $\mathcal{P}(A')$ is a starting component in Γ_{A_1} which is not generalized standard.

Suppose $X(w\beta) \notin \mathcal{P}(A_1)$ for any walk $w\beta$ in (Q_{A_1}, I_{A_1}). Then by the Skowroński–Waschbüsch algorithm $\mathcal{P}(A_1)$ is a component in Γ_A. Moreover, it is obviously a starting component in Γ_A. Hence it is as required.

Suppose now that there is a walk $w\beta$ in (Q_{A_1}, I_{A_1}) such that $X(w\beta) \in \mathcal{P}(A_1)$. Since $\mathcal{P}(A_1)$ is a starting component in Γ_{A_1}, we have $X(w) \in \mathcal{P}(A_1)$. But we have an irreducible morphism $X(w) \to X(w\beta\alpha_{i,j}^{-1} \cdots \alpha_{i,p_i-p_{i-1}}^{-1})$ in $\mathrm{mod}(A)$, where $\alpha_{i,p_i-p_{i-1}} \cdots \alpha_{i,j}$ is a path in Q'_{j_0} which starts at x. Let $\beta_r \cdots \beta_1\beta$ be a maximal path in (Q_{A_1}, I_{A_1}). Then $\mathrm{Hom}_{A_1}(X(\beta_r \cdots \beta_1\beta), X(w\beta)) \neq 0$, hence $X(\beta_r \cdots \beta_1\beta) \in \mathcal{P}(A_1)$, because $\mathcal{P}(A_1)$ is a starting component in Γ_{A_1}. Similarly $X(\beta_r \cdots \beta_1) \in \mathcal{P}(A_1)$. But there are irreducible morphisms $X(\beta_r \cdots \beta_1) \to P_x$ and $X(\alpha_{i,j+1}^{-1} \cdots \alpha_{i,p_i-p_{i-1}}^{-1}) \to P_x$ in $\mathrm{mod}(A)$. Therefore the A-modules P_x, $X(\alpha_{i,j+1}^{-1} \cdots \alpha_{i,p_i-p_{i-1}}^{-1})$ belong to the same component $\mathcal{P}(A)$. Then by the Skowroński–Waschbüsch algorithm the indecomposable projective KQ'_{j_0}-modules belong to $\mathcal{P}(A)$, and $\mathcal{P}(A)$ contains the A_1-modules from $\mathcal{P}(A_1)$. Consequently, $\mathcal{P}(A)$ is a starting component in Γ_A. Since $\mathcal{P}(A_1)$ is not generalized standard, neither is $\mathcal{P}(A)$. Moreover, if $X(w\beta) \in \mathcal{P}(A)$ then there is a walk w' such that $w'w\beta = w_0\kappa u_1 w_1 u_2 \gamma^{-1} w_2$.

Now consider the case when 0_{j_0} is not the source of any arrow in $\overline{Q}_A$. Then 0_{j_0} is the target of an arrow β. Again let A_1 be a subalgebra of A such that Q_{A_1} is obtained from Q_A by removing the vertices of Q'_{j_0} except the target x of β (treated as an arrow in Q_A), and by removing the arrows from Q'_{j_0}. We have $I_{A_1} = KQ_{A_1} \cap I_A$ and $A_1 = KQ_{A_1}/I_{A_1}$. Then A_1 contains A' as a lower minimal 2-fundamental subalgebra. We shall show that the component in Γ_A which contains the A'-modules from $\mathcal{P}(A')$ is a starting component. By the inductive assumption the component $\mathcal{P}(A_1)$ in Γ_{A_1} which contains the A'-modules from $\mathcal{P}(A')$ is a starting component which is not generalized standard.

If $X(w\beta^{-1}) \notin \mathcal{P}(A_1)$ for any walk $w\beta^{-1}$ in (Q_{A_1}, I_{A_1}) then $\mathcal{P}(A_1)$ is a component in Γ_A by the Skowroński–Waschbüsch algorithm. Furthermore, it is a starting component in Γ_A which is not generalized standard. It is also clear that for every $X(w) \in \mathcal{P}(A_1)$ there is a walk w' such that $w'w = w_0\kappa u_1 w_1 u_2 \gamma^{-1} w_2$.

Now suppose that there exists a walk $w\beta^{-1}$ in (Q_{A_1}, I_{A_1}) such that $X(w\beta^{-1}) \in \mathcal{P}(A_1)$. Then $\mathrm{Hom}_{A_1}(S_x, X(w\beta^{-1})) \neq 0$, hence $S_x \in \mathcal{P}(A_1)$, because $\mathcal{P}(A_1)$ is a starting component. Then by the inductive assumption there is a walk $u_1 w_1 u_2 \gamma^{-1} w_2 \beta^{-1}$ such that u_1 is a walk in Q'_1 and u_2 is a walk in Q'_2. It is easily seen that we can choose u_1, u_2 in such a way that they satisfy the conditions of the definition of a lower minimal 2-fundamental subalgebra. Then we have in (Q_A, I_A) the walk $u_1 w_1 u_2 \gamma^{-1} w_2 \beta^{-1} u_3$, where u_3 is a walk in Q'_{j_0} starting at a vertex which is not the target of any arrow in Q'_{j_0}. Existence of the above walk contradicts the assumption that A' is a lower minimal 2-fundamental subalgebra of A. This completes the inductive proof of the proposition.

6.7. Proposition. *Let A be a minimal multifundamental algebra which contains an upper minimal 2-fundamental subalgebra A'. Then there exists an ending component $\mathcal{I}(A)$ in Γ_A which is not generalized standard.*

Proof. Apply Lemma 6.5 and dual arguments to those in the proof of Proposition 6.6.

6.8. Theorem. *Let A be a multifundamental algebra.*

(1) *If A contains an upper minimal 2-fundamental subalgebra A' then there exists an ending component $\mathcal{I}(A)$ in Γ_A which is not generalized standard.*
(2) *If A contains a lower minimal 2-fundamental subalgebra A' then there exists a starting component $\mathcal{P}(A)$ in Γ_A which is not generalized standard.*

Proof. By Lemma 6.3 there exists a sequence of n-fundamental algebras $A_0, \dots, A_t$, $t \geq 0$, such that A_0 is a minimal n-fundamental algebra and $A_t \cong A$. Moreover, for each $0 \leq i < t$, A_{i+1} is obtained from A_i by a one-point extension or coextension by a uniserial A_i-module.

We argue by induction on t. If A' is an upper (resp. lower) minimal 2-fundamental subalgebra in A_t then it is an upper (resp. lower) minimal 2-fundamental subalgebra in A_i, $0 \leq i \leq t$, by definition.

Furthermore, Propositions 6.6 and 6.7 yield the assertion for $t = 0$. The inductive step is a consequence of Lemmas 6.4 and 6.5. This finishes the proof.

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Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: zypo@mat.uni.torun.pl

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