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OPERATOR ENTROPY INEQUALITIES

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Abstract. We investigate a notion of relative operator entropy, which develops the theory started by J. I. Fujii and E. Kamei [Math. Japonica 34 (1989), 341–348]. For two finite sequences $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ of positive operators acting on a Hilbert space, a real number q and an operator monotone function f we extend the concept of entropy by setting

$$S_q^f(\mathbf{A} \,|\, \mathbf{B}) := \sum_{j=1}^{n} A_j^{1/2}(A_j^{-1/2}B_jA_j^{-1/2})^q f(A_j^{-1/2}B_jA_j^{-1/2})A_j^{1/2},$$

and then give upper and lower bounds for $S_q^f(\mathbf{A} \,|\, \mathbf{B})$ as an extension of an inequality due to T. Furuta [Linear Algebra Appl. 381 (2004), 219–235] under certain conditions. As an application, some inequalities concerning the classical Shannon entropy are deduced.

1. Introduction and preliminaries. Throughout the paper, let $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ and I is the identity operator. When $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the full matrix algebra $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ matrices with complex entries and denote its identity by I_n. A self-adjoint operator $A \in \mathbb{B}(\mathscr{H})$ is called *positive*, written $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$. An operator A is said to be *strictly positive* (denoted by $A > 0$) if it is positive and invertible. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we write $A \leq B$ if $B - A \geq 0$.

Let f be a continuous real valued function defined on an interval J. The function f is called *operator decreasing* if $B \leq A$ implies $f(A) \leq f(B)$ for all $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J. The function f is said to be *operator concave* on J if

$$\lambda f(A) + (1-\lambda)f(B) \leq f(\lambda A + (1-\lambda)B)$$

for all self-adjoint $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J and all $\lambda \in [0,1]$.

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In 1850 Clausius [Ann. Physik (2) 79 (1850), 368–397, 500–524] introduced the notion of entropy in thermodynamics. Since then several extensions and reformulations have been developed in various disciplines (cf. [ME, LR, L, NU]). The so-called entropy inequalities have been investigated by several authors (see [BLP, BS, F2] and references therein).

A relative operator entropy of strictly positive operators A, B was introduced in noncommutative information theory by Fujii and Kamei [FK] by

$$S(A \,|\, B) = A^{1/2} \log(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

When A is positive, one may set $S(A \,|\, B) := \lim_{\epsilon \to +0} S(A + \epsilon I \,|\, B)$ if the limit exists in the strong operator topology. In the same paper, it is shown that $S(A \,|\, B) \le 0$ if $A \ge B$. There is an analogous notion called the perspective function (see [E, CK]) If $f : [0, \infty) \to \mathbb{R}$ is an operator convex function, then the *perspective function* g associated to f is defined by

$$g(B, A) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

for any self-adjoint operator B and any strictly positive operator A.

One can consider a more general case. Let $\widetilde{B} = (B_1, \ldots, B_n)$ and $\widetilde{A} = (A_1, \ldots, A_n)$ be n-tuples of self-adjoint and strictly positive operators, respectively. Then the noncommutative f-divergence functional Θ is defined by

$$\Theta(\widetilde{B}, \widetilde{A}) = \sum_{i=1}^{n} A_i^{1/2} f(A_i^{-1/2} B_i A_i^{-1/2}) A_i^{1/2}.$$

Next, recall that $X \natural_q Y$ is defined by $X^{1/2}(X^{-1/2} Y X^{-1/2})^q X^{1/2}$ for any real q and any strictly positive operators X and Y. For $p \in [0, 1]$, the operator $X \natural_p Y$ coincides with the well-known p-power mean of X, Y.

Furuta [F1] defined a parametric extension of the operator entropy by

$$S_p(A \,|\, B) = A^{1/2}(A^{-1/2} B A^{-1/2})^p \log(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where $p \in [0, 1]$ and A, B are strictly positive operators on a Hilbert space $\mathscr{H}$, and proved some operator entropy inequalities: if $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ are two sequences of strictly positive operators on a Hilbert space $\mathscr{H}$ such that $\sum_{j=1}^n A_j \natural_p B_j \le I$, then

$$\log\Big[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + t_0\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big)\Big] - (\log t_0)\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big) \tag{1.1}$$

$$\geq \sum_{j=1}^{n} S_p(A_j \,|\, B_j)$$

$$\geq -\log\Big[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big)\Big] + (\log t_0)\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big)$$

for any fixed real number $t_0 > 0$.

The object of this paper is to state an operator entropy inequality parallel to the main result of [F1] and refine some known operator entropy inequalities.

2. Operator entropy inequality. The following notion is basic in our work.

Definition 2.1. Assume that $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ are finite sequences of strictly positive operators on a Hilbert space $\mathscr{H}$. For $q \in \mathbb{R}$ and an operator monotone function $f : (0, \infty) \to [0, \infty)$ the *generalized operator Shannon entropy* is defined by

$$S_q^f(\mathbf{A} \,|\, \mathbf{B}) := \sum_{j=1}^{n} S_q^f(A_j \,|\, B_j),$$

where

$$S_q^f(A_j \,|\, B_j) = A_j^{1/2}(A_j^{-1/2} B_j A_j^{-1/2})^q f(A_j^{-1/2} B_j A_j^{-1/2}) A_j^{1/2}.$$

We recall that for $q = 0$, $f(t) = \log t$ and $A, B > 0$, we get the relative operator entropy $S_0^f(A \,|\, B) = A^{1/2} \log(A^{-1/2} B A^{-1/2}) A^{1/2} = S(A \,|\, B)$. It is interesting to point out that $S_q(A \,|\, B) = -S_{1-q}(B \,|\, A)$ for any real q, in particular, $S_1(A \,|\, B) = -S(B \,|\, A)$. In fact, since $Xf(X^*X) = f(XX^*)X$ for every $X \in \mathbb{B}(\mathscr{H})$ and every continuous function f on $[0, \|X\|^2]$, considering $X = B^{1/2} A^{-1/2}$ and $f(t) = \log t$ we get

$$\begin{aligned}
S_q(A \,|\, B) &= A^{1/2}(A^{-1/2} B A^{-1/2})^q \log(A^{-1/2} B A^{-1/2}) A^{1/2} \\
&= B^{1/2} B^{-1/2} A^{1/2}(A^{-1/2} B A^{-1/2})^q \log(A^{-1/2} B A^{-1/2}) A^{1/2} B^{-1/2} B^{1/2} \\
&= B^{1/2} X^{*-1}(X^*X)^q \log(X^*X) X^{-1} B^{1/2} \\
&= B^{1/2} X^{-1*}(X^{-1} X^{-1*})^{-q} \log(X^*X) X^{-1} B^{1/2} \\
&= B^{1/2}(X^{-1*} X^{-1})^{1-q}(X^{-1*} X^{-1})^{-1} X^{-1*} \log(X^*X) X^{-1} B^{1/2} \\
&= B^{1/2}(X^{-1*} X^{-1})^{1-q} X \log(X^*X) X^{-1} B^{1/2} \\
&= B^{1/2}(X^{-1*} X^{-1})^{1-q} \log(XX^*) X X^{-1} B^{1/2} \\
&= -B^{1/2}(X^{-1*} X^{-1})^{1-q} \log(X^{-1*} X^{-1}) B^{1/2} \\
&= -B^{1/2}(X^{*-1} X^{-1})^{1-q} \log(X^{*-1} X^{-1}) B^{1/2} = -S_{1-q}(B \,|\, A).
\end{aligned}$$

We need the following useful lemma.

Lemma 2.2 ([F1, Proposition 3.1]). *If f is a continuous real function on an interval J, then the following conditions are equivalent:*

(i) *f is operator concave.*
(ii) *$f(C^*XC + t_0(I - C^*C)) \geq C^*f(X)C + f(t_0)(I - C^*C))$ for any operator C with $\|C\| \leq 1$ and any self-adjoint operator X with $\mathrm{sp}(X) \subseteq J$, and for any fixed $t_0 \in J$.*
(iii) *$f(\sum_{j=1}^n C_j^*X_jC_j + t_0(I - \sum_{j=1}^n C_j^*C_j)) \geq \sum_{j=1}^n C_j^*f(X_j)C_j + f(t_0)(I - \sum_{j=1}^n C_j^*C_j))$ for any operators C_j with $\sum_{j=1}^n C_j^*C_j \leq I$ and self-adjoint operators X_j with $\mathrm{sp}(X_j) \subseteq J$ for $j = 1, \ldots, n$, and for any fixed $t_0 \in J$.*

For other equivalent conditions the reader may consult [FMPS, M] and references therein. Using an idea of [F1] we prove the following result.

Theorem 2.3. *Assume that f, $\mathbf{A}$ and $\mathbf{B}$ are as in Definition 2.1. Let $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I$ and let f be operator concave. Then*

$$f\Big[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0\Big(I - \sum_{j=1}^n A_j \natural_p B_j\Big)\Big] - f(t_0)\Big(I - \sum_{j=1}^n A_j \natural_p B_j\Big) \geq S_p^f(\mathbf{A} \,|\, \mathbf{B})$$

for all $p \in [0,1]$ and for any fixed $t_0 > 0$, and

$$-f\Big[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + t_0\Big(I - \sum_{j=1}^n A_j \natural_p B_j\Big)\Big] + f(t_0)\Big(I - \sum_{j=1}^n A_j \natural_p B_j\Big) \leq S_p^f(\mathbf{A} \,|\, \mathbf{B})$$

for all $p \in [2,3]$ and for any fixed $t_0 > 0$.

Proof. Since $\sum_{j=1}^n A_j \natural_q B_j \leq (\sum_{j=1}^n A_j) \natural_q (\sum_{j=1}^n B_j)$ (see [FMPS, Theorem 5.7]) for every $q \in [0,1]$, and $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I$, we have

$$\sum_{j=1}^n A_j \natural_p B_j \leq I.$$

Fix a positive real number t_0. Since f is operator concave, we get

$$\begin{aligned} &f\Big[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0\Big(I - \sum_{j=1}^n A_j \natural_p B_j\Big)\Big] \\ &= f\Big[\sum_{j=1}^n ((A_j^{-1/2}B_jA_j^{-1/2})^{p/2}A_j^{1/2})^*(A_j^{-1/2}B_jA_j^{-1/2})((A_j^{-1/2}B_jA_j^{-1/2})^{p/2}A_j^{1/2}) \\ &\qquad + t_0\Big(I - \sum_{j=1}^n A_j \natural_p B_j\Big)\Big] \end{aligned}$$

$$\geq \sum_{j=1}^{n} A_j^{1/2}(A_j^{-1/2}B_jA_j^{-1/2})^{p/2} f(A_j^{-1/2}B_jA_j^{-1/2})(A_j^{-1/2}B_jA_j^{-1/2})^{p/2}A_j^{1/2}$$

$$+ f(t_0)\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big) \quad \text{(by Lemma 2.2(iii))}$$

$$= \sum_{j=1}^{n} A_j^{1/2}(A_j^{-1/2}B_jA_j^{-1/2})^{p} f(A_j^{-1/2}B_jA_j^{-1/2})A_j^{1/2} + f(t_0)\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big)$$

$$= \sum_{j=1}^{n} S_p^f(A_j \,|\, B_j) + f(t_0)\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big),$$

whence

$$f\Big[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + t_0\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big)\Big]$$

$$\geq \sum_{j=1}^{n} S_p^f(A_j \,|\, B_j) + f(t_0)\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big).$$

Following a similar argument, we obtain

$$f\Big[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big)\Big]$$

$$\geq \sum_{j=1}^{n} S_{p-2}^f(A_j \,|\, B_j) + f(t_0)\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big).$$

Thus

$$-f\Big[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big)\Big] + f(t_0)\Big(I - \sum_{j=1}^{n} A_j \natural_p B_j\Big)$$

$$\leq -S_{p-2}^f(\mathbf{A} \,|\, \mathbf{B}).$$

Since f is a continuous nonnegative function, $X^q f(X) \geq 0$ for every $X \geq 0$ and $q \in \mathbb{R}$. Hence

$$(A_j^{-1/2}B_jA_j^{-1/2})^q f(A_j^{-1/2}B_jA_j^{-1/2}) \geq 0.$$

Consequently, $S_q^f(A_j \,|\, B_j) \geq 0$. Thus

$$S_p^f(A_j \,|\, B_j) + S_{p-2}^f(A_j \,|\, B_j) \geq 0 \quad (j = 1, \ldots, n),$$

whence $-S_{p-2}^f(\mathbf{A} \,|\, \mathbf{B}) \leq S_p^f(\mathbf{A} \,|\, \mathbf{B})$, which yields the required result. ∎

REMARK 2.4. By taking $f(t) = \log t$ in Theorem 2.3, we get (1.1).

COROLLARY 2.5. *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *and* $\mathbf{B} = (B_1, \ldots, B_n)$ *be two sequences of strictly positive operators on a Hilbert space* $\mathscr{H}$ *such that*

$\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I$. *If* $f : (0,\infty) \to [0,\infty)$ *is a function which is both operator monotone and operator concave, then*

(i) $f(\sum_{j=1}^n B_j A_j^{-1} B_j) \geq S_1^f(\mathbf{A} \,|\, \mathbf{B})$,

(ii) $f(I) \geq S_0^f(\mathbf{A} \,|\, \mathbf{B})$.

Proof. (i) Setting $p = 1$ in Theorem 2.3 and applying $\sum_{j=1}^n A_j \natural_1 B_j = \sum_{j=1}^n B_j = I$, we obtain

$$f\Big(\sum_{j=1}^n B_j A_j^{-1} B_j\Big) = f\Big(\sum_{j=1}^n A_j \natural_2 B_j\Big) \geq S_1^f(\mathbf{A} \,|\, \mathbf{B}).$$

(ii) Putting $p = 0$ in Theorem 2.3 and using $\sum_{j=1}^n A_j \natural_0 B_j = \sum_{j=1}^n A_j = I$, we get

$$f(I) = f\Big(\sum_{j=1}^n B_j\Big) = f\Big(\sum_{j=1}^n A_j \natural_1 B_j\Big) \geq S_0^f(\mathbf{A} \,|\, \mathbf{B}). \ \blacksquare$$

Next we extend the operator entropy to n strictly positive operators $A_1, \ldots, A_n \in \mathbb{B}(\mathscr{H})$ and refine the operator entropy inequality.

COROLLARY 2.6. *Let* $A_1, \ldots, A_n \in \mathbb{B}(\mathscr{H})$ *be a sequence of strictly positive operators on a Hilbert space* $\mathscr{H}$ *such that* $\sum_{j=1}^n A_j = I$. *Then*

$$\log\Big(\sum_{j=1}^n A_j^{-1}\Big) \geq (\log n)I - \frac{1}{n}\sum_{j=1}^n \log A_j. \tag{2.1}$$

Proof. Taking $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = \big(\frac{1}{n}I, \ldots, \frac{1}{n}I\big)$ and $f(t) = \log t$ in Corollary 2.5 (i), we get

$$\begin{aligned} -2(\log n)I + \log\Big(\sum_{j=1}^n A_j^{-1}\Big) &= \log\Big(\frac{1}{n^2}\sum_{j=1}^n A_j^{-1}\Big) \geq S_1^{\log}(\mathbf{A} \,|\, \mathbf{B}) \\ &= \sum_{j=1}^n \frac{1}{n} A_j^{-1/2} \log\Big(\frac{1}{n} A_j^{-1}\Big) A_j^{1/2} = \sum_{j=1}^n \frac{1}{n} \log\Big(\frac{1}{n} A_j^{-1}\Big) \\ &= -\sum_{j=1}^n \frac{1}{n}((\log n)I + \log A_j) = -(\log n)I - \frac{1}{n}\sum_{j=1}^n \log A_j, \end{aligned}$$

which yields (2.1). $\blacksquare$

COROLLARY 2.7 (Operator entropy inequality). *Assume that* $A_1, \ldots, A_n \in \mathbb{B}(\mathscr{H})$ *are positive invertible operators satisfying* $\sum_{j=1}^n A_j = I$. *Then*

$$-\sum_{j=1}^n A_j \log A_j \leq (\log n)I.$$

Proof. Letting $\mathbf{A} = (A_1, \ldots, A_n)$, $\mathbf{B} = \left(\frac{1}{n}I, \ldots, \frac{1}{n}I\right)$ and $f(t) = \log t$ in Corollary 2.5(ii), we get

$$\begin{aligned} 0 = \log I &\geq S_0^{\log}(\mathbf{A} \,|\, \mathbf{B}) \\ &= \sum_{j=1}^{n} A_j^{1/2} \log\left(\frac{1}{n}A_j^{-1}\right) A_j^{1/2} = \sum_{j=1}^{n} A_j^{1/2}(-(\log n)I - \log A_j)A_j^{1/2} \\ &= -(\log n)\sum_{j=1}^{n} A_j - \sum_{j=1}^{n} A_j^{1/2}(\log A_j)A_j^{1/2}. \ \blacksquare \end{aligned}$$

REMARK 2.8. Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be n-tuples of positive numbers such that $\sum_{j=1}^n a_j = \sum_{j=1}^n b_j = 1$. Put $A_i = [a_i]_{1\times 1} \in \mathcal{M}_1(\mathbb{C})$ and $B_i = [b_i]_{1\times 1} \in \mathcal{M}_1(\mathbb{C})$. It follows from Corollary 2.5(ii) that $0 \geq \sum_{j=1}^n a_j \log(b_j/a_j)$, which is an entropy inequality related to the Kullback–Leibler relative entropy or information divergence $S(p,q) = \sum_{j=1}^n p_j \log(p_j/q_j)$ with the convention $x \log x = 0$ if $x = 0$, and $x \log y = +\infty$ if $y = 0$ and $x \neq 0$ (cf. [KL]).

THEOREM 2.9. *Let $p \in [0,1]$ and let A, B be strictly positive operators on a Hilbert space $\mathscr{H}$ such that $A \natural_{p-2} B \leq I$ and $B^2 \leq A^2$. If $f : (0,\infty) \to [0,\infty)$ is both operator monotone and operator concave, then*

$$\begin{aligned} &f(A \natural_{p+1} B + t_0(I - A \natural_p B)) - f(t_0)(I - A \natural_p B) \\ &\quad \geq S_p^f(A \,|\, B) \geq -f(A \natural_{p-1} B + t_0(I - A \natural_p B)) + f(t_0)(I - A \natural_p B) \end{aligned}$$

for any fixed real number $t_0 > 0$.

Proof. It follows from $A \natural_{p-2} B \leq I$ that

$$\begin{aligned} A^{1/2}(A^{-1/2}BA^{-1/2})^{p-2}A^{1/2} &\leq I, \\ (A^{-1/2}BA^{-1/2})^{p-2} &\leq A^{-1}, \\ (A^{-1/2}BA^{-1/2})^{p} &\leq (A^{-1/2}BA^{-1/2})A^{-1}(A^{-1/2}BA^{-1/2}), \\ A^{1/2}(A^{-1/2}BA^{-1/2})^{p}A^{1/2} &\leq BA^{-2}B. \end{aligned}$$

Since $B^2 \leq A^2$ and the map $t \mapsto -1/t$ is operator monotone, we have $A^{1/2}(A^{-1/2}BA^{-1/2})^p A^{1/2} \leq I$, so that $A \natural_p B \leq I$. Now the same reasoning as in the proof of Theorem 2.3 (with $n = 1$ and using Lemma 2.2(ii)) yields the desired inequalities. $\blacksquare$

Recall that a map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$, where $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces, is called *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$, and it is said to be *normalized* if it preserves the identity. The paper [MMM, Lemma 5.2] includes the following refinement of the Jensen inequality for Hilbert space operators: Let $\mu = (\mu_1, \ldots, \mu_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be probability vectors. By a (discrete) *weight function* (with respect to μ and λ) we mean a mapping

$\omega : \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\} \to [0,\infty)$ such that $\sum_{i=1}^{m} \omega(i,j)\mu_i = 1$ $(j = 1,\dots,n)$ and $\sum_{j=1}^{n} \omega(i,j)\lambda_j = 1$ $(i = 1,\dots,m)$. If f is a real-valued operator concave function on an interval J, $A_1,\dots,A_n$ are self-adjoint operators with spectra in J and $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is a normalized positive map, then

$$f\Big(\sum_{j=1}^{n} \lambda_j \Phi(A_j)\Big) \geq \sum_{i=1}^{m} \mu_i f\Big(\sum_{j=1}^{n} \omega(i,j)\lambda_j\Phi(A_j)\Big) \geq \sum_{j=1}^{n} \lambda_j \Phi(f(A_j)). \tag{2.2}$$

A matrix $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{C})$ is said to be *doubly stochastic* if $a_{ij} \geq 0$ $(i,j = 1,\dots,n)$ and $\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1$. Now we introduce a refinement of the operator Jensen inequality.

THEOREM 2.10. *Suppose that f is a real-valued operator concave function on an interval J and $A_1,\dots,A_n$ are self-adjoint operators with spectra in J. Assume that $B = [b_{ij}]$ and $C = [c_{ij}]$ are $n \times n$ doubly stochastic matrices, ω_1 and ω_2 are weight functions with respect to the same probability vector, and $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is a normalized positive map. If the operator-valued functions F_{ω_1,ω_2} and $F_{B,C}$ are defined by*

$$F_{\omega_1,\omega_2}(t) := \sum_{i=1}^{m} \mu_i f\Big(\sum_{j=1}^{n}[(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j\Phi(A_j)\Big) \qquad (0 \leq t \leq 1)$$

and

$$F_{B,C}(t) := \frac{1}{n}\sum_{i=1}^{n} f\Big(\sum_{j=1}^{n}[(1-t)b_{ij} + tc_{ij}]\Phi(A_j)\Big) \qquad (0 \leq t \leq 1), \tag{2.3}$$

then

(i)

$$f\Big(\sum_{j=1}^{n} \lambda_j\Phi(A_j)\Big) \geq F_{\omega_1,\omega_2}(t) \geq \sum_{j=1}^{n} \lambda_j\Phi(f(A_j)) \qquad (0 \leq t \leq 1). \tag{2.4}$$

In particular,

$$f\left(\frac{1}{n}\sum_{j=1}^{n} \Phi(A_j)\right) \geq F_{B,C}(t) \geq \frac{1}{n}\sum_{j=1}^{n} \Phi(f(A_j)) \qquad (0 \leq t \leq 1).$$

(ii) *For any $i = 1,\dots,n$, the maps*

$$t \mapsto f\Big(\sum_{j=1}^{n}[(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j\Phi(A_j)\Big) \qquad (0 \leq t \leq 1),$$

as well as the function F_{ω_1,ω_2}, are operator concave. In particular, $F_{B,C}$ is concave on $[0,1]$.

Proof. (i) Since for every t in $[0,1]$, the map

$$(i,j) \mapsto (1-t)\omega_1(i,j) + t\omega_2(i,j) \quad (1 \le i \le m,\ 1 \le j \le n)$$

is a weight function, (2.4) follows from (2.2). By taking $m = n$, $\lambda_j = \mu_i = 1/n$, $\omega_1(i,j) = nb_{ij}$, $\omega_2(i,j) = nc_{ij}$ in $F_{\omega_1,\omega_2}(t)$, we obtain the second part.

(ii) Let $\eta_1, \eta_2 \ge 0$ with $\eta_1 + \eta_2 = 1$ and let $t_1, t_2 \in [0,1]$. For every i with $1 \le i \le m$, we have

$$\begin{aligned}
&f\Big(\sum_{j=1}^{n}[(1-\eta_1 t_1 - \eta_2 t_2)\omega_1(i,j) + (\eta_1 t_1 + \eta_2 t_2)\omega_2(i,j)]\lambda_j \Phi(A_j)\Big)\\
&= f\Big(\eta_1 \sum_{j=1}^{n}[(1-t_1)\omega_1(i,j) + t_1\omega_2(i,j)]\lambda_j \Phi(A_j)\\
&\quad + \eta_2 \sum_{j=1}^{n}[(1-t_2)\omega_1(i,j) + t_2\omega_2(i,j)]\lambda_j \Phi(A_j)\Big)\\
&\ge \eta_1 f\Big(\sum_{j=1}^{n}[(1-t_1)\omega_1(i,j) + t_1\omega_2(i,j)]\lambda_j \Phi(A_j)\Big)\\
&\quad + \eta_2 f\Big(\sum_{j=1}^{n}[(1-t_2)\omega_1(i,j) + t_2\omega_2(i,j)]\lambda_j \Phi(A_j)\Big) \quad \text{(by concavity of } f\text{)},
\end{aligned}$$

which implies (ii). The concavity of $F_{B,C}$ over $[0,1]$ is clear. ■

By taking $f(t) = -t\log t$ and $\Phi(t) = t$ in (2.3) and by using Theorem 2.10, we obtain the following result:

Corollary 2.11 (Refinement of an operator entropy inequality). *Assume that $A_1, \ldots, A_n$ are positive self-adjoint invertible operators with spectra in an interval J and $\sum_{j=1}^{n} A_j = I$. If $B = [b_{ij}]$ and $C = [c_{ij}]$ are $n \times n$ doubly stochastic matrices, then*

$$\begin{aligned}
(\log n)I &\ge \sum_{i=1}^{n}\Big[-\Big(\sum_{j=1}^{n}[(1-t)b_{ij} + tc_{ij}]A_j\Big)\log\Big(\sum_{j=1}^{n}[(1-t)b_{ij} + tc_{ij}]A_j\Big)\Big]\\
&\ge -\sum_{j=1}^{n} A_j \log A_j \quad (0 \le t \le 1).
\end{aligned}$$

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