NON-MEAGER P-FILTERS ARE COUNTABLE DENSE HOMOGENEOUS

BY

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Abstract. We prove that if \( F \) is a non-meager \( P \)-filter, then both \( F \) and \( \omega F \) are countable dense homogeneous spaces.

1. Introduction. All spaces considered are separable and metrizable.

A separable space \( X \) is countable dense homogeneous (CDH for short) if whenever \( D \) and \( E \) are countable dense subsets of \( X \), there exists a homeomorphism \( h : X \to X \) such that \( h[D] = E \). Using the now well-known back-and-forth argument, Cantor [3] gave the first example of a CDH space: the real line. In fact, many other important spaces are CDH, e.g. the Euclidean spaces, the Hilbert cube and the Cantor set. Results from [2] and [13] provide general classes of CDH spaces that include the examples mentioned. In [6] and [11] the reader can find summaries of past research and bibliography about CDH spaces.

In [5], Fitzpatrick and Zhou posed the following problems.

1.1. Question. Does there exist a CDH metrizable space that is not completely metrizable?

1.2. Question. For which 0-dimensional subsets \( X \) of \( \mathbb{R} \) is \( \omega X \) CDH?

Concerning these two problems, the following results have been obtained.

1.3. Theorem ([6]). Let \( X \) be a separable metrizable space.

- If \( X \) is CDH and Borel, then \( X \) is completely metrizable.
- If \( \omega X \) is CDH, then \( X \) is a Baire space.

1.4. Theorem ([4]). There is a CDH set of reals \( X \) of size \( \omega_1 \) that is a \( \lambda \)-set \(^{(1)}\) and thus not completely metrizable.

The techniques used in the proof of Theorem [1.4] produce spaces that are not Baire spaces, so by Theorem [1.3] they cannot answer Question [1.2].

2010 Mathematics Subject Classification: 54D80, 54H05, 54B10, 54A35.

Key words and phrases: countable dense homogeneous, non-meager \( P \)-filter.

\(^{(1)}\) Recall that a set of reals \( X \) is a \( \lambda \)-set if every countable subset of \( X \) is a relative \( G_δ \) set.
There is a natural bijection between the Cantor set $\omega_2$ and $\mathcal{P}(\omega)$ via characteristic functions. In this way we may identify $\mathcal{P}(\omega)$ with the Cantor set. Thus, any subset of $\mathcal{P}(\omega)$ can be thought of as a separable metrizable space.

Recall that a set $F \subset \mathcal{P}(\omega)$ is a filter (on $\omega$) if (a) $X \in F$ and $\emptyset \notin F$; (b) if $A, B \in F$, then $A \cap B \in F$ and (c) if $A \in F$ and $A \subset B \subset \omega$, then $B \in F$. For $\mathcal{X} \subset \mathcal{P}(\omega)$, let $\mathcal{X}^* = \{x \subset \omega : \omega \setminus x \in \mathcal{X}\}$. Then a set $I \subset \mathcal{P}(\omega)$ is an ideal if and only if $I^*$ is a filter. We will assume that all filters contain the Fréchet filter $\{x \subset \omega : \omega \setminus x \text{ is finite}\}$ (dually, all ideals contain the set of finite subsets of $\omega$). An ultrafilter is a maximal filter with respect to inclusion.

If $A, B$ are sets, $A \subset^* B$ means that $A \setminus B$ is finite. A filter $\mathcal{F}$ is called a $P$-filter if given $\{X_n : n < \omega\} \subset \mathcal{F}$ there exists $X \in \mathcal{F}$ such that $X \subset^* X_n$ for all $n < \omega$. Such an $X$ is called a pseudo-intersection of $\{X_n : n < \omega\}$. Dually, $\mathcal{I}$ is a $P$-ideal if $\mathcal{I}^*$ is a $P$-filter. An ultrafilter that is a $P$-filter is called a $P$-point.

Considering ultrafilters as topological spaces, the following results were obtained recently by Medini and Milovich.

1.5. Theorem ([10, Theorems 15, 21, 24, 41, 43 and 44]). Assume $MA(\text{countable})$. Then there are ultrafilters $\mathcal{U} \subset \mathcal{P}(\omega)$ with any of the following properties:

(a) $\mathcal{U}$ is CDH and a $P$-point,
(b) $\mathcal{U}$ is CDH and not a $P$-point,
(c) $\mathcal{U}$ is not CDH and not a $P$-point, and
(d) $\omega \mathcal{U}$ is CDH.

Since ultrafilters do not even have the Baire property ([11, 4.1.1]), Theorem 1.5 gives a consistent answer to Question 1.1 and a consistent example for Question 1.2.

The purpose of this note is to extend these results on ultrafilters to a wider class of filters on $\omega$. In particular, we prove the following result, which answers Questions 3, 5 and 11 of [10].

1.6. Theorem. Let $\mathcal{F}$ be a non-meager $P$-filter on $\mathcal{P}(\omega)$ extending the Fréchet filter. Then both $\mathcal{F}$ and $\omega \mathcal{F}$ are CDH.

It is known that non-meager filters do not have the Baire property ([11, 4.1.1]). However, the existence of non-meager $P$-filters is an open question (in ZFC). It is known that the existence of non-meager $P$-filters follows from $\text{cof}([\mathcal{d}]^{\omega}) = \mathcal{d}$ (where $\mathcal{d}$ is the dominating number, see [11, 1.3.A]). Hence, if all $P$-filters are meager then there is an inner model with large cardinals. See [11, 4.4.C] or [7] for a detailed description of this problem.

Note that every CDH filter has to be non-definable in the following sense.
1.7. Proposition. Let \( F \) be a filter on \( \mathcal{P}(\omega) \) extending the Fréchet filter. If one of \( F \) or \( \omega F \) is CDH, then \( F \) is non-meager.

Proof. If \( \omega F \) is CDH, then \( F \) is non-meager by [6, Theorem 3.1]. Assume that \( F \) is CDH. If \( F \) is the Fréchet filter, then \( F \) is countable, hence not CDH. If \( F \) is not the Fréchet filter, there exists \( x \in F \) such that \( \omega \setminus x \) is infinite. Thus, \( C = \{ y : x \subset y \subset \omega \} \) is a copy of the Cantor set contained in \( F \).

If \( F \) were meager, we arrive at a contradiction as follows: Let \( D \subset F \) be a countable dense subset of \( F \) such that \( D \cap C \) is dense in \( C \). Since \( F \) is meager in itself, by [6, Lemma 2.1], there is a countable dense subset \( E \) of \( F \) that is a \( G_\delta \) set relative to \( F \). Let \( h : F \to F \) be a homeomorphism such that \( h[D] = E \). Then \( h[D \cap C] \) is a countable dense subset of the Cantor set \( h[C] \) that is a relative \( G_\delta \) subset of \( h[C] \), which is impossible. So \( F \) is non-meager and the proof is complete. \( \blacksquare \)

Notice that \((\mathcal{P}(\omega), \triangle, \emptyset)\) is a topological group (where \( A \triangle B \) denotes the symmetric difference of \( A \) and \( B \)) as it corresponds to addition modulo 2 in \( \omega 2 \). Given a filter \( F \subset \mathcal{P}(\omega) \), the dual ideal \( F^* \) is homeomorphic to \( F \) by means of the map that sends each subset of \( \omega \) to its complement. Notice that \( \emptyset \in F^* \) and \( F^* \) is closed under \( \triangle \). Moreover, for each \( x \in \mathcal{P}(\omega) \), the function \( y \mapsto y \triangle x \) is an autohomeomorphism of \( \mathcal{P}(\omega) \). From this it is easy to see that \( F \) is homogeneous. Thus, by [10, Proposition 3], “non-meager” in Proposition 1.7 can be replaced by “a Baire space”.

In [9, Theorem 1.2] it is proved that a filter \( F \) is hereditarily Baire if and only if \( F \) is a non-meager \( \mathcal{P} \)-filter. By Theorem 1.5 it is consistent that not all CDH ultrafilters are \( \mathcal{P} \)-points so it is consistent that there are CDH filters that are not hereditarily Baire. These observations answer Question 4 in [10].

Recall that Theorem 1.5 also shows that it is consistent that there exist non-CDH non-meager filters.

1.8. Question. Is there a combinatorial characterization of CDH filters?

1.9. Question. Is there a CDH filter (ultrafilter) in ZFC? Is there a non-CDH and non-meager filter (ultrafilter) in ZFC?

2. Proof of Theorem 1.6. For any set \( X \), let \( [X]^{<\omega} \) and \( [X]^\omega \) denote the sets of its finite and countably infinite subsets, respectively. Also \( ^{<\omega}X = \bigcup\{ ^nX : n < \omega \} \).

Since any filter is homeomorphic to its dual ideal, we may alternatively speak about a filter or its dual ideal. In particular, the following result is better expressed in the language of ideals. Its proof follows from [10, Lemma 20], we include it for the sake of completeness.
2.1. Lemma. Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal, $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ a continuous function and $D$ a countable dense subset of $\mathcal{I}$. If there exists $x \in \mathcal{I}$ such that $\{d \triangle f(d) : d \in D\} \subset \mathcal{P}(x)$, then $f[\mathcal{I}] = \mathcal{I}$.

Proof. Since $D$ is dense in $\mathcal{P}(\omega)$ and $d \triangle f(d) \subset x$ for all $d \in D$, by continuity it follows that $y \triangle f(y) \subset x$ for all $y \in \mathcal{P}(\omega)$. Then $y \triangle f(y) \in \mathcal{I}$ for all $y \in \mathcal{P}(\omega)$. Since $\mathcal{I}$ is closed under $\triangle$ and $a \triangle a = \emptyset$ for all $a \in \mathcal{P}(\omega)$, it is easy to see that $y \in \mathcal{I}$ if and only if $f(y) \in \mathcal{I}$ for all $y \in \mathcal{P}(\omega)$. ■

Let $X \subset [\omega]^\omega$. A tree $T \subset <\omega([\omega]<\omega)$ is called an $X$-tree of finite sets if for each $s \in T$ there is $X_s \in X$ such that for every $a \in [X_s]<\omega$ we have $s \sim a \in T$. It turns out that non-meager $P$-filters have a very useful combinatorial characterization as follows.

2.2. Lemma ([8] Lemma 1.3]). Let $\mathcal{F}$ be a filter on $\mathcal{P}(\omega)$ that extends the Fréchet filter. Then $\mathcal{F}$ is a non-meager $P$-filter if and only if every $\mathcal{F}$-tree of finite sets has a branch whose union is in $\mathcal{F}$.

Next we prove a combinatorial property that will allow us to construct autohomeomorphisms of the Cantor set that restrict to a given ideal. For $x \in \mathcal{P}(\omega)$, let $\chi(x) \in \omega^2$ be its characteristic function.

2.3. Lemma. Let $\mathcal{I}$ be a non-meager $P$-ideal and $D_0$, $D_1$ be two countable dense subsets of $\mathcal{I}$. Then there exists $x \in \mathcal{I}$ such that

(i) for each $d \in D_0 \cup D_1$, $d \subset^* x$ and
(ii) for each $i \in 2$, $d \in D_i$, $n < \omega$ and $t \in n \cap 2$, there exists $e \in D_1 - i$ such that $d \setminus x = e \setminus x$ and $\chi(e)|n \cap x = t$.

Proof. Let $\mathcal{F} = \mathcal{I}^*$. We will construct an $\mathcal{F}$-tree of finite sets $T$ and use Lemma 2.2 to find $x \in \mathcal{I}$ with the properties listed. Let us give an enumeration $(D_0 \cup D_1) \times <\omega^2 = \{(d_n, t_n) : n < \omega\}$ such that $\{d_n : n \equiv i \pmod{2}\} = D_i$ for $i \in 2$.

The definition of $T$ will be by recursion. For each $s \in T$ we also define $n(s) < \omega$, $F_s \in \mathcal{F}$ and $\phi_s : \text{dom}(s) \to D_0 \cup D_1$ so that the following properties hold:

(1) $\forall s, t \in T \ (s \subseteq t \Rightarrow n(s) < n(t))$,
(2) $\forall s \in T \ \forall k < \text{dom}(s) \ (s(k) \subset n(s|_{k+1}) \setminus n(s|_{k}))$,
(3) $\forall s, t \in T \ (s \subset t \Rightarrow F_t \subset F_s)$,
(4) $\forall s \in T \ (F_s \subset \omega \setminus n(s))$,
(5) $\forall s, t \in T \ (s \subset t \Rightarrow \phi_s \subset \phi_t)$,
(6) $\forall s \in T \ (k = \text{dom}(s) \ ((d_{k-1} \cup \phi_s(k-1)) \setminus n(s) \subset \omega \setminus F_s))$.

Since $\emptyset \in T$, let $n(\emptyset) = 0$ and $F_\emptyset = \omega$. Assume we have $s \in T$ and $a \in F_s$, we have to define everything for $s \sim a$. Let $k = \text{dom}(s)$. We start by defining $n(s \sim a) = \max \{k, \max (a), \text{dom}(t_k)\} + 1$. Next we define $\phi_s \sim a$. We only have to do it at $k$ because of (5). We have two cases.
Case 1: There exists \( m < \text{dom}(t_k) \) with \( t_k(m) = 1 \) and \( m \in s(0) \cup \cdots \cup s(k - 1) \). We simply declare \( \phi_{s \sim a}(k) = d_k \).

Case 2: Not Case 1. We define \( r_{s \sim a} \in n(s \sim a)2 \) in the following way.

\[
r_{s \sim a}(m) = \begin{cases} 
  d_k(m) & \text{if } m \in s(0) \cup \cdots \cup s(k - 1) \cup a, \\
  t_k(m) & \text{if } m \in \text{dom}(t_k) \setminus (s(0) \cup \cdots \cup s(k - 1) \cup a), \\
  1 & \text{in any other case.}
\end{cases}
\]

Let \( i \in 2 \) be such that \( i \equiv k \mod 2 \). So \( d_k \in D_i \), let \( \phi_{s \sim a}(k) \in D_{1-i} \) be such that \( \phi_{s \sim a}(k) \cap n(s \sim a) = (r_{s \sim a})^{-1}(1) \), this is possible because \( D_{1-i} \) is dense in \( \mathcal{P}(\omega) \). Finally, define

\[
F_{s \sim a} = (F \cap (\omega \setminus d_{k-1})) \cap (\omega \setminus \phi_{s \sim a}(k - 1)) \setminus n(s \sim a).
\]

Clearly, \( F_{s \sim a} \in \mathcal{F} \) and it is easy to see that conditions (1)–(6) hold.

By Lemma 2.2, there exists a branch \( \{(y_0, \ldots, y_n) : n < \omega \} \) of \( T \) whose union \( y = \bigcup \{y_n : n < \omega \} \) is in \( \mathcal{F} \). Let \( x = \omega \setminus y \in \mathcal{I} \). We prove that \( x \) is the element we were looking for. It is easy to prove that (6) implies (i).

We next prove that (ii) holds. Let \( i \in 2 \), \( n < \omega \), \( t \in n^{n \cap \omega}2 \) and \( d \in D_i \). Let \( k < \omega \) be such that \( (d_k, t_k) = (d, t') \), where \( t' \in n2 \) is such that \( t'|_{n \cap \omega} = t \) and \( t'|_{n-x} = 0 \). Consider step \( k + 1 \) in the construction of \( y \), that is, the step when \( y(k + 1) \) was defined. Notice that we are in Case 2 of the construction and \( r_{y|_{k+1}} \) is defined. Then \( \phi_{y|_{k+1}}(k) = e \) is an element of \( D_{1-i} \). It is not hard to see that \( d \setminus x = e \setminus x \) and \( \chi(e)|_{n \cap \omega} = t \). This completes the proof of the lemma.

We will now show that it is enough to prove Theorem 1.6 for \( \mathcal{F} \). Recall the following characterization of non-meager filters.

2.4. Lemma ([1] Theorem 4.1.2]). Let \( \mathcal{F} \) be a filter. Then \( \mathcal{F} \) is non-meager if and only if for every partition of \( \omega \) into finite sets \( \{J_n : n < \omega \} \), there is \( X \in \mathcal{F} \) such that \( \{n < \omega : X \cap J_n = \emptyset \} \) is infinite.

The following was originally proved by Shelah (see [12], Fact 4.3, p. 327]). We include a proof for the convenience of the reader.

2.5. Lemma. If \( \mathcal{F} \) is a non-meager P-filter, then \( \omega \mathcal{F} \) is homeomorphic to a non-meager P-filter.

Proof. Let

\[
\mathcal{G} = \{A \subset \omega \times \omega : \forall n < \omega \ (A \cap (\{n\} \times \omega) \in \mathcal{F})\}.
\]

Notice that \( \mathcal{G} \) is homeomorphic to \( \omega \mathcal{F} \). It is easy to see that \( \mathcal{G} \) is a filter on \( \omega \times \omega \). We next prove that \( \mathcal{G} \) is a non-meager P-filter.

Let \( \{A_k : k < \omega \} \subset \mathcal{G} \). For each \( \{k, n\} \subset \omega \), we define \( A_k^n = \{x \in \omega : (n, x) \in A_k \} \in \mathcal{F} \). Since \( \mathcal{F} \) is a P-filter, there is \( A \in \mathcal{F} \) such that \( A \subset^* A_k^n \)
for all \( \{k,n\} \subset \omega \). Let \( f : \omega \to \omega \) be such that \( A \setminus f(n) \subset A_k^n \) for all \( k \leq n \). Let
\[
B = \bigcup \{ \{n\} \times (A \setminus f(n)) : n < \omega \}.
\]
Then it is easy to see that \( B \in \mathcal{G} \) and \( B \) is a pseudointersection of \( \{A_n : n < \omega\} \). So \( \mathcal{G} \) is a \( P \)-filter.

Let \( \{J_k : k < \omega\} \) be a partition of \( \omega \times \omega \) into finite subsets. Recursively, we define a sequence \( \{F_n : n < \omega\} \subset \mathcal{F} \) and a sequence \( \{A_n : n < \omega\} \subset [\omega]^{\omega} \) such that \( A_{n+1} \subset A_n \) and \( A_n \subset \{k < \omega : J_k \cap (\{n\} \times F_n) = \emptyset\} \) for all \( n < \omega \).

For \( n = 0 \), since \( \mathcal{F} \) is non-meager, by Lemma 2.4 there is \( F_0 \in \mathcal{F} \) such that \( \{k < \omega : J_k \cap (\{0\} \times F_0) = \emptyset\} \) is infinite; call this last set \( A_0 \). Assume that we have the construction up to \( m < \omega \), then \( B = \{J_k \cap (\{m+1\} \times \omega) : k \in A_m\} \) is a family of pairwise disjoint finite subsets of \( \{m+1\} \times \omega \). If \( \bigcup B \) is finite, let \( F_{m+1} \in \mathcal{F} \) be such that \( F_{m+1} \cap \bigcup B = \emptyset \) and let \( A_{m+1} = A_m \). If \( \bigcup B \) is infinite, let \( \{B_k : k \in A_m\} \) be any partition of \( \{m+1\} \times \omega \) \( \bigcup B \) into finite subsets (some possibly empty). For each \( k \in A_m \), let \( C_k = (J_k \cap (\{m+1\} \times \omega)) \cup B_k \). Then \( \{C_k : k \in A_m\} \) is a partition of \( \{m+1\} \times \omega \) into finite sets, so by Lemma 2.4 there is \( F_{m+1} \in \mathcal{F} \) such that \( \{k \in A_m : C_k \cap (\{m+1\} \times F_{m+1}) = \emptyset\} \) is infinite; call this set \( A_{m+1} \). This completes the recursion.

Define an increasing function \( s : \omega \to \omega \) such that \( s(0) = \min A_0 \) and \( s(k+1) = \min(A_{k+1} \setminus \{s(0), \ldots, s(k)\}) \) for \( k < \omega \). Also, define \( t : \omega \to \omega \) such that \( t(0) = 0 \) and \( t(n+1) = \min\{m < \omega : (J_{s(0)} \cup \cdots \cup J_{s(n)}) \cap (\{n+1\} \times \omega) \subset \{n+1\} \times m\} \).

Finally, let
\[
G = \bigcup \{\{n\} \times (F_n \setminus t(n)) : n < \omega\}.
\]
Then \( G \in \mathcal{G} \) and for all \( k < \omega \), \( G \cap J_{s(k)} = \emptyset \). Thus, \( \mathcal{G} \) is non-meager by Lemma 2.4.

We now have everything ready to prove our result.

Proof of Theorem 1.6. By Lemma 2.5 it is enough to prove that \( \mathcal{F} \) is CDH, equivalently that \( \mathcal{I} = \mathcal{F}^* \) is CDH. Let \( D_0 \) and \( D_1 \) be two countable dense subsets of \( \mathcal{I} \) and let \( x \in \mathcal{I} \) be given by Lemma 2.3.

We will construct a homeomorphism \( h : \mathcal{P}(\omega) \to \mathcal{P}(\omega) \) such that \( h[D_0] = D_1 \) and
\[
(\ast) \quad \forall d \in D \ (d \triangle h(d) \subset x).
\]

By Lemma 2.1 \( h[\mathcal{I}] = \mathcal{I} \) and we will have finished.

We shall define \( h \) by approximations. By this we mean the following. We will give a strictly increasing sequence \( \{n(k) : k < \omega\} \subset \omega \) and in step \( k < \omega \) a homeomorphism (permutation) \( h_k : \mathcal{P}(n(k)) \to \mathcal{P}(n(k)) \) such that
\[
(\ast) \quad \forall j < k \omega \forall a \in \mathcal{P}(n(k)) \ (h_k(a) \cap n(j) = h_j(a \cap n(k))).
\]
By (\(\ast\)), we can define \(h : P(\omega) \rightarrow P(\omega)\) to be the inverse limit of \(\{h_k : k < \omega\}\), which is a homeomorphism.

Let \(D_0 \cup D_1 = \{d_n : n < \omega\}\) in such a way that \(\{d_n : n \equiv i \mod 2\}\) = \(D_i\) for \(i \in 2\). To make sure that \(h[D_0] = D_1\), in step \(k\) we have to decide the value of \(h(d_k)\) when \(k\) is even and the value of \(h^{-1}(d_k)\) when \(k\) is odd. We do this by approximating a bijection \(\pi\) for \(h\) in \(\omega\) steps by a chain of finite bijections \(\{\pi_k : k < \omega\}\) and letting \(\pi = \bigcup\{\pi_k : k < \omega\}\). In step \(k < \omega\), we would like to have \(\pi_k\) defined on some finite set so that the following conditions hold whenever \(\pi_k \subset \pi\):

\[(a)_k\] if \(j < k\) is even, then \(h_k(d_j \cap n(k)) = \pi(d_j) \cap n(k)\), and

\[(b)_k\] if \(j < k\) is odd, then \(h_k(d_j \cap n(k)) = \pi^{-1}(d_j) \cap n(k)\).

Notice that once \(\pi\) is completely defined, if \((a)_k\) and \((b)_k\) hold for all \(k < \omega\), then \(h[D] = E\). During the construction, we need to make sure that the following two conditions hold:

\[(c)_k\] \(\forall i \in n(k) \setminus x \forall a \in P(n(k))\) \((i \in a \iff i \in h_k(a))\), and

\[(d)_k\] \(\forall d \in \text{dom}(\pi_k)\) \((d \setminus x = \pi_k(d) \setminus x)\).

Condition \((c)_k\) is a technical condition that will help us carry out the recursion. Notice that if we have condition \((d)_k\) for all \(k < \omega\), then \((\ast)\) will hold.

Assume that we have defined \(n(0) < \cdots < n(s - 1)\), \(h_0, \ldots, h_{s-1}\) and a finite bijection \(\pi_s \subset D_0 \times D_1\) with \(\{d_r : r < s\} \subset \text{dom}(\pi_s) \cup \text{dom}(\pi_s^{-1})\) in such a way that if \(\pi \supset \pi_s\), then \((a)_{s-1} - (d)_{s-1}\) hold. Let us consider the case when \(s\) is even; the other case can be treated in a similar fashion.

If \(d_s = \pi_s^{-1}(d_r)\) for some odd \(r < s\), let \(n(s) = n(s - 1) + 1\). If we let \(\pi_{s+1} = \pi_s\), it is easy to define \(h_s\) so that it is compatible with \(h_{s-1}\) in the sense of \((\ast)\), in such a way that \((a)_{s} - (d)_{s}\) hold for any \(\pi \supset \pi_{s+1}\). So we may assume this is not the case.

Notice that the set \(S = \{d_r : r < s + 1\} \cup \{\pi_s(d_r) : r < s, r \equiv 0 \mod 2\} \cup \{\pi_s^{-1}(d_r) : r < s, r \equiv 1 \mod 2\}\) is finite. Choose \(p < \omega\) so that \(d_s \setminus p \subset x\). Let \(r_0 = h_{s-1}(d_s \cap n(s - 1)) \in P(n(s - 1))\). Choose \(n(s - 1) < m < \omega\) and \(t \in m\cap 2\) in such a way that \(t^{-1}(1) \cap n(s - 1) = r_0 \cap n(s - 1) \cap x\) and \(t\) is not extended by any element of \(\{\chi(a) : a \in S\}\). By Lemma 2.3 there exists \(e \in E\) such that \(d_s \setminus x = e \setminus x\) and \(\chi(e)|_{m\cap x} = t\). Notice that \(e \notin S\) and \(\chi(e)|_{n(s-1)} = r_0\). We define \(\pi_{s+1} = \pi_s \cup \{(d_s, e)\}\). Notice that \((d)_s\) holds in this way.

Now that we have decided where \(\pi\) will send \(d_s\), let \(n(s) > \max\{p, m\}\) be such that there are no two distinct \(a, b \in S \cup \{\pi_{s+1}(d_s)\}\) with \(a \cap n(s) = b \cap n(s)\). Topologically, all elements of \(S \cup \{\pi_{s+1}(d_s)\}\) are contained in distinct basic open sets of measure \(1/(n(s) + 1)\).
Finally, we define the bijection $h_s : \mathcal{P}(n(s)) \to \mathcal{P}(n(s))$. For this part of the proof we will use characteristic functions instead of subsets of $\omega$ (otherwise the notation would become cumbersome). Therefore, we may say $h_r : n(r)2 \to n(r)2$ is a homeomorphism for $r < s$.

Let $(q, q') \in n(s-1)2 \times n(s)\setminus x2$ be a pair of compatible functions. Notice that $(h_{s-1}(q), q')$ are also compatible by $(c)_{s-1}$. Consider the following condition:

$\nabla(q, q') : \forall a \in n(s)2 \ (q \cup q' \subset a \iff h_{s-1}(q) \cup q' \subset h_s(a))$.

Notice that if we define $h_s$ so that $\nabla(q, q')$ holds for each pair $(q, q') \in n(s-1)2 \times n(s)\setminus x2$ of compatible functions, then $(\ast)$ and $(c)_s$ hold as well.

Then for each pair $(q, q') \in n(s-1)2 \times n(s)\setminus x2$ of compatible functions we only have to find a bijection $g : T2 \to T2$, where $T = (n(s) \cap x) \setminus n(s-1)$ (this bijection will depend on the pair) and define $h_s : n(s)2 \to n(s)2$ as

$$h_s(a) = h_{s-1}(q) \cup q' \cup g(f \upharpoonright T)$$

whenever $a \in n(s)2$ and $q \cup q' \subset a$. There is only one restriction in the definition of $g$ and it is imposed by conditions $(a)_s$ and $(b)_s$; namely that $g$ is compatible with the bijection $\pi_{s+1}$ already defined. However by the choice of $n(s)$ this is not hard to achieve. This finishes the inductive step and the proof. ■

Acknowledgements. This paper is part of the first-named author’s doctoral dissertation. Research was supported by CONACyT scholarship for Doctoral Students. The second-named author acknowledges support from PAPIIT grant IN102311 and CONACyT grant 177758.

REFERENCES


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Received 27 April 2012;
revised 28 February 2013  (5676)