COLLOQUIUM MATHEMATICUM

VOL. 117 2009 NO. 2

INDUCED SUBSYSTEMS ASSOCIATED TO A CANTOR MINIMAL SYSTEM

BY

HEIDI DAHL (Trondheim) and MATS MOLBERG (Elverum)

Abstract. Let (X,T) be a Cantor minimal system and let $(R,\mathcal{T})$ be the associated étale equivalence relation (the orbit equivalence relation). We show that for an arbitrary Cantor minimal system (Y,S) there exists a closed subset Z of X such that (Y,S) is conjugate to the subsystem $(Z,\widetilde{T})$, where $\widetilde{T}$ is the induced map on Z from T. We explore when we may choose Z to be a T-regular and/or a T-thin set, and we relate T-regularity of a set to R-étaleness. The latter concept plays an important role in the study of the orbit structure of minimal $\mathbb{Z}^d$-actions on the Cantor set by T. Giordans et al. [J. Amer. Math. Soc. 21 (2008)].

1. Main results. We state the two main theorems of this paper, postponing the proofs till later. In the next two sections we will give definitions of pertinent concepts, and state some properties and results that will be relevant for the proofs.

THEOREM 1.1. *Let (X,T) be a Cantor minimal system, and let (Y,S) be an arbitrarily given Cantor minimal system. There exists a closed subset $Z\subset X$ such that all points in Z have finite return times under the action of T, and if $\widetilde{T}: Z\to Z$ is the induced map (i.e. $\widetilde{T}z = T^m z$, where $m=\inf\{k\in\mathbb{N}\mid T^k z\in Z\}$), then $(Y,S)\simeq(Z,\widetilde{T})$. We can choose $Z\subset X$ to be a T-regular set if we allow one point x in Z to have infinite return time, appropriately defining $\widetilde{T}x$. Moreover, we can always choose Z to be a T-thin set in X, i.e. $\mu(Z)=0$ for all T-invariant probability measures μ.*

THEOREM 1.2. *Let (X,T) be a Cantor minimal system and let $(R,\mathcal{T})$ be the associated étale equivalence relation. Let Z be a non-empty closed subset of X. The following are equivalent:*

(i) *Z is T-regular, i.e. the (forward and backward) return time maps (with respect to Z) are continuous.*
(ii) *Z is R-étale, i.e. $R\cap(Z\times Z)$ is an étale equivalence relation.*

2010 *Mathematics Subject Classification*: 54H20, 37B10, 19K14.
Key words and phrases: Cantor minimal systems, AF-equivalence relations, étale equivalence relations, Bratteli diagrams.

DOI: 10.4064/cm117-2-4

(iii) *Z is $R_{\{x\}}$-étale for all $x \in X$, where $R_{\{x\}}$ is obtained from R by splitting the T-orbit of x in the forward and backward T-orbits.*

2. Basic concepts. In this and the next section we will recall some basic definitions and results that we will need concerning Cantor minimal systems and étale equivalence relations. For details we refer to [3], [4], [5] and the survey article [8].

Let X be a locally compact and second countable (hence metrizable) Hausdorff space. An *étale equivalence relation* $R\,(\subset X\times X)$ on X is a countable equivalence relation (i.e. every equivalence class is at most countable) which has a topology $\mathcal{T}$ making it a locally compact topological groupoid, and with the additional property that the range map, $r : R \to X$, defined by $r((x,y)) = x$, is a local homeomorphism. Recall that r is a *local homeomorphism* if for all $(x,y) \in R$ there exists an open neighborhood $U_{(x,y)} \subset R$ of (x,y) such that

(i) $r(U_{(x,y)})$ is open in X;
(ii) $r : U_{(x,y)} \to r(U_{(x,y)})$ is a homeomorphism.

Recall that the product of *composable pairs* $(x,y),(y,z) \in R$ is $(x,y)\cdot(y,z) = (x,z)$, and the *inverse* $(x,y)^{-1}$ of $(x,y) \in R$ is (y,x). We will denote an étale equivalence relation by $(R,\mathcal{T})$, or simply by R. We say that R is *minimal* if $[x]_R$ is dense in X for every $x \in X$, where $[x]_R = \{y \in X \mid (x,y) \in R\}$ is the equivalence class of x. The diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a clopen subset of R (cf. [7, Prop. 2.8]), and is homeomorphic to X. It should be remarked that only rarely does the topology $\mathcal{T}$ on R coincide with the relative topology from $X \times X$. In general, $\mathcal{T}$ is finer than the relative topology. We will refer to $U_{(x,y)}$ as an *étale neighborhood*, and the local homeomorphism condition as the *étale condition*. It is easily seen that if S $(\subset X \times X)$ is an open sub-equivalence relation of R, then S is étale in the relative topology. A (Borel) probability measure μ on X is said to be *R-invariant* if

$$\mu(r(E)) = \mu(s(E))$$

for every étale neighborhood $U_{(x,y)}$ and Borel set $E \subset U_{(x,y)}$. Here $s : R \to X$ denotes the *source map*, defined by $s((x,y)) = y$. We denote by $M(X,R)$ the set of all R-invariant (Borel) probability measures on X.

Let $(R_i,\mathcal{T}_i)$ be étale equivalence relations on X_i, $i = 1,2$. We say that $(R_1,\mathcal{T}_1)$ is *isomorphic* to $(R_2,\mathcal{T}_2)$, and write $(R_1,\mathcal{T}_1) \cong (R_2,\mathcal{T}_2)$, if there exists a homeomorphism $F : X_1 \to X_2$ such that

(i) $(x,y) \in R_1 \Leftrightarrow (F(x),F(y)) \in R_2$;
(ii) $F\times F : (R_1,\mathcal{T}_1) \to (R_2,\mathcal{T}_2)$, defined by $F\times F((x,y)) = (F(x),F(y))$ for $(x,y) \in R_1$, is a homeomorphism.

If condition (i) is satisfied, we say that $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ are *orbit equivalent*. (Note that condition (i) is equivalent to $F([x]_{R_1}) = [F(x)]_{R_2}$ for each $x \in X$.)

By an *action* of a countable (discrete) group G on a locally compact, second countable space X we mean a homomorphism $\alpha : G \to \mathrm{Homeo}(X)$. When the action is *free*, i.e. $\alpha_g(x) = x$ for some $x \in X$, some $g \in G$ implies that g is the identity element of G, this gives rise to an étale equivalence relation R_G on X. That is, we let R_G be the *orbit equivalence relation* induced by α, where the equivalence class of $x \in X$ is the orbit $[x]_G = \{\alpha_g(x) \mid g \in G\}$. We give R_G the topology $\mathcal{T}_G$, which is obtained by transferring the (product) topology from the product space $X \times G$ using the map $(x, g) \to (\alpha_g(x), x)$. (This map is bijective since the action α is free.) The resulting space $(R_G, \mathcal{T}_G)$ will be an étale equivalence relation on X.

We will be concerned with the following, which falls under the general scheme described above: Let (X, T) be a *Cantor minimal system*, i.e. X is the Cantor set and $T : X \to X$ is a minimal homeomorphism, where minimality means that the orbit $[x]_T = \{T^n x \mid n \in \mathbb{Z}\}$ is dense in X for all $x \in X$. By viewing (X, T) as a (free) $\mathbb{Z}$-action on X, where $1 \in \mathbb{Z}$ corresponds to T, we get an étale equivalence relation on X as described above.

Two Cantor minimal systems (X, T) and (Y, S) are *conjugate*, written $(X, T) \simeq (Y, S)$, if there exists a homeomorphism $h : X \to Y$ such that $h \circ T = S \circ h$. Conjugate Cantor minimal systems give isomorphic orbit equivalence relations.

Let (X, T) be a Cantor minimal system. For a closed, non-empty subset Z of X, define $\lambda^+, \lambda^- : Z \to \mathbb{N} \cup \{\infty\}$, where $\mathbb{N} \cup \{\infty\}$ is given the one-point compactification topology, by

$$\lambda^+(z) = \inf\{k \geq 1 \mid T^k z \in Z\},$$
$$\lambda^-(z) = \inf\{k \geq 1 \mid T^{-k} z \in Z\}.$$

(We use the convention that inf of the empty set is ∞.) These maps are called the *forward* and *backward return time maps* with respect to Z. We say that $Z \subset X$ is *regular* with respect to T (or T-regular) if both maps λ^+ and λ^- are continuous.

Remark 2.1. The maps λ^+ and λ^- are lower semicontinuous. To see this, just observe that if $\lambda^+(z) = k$, then $T^i z$ is not in Z for $i = 1, \ldots, k-1$, and since $X \setminus Z$ is open, there are open neighborhoods $U_{T^i z} \subset X$ around each of these points, each $U_{T^i z}$ disjoint from Z. Hence $V = \bigcap_{i=1}^{k-1} T^{-i}(U_{T^i z}) \cap Z$ is an open neighborhood of z in Z, and for all $z' \in V$ we have $\lambda^+(z') \geq k$. With a slight modification the argument goes through also for $\lambda^+(z) = \infty$, by considering $V_N = \bigcap_{i=1}^{N} T^{-i}(U_{T^i z}) \cap Z$ for each $N \in \mathbb{N}$. A similar proof can be given for λ^-.

Given an étale equivalence relation $(R, \mathcal{T})$ on a locally compact second countable space X and a (non-empty) closed subset Z of X we define $R|_Z = R \cap (Z \times Z)$ as an equivalence relation on Z. We say that Z is *R-étale* if $R|_Z$, given the relative topology $\mathcal{T}|_Z$ from $\mathcal{T}$, is étale. It is said to be *R-thin* if $\mu(Z) = 0$ for every $\mu \in M(X, R)$.

3. Bratteli diagrams as models for Cantor minimal systems and for AF-equivalence relations. The concept of a *Bratteli diagram* will be important for us, because it serves as a model, and as such as a crucial tool, for both AF-equivalence relations and for Cantor minimal systems (when an ordering is introduced). A Bratteli diagram (V, E) is a special directed infinite graph, consisting of a vertex set V, an edge set E and two maps $i, t : E \to V$ such that

(i) V is an infinite union of disjoint, non-empty finite sets, $V = \bigcup_{n=0}^{\infty} V_n$, and V_0 is a one-point set, $V_0 = \{v_0\}$;
(ii) E is an infinite union of disjoint, non-empty finite sets, $E = \bigcup_{n=1}^{\infty} E_n$;
(iii) the *source* (or *initial*) map i satisfies $i(E_n) \subset V_{n-1}$ for all $n \geq 1$, and $i^{-1}(v) \neq \emptyset$ for all $v \in V$;
(iv) the *range* (or *terminal*) map t satisfies $t(E_n) \subset V_n$ for all $n \geq 1$, and $t^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.

For a Bratteli diagram (V, E) we denote by $X_{(V,E)}$ the set of all infinite paths in (V, E), where a *path* $x = (e_n)_{n=1}^{\infty}$ is a sequence of edges $e_1, e_2, \ldots$ such that $e_n \in E_n$ and $t(e_n) = i(e_{n+1})$ for all n. We can also talk about (finite) paths between a vertex $v \in V_n$ and a vertex $w \in V_m$, $m > n$, and it is obvious what we mean by that. If there exists an edge $e \in E_n$ with source $v \in V_{n-1}$ and range $u \in V_n$, we say that v is *connected* to u. We say that two paths $x = (e_n)_{n=1}^{\infty}$, $y = (f_n)_{n=1}^{\infty}$ in $X_{(V,E)}$ are *cofinal* if there exists an $N \in \mathbb{N}$ such that $e_n = f_n$ for all $n > N$. Henceforth we will only consider non-trivial Bratteli diagrams (V, E), i.e. $X_{(V,E)}$ is an infinite set.

We describe two important operations that we can perform on a Bratteli diagram, turning it into new Bratteli diagrams that retain the basic properties of the original. These are *telescoping* and *symbol splitting*. Let (V, E) be a Bratteli diagram. Let $0 = t_0 < t_1 < t_2 < \cdots$ be a sequence of natural numbers. Define a new Bratteli diagram (V', E') by setting $V'_n = V_{t_n}$ and $E'_n = \{$all finite paths between $V_{t_{n-1}}$ and $V_{t_n}\}$. The range and source maps are the obvious ones. We say that (V', E') is a *telescope* of (V, E).

By the operation of symbol splitting we create a new diagram (V', E') from (V, E) by inserting new vertex levels. Let $V'_{2k} = V_k$ for $k \geq 0$, and let $|V'_{2k-1}| = |E_k|$ for $k \geq 1$. There is an obvious way of defining E'_{2k-1} and E'_{2k} such that by telescoping between levels $2k-1$ and $2k$ we get E_k. In other words, each edge in E_k is split in two by introducing a vertex in V'_{2k-1}.

(See Figure 1 and Figure 2 for examples of telescoping and symbol splitting, respectively. Disregard the ordering of the edges for the time being.)

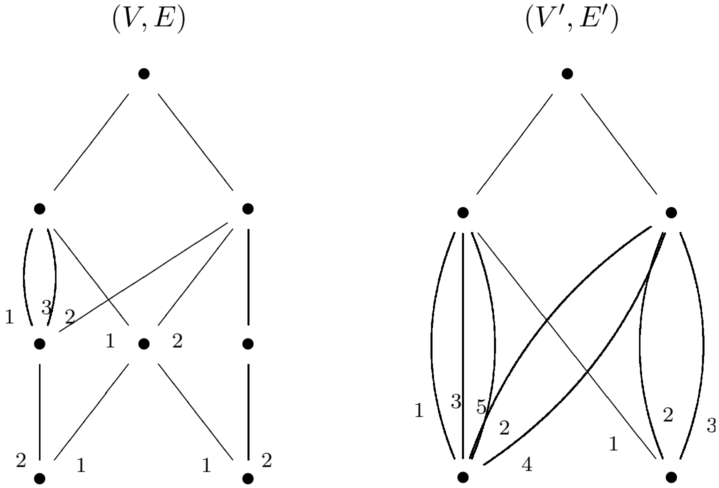

Fig. 1. An illustration of telescoping

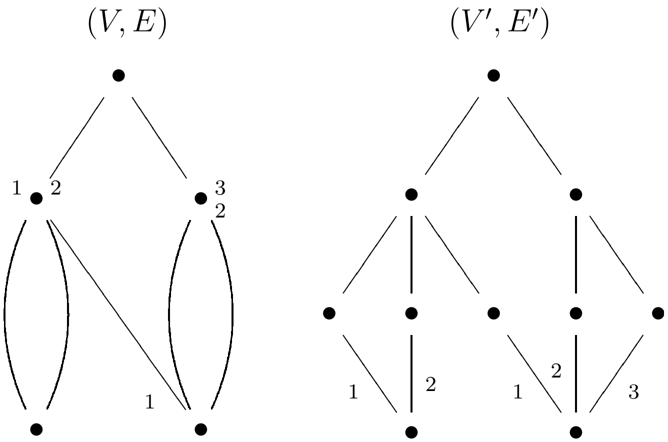

Fig. 2. An illustration of symbol splitting

A diagram is *simple* if it can be telescoped into a diagram (V', E') where each $v \in V'_{n-1}$ is connected to each $u \in V'_n$ for all $n > 0$. For a simple Bratteli diagram (V, E), the path space $X_{(V,E)}$ becomes a Cantor set, where the *cylinder sets* $\{C_n(x) \mid x = (e_1, e_2, \ldots) \in X_{(V,E)}, n \in \mathbb{N}\}$ form a clopen basis for the topology. Here $C_n(x) = \{y = (f_1, f_2, \ldots) \in X_{(V,E)} \mid f_1 = e_1, f_2 = e_2, \ldots, f_n = e_n\}$. We remark here that if we drop the condition that (V, E) is simple, then $X_{(V,E)}$ is still a zero-dimensional space, i.e. $X_{(V,E)}$ has a countable basis of clopen sets (consisting of the cylinder sets $C_n(x)$).

We can give a partial order to the edge set by giving a linear order to the set of edges $t^{-1}(v)$ for each vertex $v \in V \setminus V_0$. Let $(V, E, \geq)$ denote a Bratteli diagram equipped with a partial order $\geq$ on E; we call it an *ordered Bratteli diagram*. This induces a (partial) lexicographic order on the path space $X_{(V,E)}$. Specifically, $(e_i)_{i=1}^{\infty} > (f_i)_{i=1}^{\infty}$ if there exists $n \in \mathbb{N}$ such that $r(e_n) = r(f_n)$, $e_k = f_k$ for all $k > n$ and $e_n > f_n$. A path $x = (e_1, e_2, \ldots) \in X_{(V,E)}$ is a *maximal* (resp. *minimal*) *path* if all edges e_n are maximal (resp. minimal) in the linearly ordered set $r^{-1}(r(e_n))$.

Given an ordered Bratteli diagram $(V, E, \geq)$, the *Vershik map* $T_{(V,E)} : X_{(V,E)} \to X_{(V,E)}$ is defined so that a non-maximal path is mapped to its successor in the lexicographic order, while a maximal path is mapped to a minimal path. Let $(V, E, \geq)$ be an ordered Bratteli diagram, where (V, E) is simple. We say that $(V, E, \geq)$ is *properly ordered* if there exist exactly one maximal path and one minimal path. Then $(X_{(V,E)}, T_{(V,E)})$ is a Cantor minimal system, and we call such a system a *Bratteli–Vershik system*.

We state a basic theorem that we shall need, which we may call the model theorem for Cantor minimal systems, and we will refer to the properly ordered Bratteli diagram $(V, E \geq)$ occurring in the theorem as a *Bratteli–Vershik model* (for the given Cantor minimal system (X, T)).

Theorem 3.1 ([5, Theorem 4.7], [8, Theorem 4]). *Let (X, T, x) be a (pointed) Cantor minimal system, where $x \in X$. There exists a properly ordered Bratteli diagram $(V, E, \geq)$ such that (X, T, x) is (pointedly) conjugate to $(X_{(V,E)}, T_{(V,E)}, x_{\min})$, where $x_{\min}$ is the unique minimal path in $X_{(V,E)}$. This means that the conjugating map $h : X \to X_{(V,E)}$ maps x to $x_{\min}$.*

Remark 3.2. There is a natural way to introduce an ordering on a Bratteli diagram which is obtained from an ordered Bratteli diagram by either telescoping or symbol splitting (cf. [3, Section 3]). Both telescoping and symbol splitting yield natural homeomorphisms, preserving cofinality, between the original path space and the new path space, such that $(X_{(V,E)}, T_{(V,E)})$ is conjugate to $(X_{(V',E')}, T_{(V',E')})$, where $(V', E', \geq)$ is the ordered Bratteli diagram obtained from $(V, E, \geq)$ by a finite number of telescopings and/or symbol splittings.

The Bratteli diagram (V, E) induces an equivalence relation on $X_{(V,E)}$, denoted by $\mathrm{AF}(V, E)$, namely, two paths are equivalent if and only if they are cofinal. Topologized appropriately (cf. [4, Example 2.7 (ii)]), $\mathrm{AF}(V, E)$ becomes a so-called *AF-equivalence relation*, according to the following definition.

Definition 3.3. An *AF-equivalence relation* R on a zero-dimensional space X is an étale equivalence relation $(R, \mathcal{T})$ such that $R = \bigcup_{n=1}^{\infty} R_n$, where $R_1 \subset R_2 \subset \cdots$ is an increasing sequence of sub-equivalence relations of R such that $R_n = (R_n, \mathcal{T}_n)$ is a compact étale equivalence relation (CEER) for all n. The topology of R is the inductive limit topology, i.e. $U \subset \mathcal{T}$ iff $U \cap R_n \in \mathcal{T}_n$ for all n. We write $(R, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$.

Just as ordered Bratteli diagrams serve as models for Cantor minimal systems, (unordered) Bratteli diagrams serve as models for AF-equivalence relations as stated in the following theorem.

THEOREM 3.4 ([4, Theorem 3.9]). *Let $(R, T) = \varinjlim(R_n, \mathcal{T}_n)$ be an AF-equivalence relation on the zero-dimensional space X. There exists a Bratteli diagram (V, E) such that $(R, \mathcal{T})$ is isomorphic to the AF-equivalence relation* $\mathrm{AF}(V, E)$ *associated to (V, E). Furthermore, (V, E) is simple if and only if $(R, \mathcal{T})$ is minimal.*

Combining Theorems 3.1 and 3.4 we get the following corollary.

COROLLARY 3.5 ([4, Theorem 2.4]). *Let (X, T) be a Cantor minimal system and let $(R, \mathcal{T})$ be the associated étale equivalence relation as described in Section* 2. *Let x be an arbitrary point in X. The sub-equivalence relation $R_{\{x\}}$ of R whose equivalence classes are the full T-orbits, except that the T-orbit of x is split in two at x (the forward orbit $\{T^n x \mid n \geq 1\}$ and the backward orbit $\{T^n x \mid n \leq 0\}$), is open in R. Furthermore, $(R_{\{x\}}, \mathcal{T}_{\{x\}})$ is an AF-equivalence relation on X, where $\mathcal{T}_{\{x\}}$ is the relative topology.*

REMARK 3.6. It is noteworthy that if x_1 and x_2 are any two points in X, then $(R_{\{x_1\}}, \mathcal{T}_{\{x_1\}}) \cong (R_{\{x_2\}}, \mathcal{T}_{\{x_2\}})$. This follows from [4, Lemma 4.13] and [5, Theorem 5.3].

Let (V, E) be a Bratteli diagram. By a *subdiagram* of (V, E) we mean a Bratteli diagram (W, F) such that $W \subset V$, $F \subset E$ and $t(F) \cup \{v_0\} = i(F)$. The range and source maps of (W, F) are the restrictions of the range and source maps of (V, E). Note that a subdiagram (W, F) of a Bratteli diagram (V, E) is being telescoped or symbol split in an obvious way simultaneously as these operations are applied to (V, E). If $(V, E, \geq)$ is an ordered Bratteli diagram, a subdiagram (W, F) of (V, E) will inherit the order in an obvious way. Note that if (W, F) is a subdiagram of (V, E), then the topology of $\mathrm{AF}(W, F)$ coincides with the relative topology from $\mathrm{AF}(V, E)$, and so $\mathrm{AF}(V, E)|_{X_{(W,F)}}$ is AF, and hence étale. With the terminology we have introduced we can say that $X_{(W,F)}$ is $\mathrm{AF}(V, E)$-*étale*.

We include the following result concerning subdiagrams and $\mathrm{AF}(V, E)$-invariant probability measures.

PROPOSITION 3.7 ([6, Theorem 2.21]). *Let (V, E) be a Bratteli diagram and (W, F) a subdiagram. Suppose that there exists a positive constant M and $N > 1$ such that, for all $w \in W_{N-1}$ and $w' \in W_N$, we* *have* $M\#F_N(w,w') \le \#E_N(w,w')$. *Then* $\mu(X_{(W,F)}) \le M^{-1}$ *for all* μ *in* $M(X_{(V,E)}, \mathrm{AF}(V,E))$.

(Here $\#F_N(w,w')$ denotes the number of edges in F_N with source w and range w'.)

4. Proof of Theorem 1.1. The key to the proof is the following lemma, which illustrates what a useful tool Bratteli diagram models can be. In fact, by elementary and easy manipulations on a given Bratteli diagram (ordered or unordered) one can set the stage for proving non-trivial results that seem to be inaccessible otherwise.

LEMMA 4.1. *Let* (X,T) *be a Cantor minimal system, and let* $\{(l_k,n_k)\}_{k=1}^{\infty}$ *be a sequence of pairs of natural numbers, where* $l_k \ge 2$. *There exists a Bratteli–Vershik model* $(V,E,\ge)$ *for* (X,T) *such that*

(i) $|V_k| \ge l_k$ *for all* $k \ge 1$;
(ii) $x_{\min}$ *and* $x_{\max}$ *do not pass through the same vertex at any level* $k \ge 1$ *of* (V,E), *where* $x_{\min}$ *and* $x_{\max}$ *denote the unique minimal and maximal paths, respectively, in* $X_{(V,E)}$;
(iii) *if* $V_{k-1} = \{v_1,\dots,v_{m_{k-1}}\}$, $k \ge 1$, *then for all vertices* $w \in V_k$ *and for all* $v_i \in V_{k-1}$ *we can choose* n_k *edges* $\{e_{(i,j)}\}_{j=1}^{n_k}$ *in* E_k *connecting* w *to* v_i; *furthermore, the ordering of these edges is as follows*:

$$e_{(1,1)} < e_{(2,1)} < \dots < e_{(m_{k-1},1)} < e_{(1,2)} < e_{(2,2)} < \dots < e_{(m_{k-1},2)} < \dots < e_{(1,n_k)} < \dots < e_{(m_{k-1},n_k)}.$$

Proof. Let $(V,E,\ge)$ be a Bratteli–Vershik model for (X,T). It is easy to see that by a succession of telescoping, symbol splitting and telescoping, in that order, one may satisfy conditions (i) and (ii). Indeed, more can be achieved by the same token. If $(V',E',\ge)$ denotes the new ordered Bratteli diagram obtained, which by Remark 3.2 is again a Bratteli–Vershik model for (X,T), then we can assume that $|V'_{k-1}| \le |V'_k|$ for all $k \ge 1$. Furthermore, we may assume that (V',E') is totally connected, that is, between any $v \in V'_{k-1}$ and $w \in V'_k$, there exists an edge $e \in E'_k$ connecting the two, i.e. $i(e) = v$ and $t(e) = w$. We observe that all the properties of $(V'E',\ge)$ listed above are preserved under telescopings of $(V',E',\ge)$.

So we may at start assume that $(V,E,\ge)$ has all the above-mentioned properties. We want to show that condition (iii) can be obtained by telescoping $(V,E,\ge)$, and this will finish the proof by the above remarks. Clearly, we can choose a level $k_1 \ge 1$ such that if we telescope between levels $k_0 = 0$ and k_1 of $(V,E,\ge)$, then (iii) is satisfied for $k=1$. Assume we have telescoped $(V,E,\ge)$ between the levels $0 = k_0 < k_1 < \dots < k_l$, so that (iii) is satisfied for $k = 1,\dots,l$. Now choose an arbitrary vertex $u \in V_{k_l+1}$.

By our assumption on total connection, there exists an edge e'_i ranging at u and sourcing at v'_i for every $i \in \{1, \dots, s\}$, where $V_{k_l} = \{v'_1, \dots, v'_s\}$. To simplify the notation, rearrange the vertices in V_{k_l} according to the linear order of the set $\{e'_i\}_{i=1}^s$, i.e. if $\{e'_i\}_{i=1}^s = \{e_1 < \cdots < e_s\}$, then $V_{k_l} = \{v'_i\}_{i=1}^s = \{v_1, \dots, v_s\}$, and the source of e_i is v_i. There exists a level $k_{l+1} > k_l + 1$ such that the number of paths from u to any vertex in $V_{k_{l+1}}$ is at least n_{k+1}. We telescope between level k_l and k_{l+1} of our diagram, and for each $v_i \in V_{k_l}$ and $w \in V_{k_{l+1}}$ we choose n_{k+1} paths arbitrarily between v_i and w, except that we require the first edge of each path to be e_i. Now it is easy to see, using the lexicographic order on paths, that we may arrange these n_{k+1} paths, which become edges after telescoping, in such a way that they satisfy condition (iii) for $k = k_{l+1}$. Telescoping $(V, E, \geq)$ to levels $0 = k_0 < k_1 < k_2 < \cdots$, we get a diagram satisfying the three conditions of the lemma. ■

Proof of Theorem 1.1. The idea is to use Lemma 4.1 to construct a Bratteli–Vershik model $(V, E, \geq)$ for (X, T), in which we can imbed a Bratteli–Vershik model $(W, F, \geq)$ for (Y, S), such that the ordering on $(W, F \geq)$ coincides with the one induced from $(V, E, \geq)$. This will obviously give a conjugacy $h : (Y, S) \to (Z, \widetilde{T})$, where $Z = X_{(W,F)}$, and $\widetilde{T} : Z \to Z$ is the induced map as described in the theorem.

Let $(W, F, \geq)$ be a Bratteli–Vershik model for (Y, S), where the maximal and the minimal paths pass through different vertices at each level. Let $l_k = |W_k|$ and let n_k be the maximal number of edges ranging at a vertex at level W_k, i.e. $n_k = \max\{|t^{-1}(w)| \mid w \in W_k\}$. Let $(V, E, \geq)$ be a Bratteli–Vershik model for (X, T) satisfying the conditions of Lemma 4.1 with respect to the sequence $\{(l_k, n_k)\}_{k=1}^\infty$. We describe how to define a copy of (W, F) as a subdiagram of (V, E), such that the order of $(W, F, \geq)$ coincides with the induced order from $(V, E, \geq)$.

For $k > 0$, choose $l_k = |W_k|$ vertices $\{v_1^{(k)}, v_2^{(k)}, \dots, v_{l_k}^{(k)}\} \subset V_k$, including the ones that the unique maximal and minimal paths pass through, denoting these by $v_{\max}^{(k)}$ and $v_{\min}^{(k)}$, respectively. We denote the corresponding vertices in W_k by $w_{\max}^{(k)}$ and $w_{\min}^{(k)}$, respectively. Let g_k be a bijection between W_k and $\{v_i^{(k)}\}_{i=1}^{l_k}$ such that $g_k(w_{\max}^{(k)}) = v_{\max}^{(k)}$ and $g_k(w_{\min}^{(k)}) = v_{\min}^{(k)}$. We define $g_0(w_0) = v_0$, where $W_0 = \{w_0\}$ and $V_0 = \{v_0\}$.

Next we define an injective map h_k from F_k into E_k. For a vertex $w^{(k)} \in W_k$, let $\{f_s\}_{s=1}^m$ be the linearly ordered edges in F_k with range $w^{(k)}$, i.e. $f_1 < \cdots < f_m$. Let $v^{(k)} = g_k(w^{(k)})$, and let $\{e_t\}_{t=1}^n$ be the linearly ordered edges in E_k with range $v^{(k)}$. Define $h_k(f_1) = e_{t_1}$, where e_{t_1} is the minimal edge in $\{e_t\}_{t=1}^n$ such that $g_{k-1}(i(f_1)) = i(e_{t_1})$. Note that if f_1 is an edge of the unique minimal path in $X_{(W,F)}$, then e_{t_1} is an edge of the unique minimal path in $X_{(V,E)}$. After having defined $h_k(f_l) = e_{t_l}$, for $l < m$ and if $l+1 < m$, we define $h_k(f_{l+1}) = e_{t_{l+1}}$, where $e_{t_{l+1}}$ is the minimal edge in $\{e_t\}_{t=1}^n$ greater than e_{t_l}, such that $g_{k-1}(i(f_{l+1})) = i(e_{t_{l+1}})$. If $l+1 = m$, we define $h_k(f_m)$ to be the maximal edge e in $\{e_t\}_{t=1}^n$ such that $g_{k-1}(i(f_m)) = i(e)$. Note that if f_m is an edge of the unique maximal path in $X_{(W,F)}$ then $h_k(f_m)$ will be an edge of the unique maximal path in $X_{(V,E)}$. The properties satisfied by the ordered Bratteli diagram $(V, E, \geq)$ entail that the map h_k is well-defined for $k = 1, 2, \ldots$.

Let

$$W' = \{g_k(w) \mid w \in W_k,\, k = 0, 1, \ldots\},$$
$$F' = \{h_k(f) \mid f \in F_k,\, k = 1, 2, \ldots\}.$$

It is easy to see that (W', F') is a subdiagram of (V, F), and that (W', F') is in an obvious way isomorphic to (W, F). Transferring the order from $(W, F \geq)$ to (W', F') by this isomorphism we get a copy, $(W', F', \geq)$, of $(W, F, \geq)$. Hence the two associated Bratteli–Vershik systems are conjugate, with conjugating map $h : X_{(W,F)} \to X_{(W',F')}$ being defined by $h(x) = (h_1(f_1), h_2(f_2), \ldots) \in X_{(W',F')}$, where $x = (f_1, f_2, \ldots) \in X_{(W,F)}$. Furthermore, it follows by our definition of the pair of maps (g_{k-1}, h_k), for $k = 1, 2, \ldots$, that the ordering on $(W', F', \geq)$ coincides with the induced ordering from $(V, E, \geq)$, i.e. if $f_1, f_2 \in F'$ and $t(f_1) = t(f_2)$, then $f_1 < f_2$ in $(W', F', \geq)$ if and only if $f_1 < f_2$ in $(V, E, \geq)$. We conclude from all this that the proof of the first statement of the theorem is complete if we show that the return time to $Z = X_{(W',F')}$ is finite. But this follows from the fact that by our set-up the unique maximal and minimal paths in $X_{(W',F')}$ coincide with the unique maximal and minimal paths, respectively, in $X_{(V,E)}$. We omit the easy details.

The set $Z = X_{(W',F')}$ may not be regular since the forward return time map at $x_{\max}$, and the backward return time map at $x_{\min}$, may not be continuous. (Here $x_{\max}$ and $x_{\min}$ denote the (coinciding) unique maximal and minimal paths, respectively, in $X_{(W',F')}$ and $X_{(V,E)}$.) At all other paths it is easy to see that the two return time maps are continuous. (An example of a closed subset $Z \subset X$ which is not regular is shown in Figure 5.) By a slight modification of the construction of $(W', F', \geq)$ we can achieve that $Z = X_{(W',F')}$ is regular, but we pay the price that at the unique maximal and minimal paths in $X_{(W',F')}$ (which no longer coincide with the corresponding ones in $X_{(V,E)}$) the return time is no longer finite. Referring to the notation used above, we let $l_k = |W_k| + 2$. This time we avoid $v_{\max}^{(k)}$ and $v_{\min}^{(k)}$ for all k in the construction of the subdiagram (W', F'). It is easy to see that $Z = X_{(W',F')}$ is T-regular, and that $(Y, S) \simeq (Z, \widetilde{T})$, where we define $\widetilde{T}(y_{\max}) = y_{\min}$. (Here $y_{\max}$ and $y_{\min}$ denote the unique maximal and minimal paths, respectively, in $X_{(W',F')}$.) We see that in this case the forward return time of $y_{\max}$ and the backward return time of $y_{\min}$ are both infinite. However, the return time maps at both these points are continuous. We omit the details.

The last assertion of the theorem is easy to obtain. In fact, we can telescope (V,E), before we define the subdiagram (W',F'), so that the ratio of the number of paths from the top vertex in (V,E) to any vertex at level k to the corresponding number for (W,F), tends to zero as k goes to infinity. Then $Z = X_{(W',F')}$ is going to be a thin subset of $X_{(V,E)}$, a fact that is easily seen by recalling Proposition 3.7. ∎

We give a simple example to illustrate the construction in the proofs of Lemma 4.1 and Theorem 1.1. We keep the above notation.

EXAMPLE 4.2. Let (Y,S) be the Sturmian flow with rotation number equal to the golden mean. The simplest Bratteli–Vershik model $(W,F,\geq)$ for (Y,S) is shown in Figure 3 (cf. [8, 3.3]). In this case the parameters

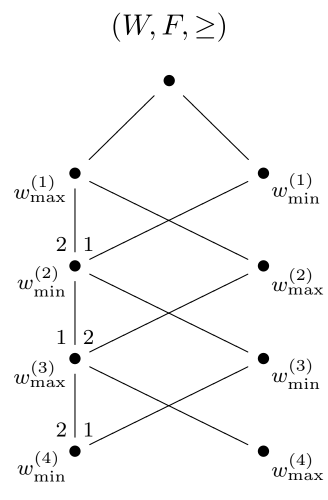

Fig. 3. A Bratteli–Vershik model for (Y,S)

are $l_k = |W_k| = 2$ for all $k \geq 1$, and $n_k = 2$ for $k \geq 2$. Let (X,T) be the 2-odometer. The simplest Bratteli–Vershik model for (X,T) is shown on the left of Figure 4. Also in Figure 4 we indicate the manipulations done in order to get a Bratteli–Vershik model $(V,E,\geq)$ for (X,T) that is adapted for the construction of a copy $(W',F',\geq)$ of $(W,F,\geq)$ as a subdiagram. (Note that in this case we only need the operations of symbol splitting and telescoping, in that order.) The edges belonging to $(W',F',\geq)$ are drawn solid.

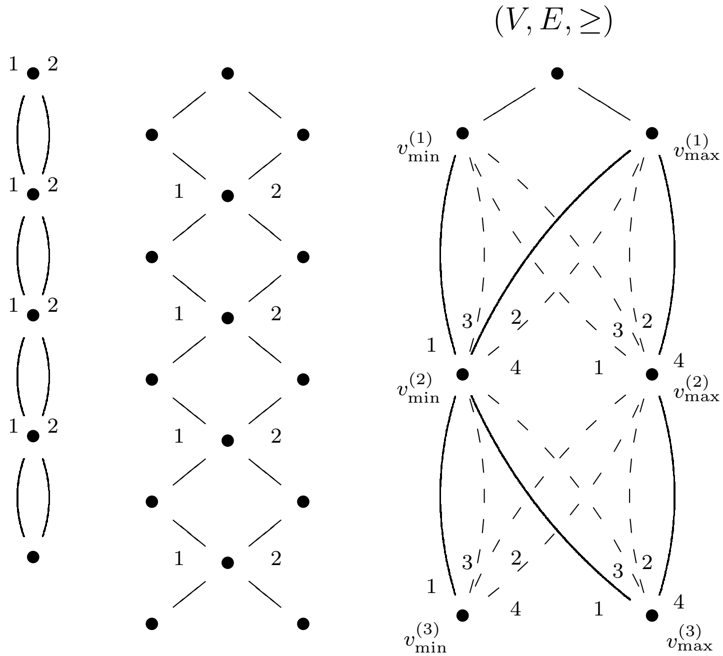

Fig. 4. Starting with the two-odometer, we first do symbol splitting between every level, then we telescope to the sequence $0 < 1 < 5 < 9 < 13 < \cdots$.

5. Proof of Theorem 1.2. We start by showing (i)⇒(ii). For $k \in \mathbb{N}$, define the maps $\lambda_k^+, \lambda_k^- : Z \to \mathbb{N} \cup \{\infty\}$ by

$$\lambda_k^+(z) = \inf\{l \geq 1 \mid T^{l_i}(z) \in Z \text{ for } k \text{ distinct numbers } 1 \leq l_i \leq l\},$$
$$\lambda_k^-(z) = \inf\{l \geq 1 \mid T^{-l_i}(z) \in Z \text{ for } k \text{ distinct numbers } 1 \leq l_i \leq l\}.$$

Again we use the convention that inf of the empty set is ∞. We call these maps the *positive* and *negative kth return time maps* with respect to Z. We claim that λ_k^+ and λ_k^- are continuous for every $k \in \mathbb{N}$ if and only if λ^+ and λ^-, respectively, are continuous. This follows by induction on k, and the observation that for all $z \in Z$ we have $\lambda_{k+1}^+(z) = \lambda_k^+(z) + \lambda^+(T^{\lambda_k^+(z)}(z))$. The claim concerning λ_k^- is completely analogous. (The modifications needed if $\lambda_k^{\pm}(y) = \infty$ are obvious.)

Suppose Z is regular. Let $z_0 \in Z$, and assume that the equivalence class of z_0 in Z is $[z_0]_{R|Z} = \{\ldots, z_{-1}, z_0, z_1, \ldots\}$, where $z_i = T^{m_i} z_0$, and we have arranged the points in such a way that $\cdots < m_{-1} < m_0 = 0 < m_1 < \cdots$. We want to find an étale neighborhood $U_{(z_0,z)} \subset R|_Z$ for all possibilities of $z \in [z_0]_{R|Z}$. If $z = z_0$, then obviously $\Delta_X \cap (Z \times Z)$ is an étale neighborhood containing (z_0, z_0), where $\Delta_X = \{(x,x) \mid x \in X\}$ is the diagonal of R. Next assume z is in the positive orbit of z_0. Note that if $\lambda^+(z_0) = \infty$ then there are no points to check. Suppose $z = z_k$, i.e. $z = T^{m_k} z_0$. This means that

$z_0 \in (\lambda_k^+)^{-1}(m_k)$, which is open in Z by continuity of λ_k^+. Let $V_{(z_0,z)}$ be an étale neighborhood of (z_0, z) in R, and let $U_{z_0} = (\lambda_k^+)^{-1}(m_k) \cap r(V_{(z_0,z)}) \subset Z$. Define $U_{(z_0,z)} = \{(x, T^{m_k}x) \mid x \in U_{z_0}\}$. This is an étale neighborhood of (z_0, z) in $R|_Z$, a fact that is easily verified. If z is in the negative orbit the argument is analogous, using the continuity of λ_k^-.

To prove (ii)⇒(iii), let $x \in X$. By Corollary 3.5, $R_{\{x\}}$ is open in R, and hence $R_{\{x\}}|_Z$ is open in $R|_Z$. As Z is R-étale, it follows that Z is $R_{\{x\}}$-étale, since every open sub-equivalence relation of an étale equivalence relation is étale.

To show (iii)⇒(i), suppose that Z is *not* T-regular. We want to show that this implies that there exists $x \in X$ such that Z is not $R_{\{x\}}$-étale. In fact, we will show that $x = z \in Z$, where z is a point of discontinuity of either λ^+ or λ^-. So let $z \in Z$, and suppose λ^+ is not continuous at z. (A similar argument applies for λ^-.) As λ^+ is always lower semicontinuous, it is not upper semicontinuous at z. So there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset Z$ such that $z_n \to z$ and $\lim_n \lambda^+(z_n) > \lambda^+(z)$. [Note: $\lim_n \lambda^+(z_n)$ may be infinite, but if so, the argument is unchanged.] Assuming now, to the contrary, that Z is $R_{\{z\}}$-étale, there exists an étale neighborhood $U_{(z,T^{\lambda^+(z)}z)}$ of $(z, T^{\lambda^+(z)}z)$ in $R_{\{z\}}|_Z$. Choose an open neighborhood V_z of z in X. Put $V_{(z,T^{\lambda^+(z)}z)} = \{(x, T^{\lambda^+(z)}x) \mid x \in V_z\}$. Then $V_{(z,T^{\lambda^+(z)}z)}$ is an étale neighborhood of $(z, T^{\lambda^+(z)}z)$ with respect to R, and this implies that $U_{(z,T^{\lambda^+(z)}z)} \cap V_{(z,T^{\lambda^+(z)}z)}$ is another étale neighborhood of $(z, T^{\lambda^+(z)}z)$ with respect to $R_{\{z\}}|_Z$. This is clear since $R_{\{z\}}|_Z$ has the relative topology from R. However, $\{z_n\}_{n=1}^{\infty} \cap r(U_{(z,T^{\lambda^+(z)}z)} \cap V_{(z,T^{\lambda^+(z)}z)}) = \emptyset$, and so $r(U_{(z,T^{\lambda^+(z)}z)} \cap V_{(z,T^{\lambda^+(z)}z)})$ cannot be open in Z, and hence $U_{(z,T^{\lambda^+(z)}z)} \cap V_{(z,T^{\lambda^+(z)}z)}$ is not an étale neighborhood of $(z, T^{\lambda^+(z)}z)$ in $R_{\{z\}}|_Z$. This contradiction finishes the proof of (iii)⇒(i), and hence the proof of Theorem 1.2.

COROLLARY 5.1. *Let (X,T) be a Cantor minimal system, and let $(R, \mathcal{T})$ be the associated étale equivalence relation. Let Z be an R-étale subset of X. For any $x \in X$ there exists a simple Bratteli diagram (V,E) and a homeomorphism $h : X_{(V,E)} \to X$ implementing an isomorphism $h \times h : \mathrm{AF}(V,E) \to R_{\{x\}}$ such that $h^{-1}(Z) = X_{(W,F)}$ for some subdiagram (W,F) of (V,E).*

Proof. By Corollary 3.5, $R_{\{x\}}$ is an AF-equivalence relation for $x \in X$. By Theorem 1.2 we find that Z is $R_{\{x\}}$-étale. By [4, Theorem 3.11] we get the result. ∎

The example we will now exhibit is somewhat related to the above corollary, even though it illustrates a different aspect of the theory. Figure 5 shows

$(V, E, \geq)$

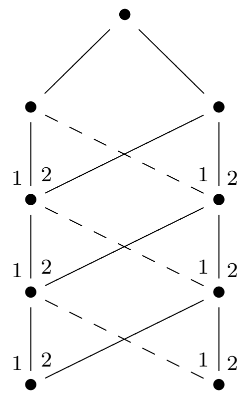

Fig. 5. The subdiagram (W, F) of the Bratteli diagram (V, E) is obtained by deleting the dotted edges.

that not every subdiagram of a properly ordered, simple Bratteli diagram is regular with respect to the Vershik map, and hence not all subdiagrams give rise to étale sub-equivalence relations by Theorem 1.2. Let $(X_{(V,E)}, T_{(V,E)})$ be the Bratteli–Vershik system associated to $(V, E, \geq)$ in Figure 5, and let $x_{\min}$ and $x_{\max}$ be the unique minimal and maximal paths, respectively, in $X_{(V,E)}$. Let $Z = X_{(W,F)}$. Now $\lambda^+(x_{\max}) = 1$, but there exists a sequence $\{z_n\}$ in Z converging to $x_{\max}$, such that $\lambda^+(z_n) \to \infty$, which shows that λ^+ is not continuous at $x_{\max}$.

Note also, referring again to Figure 5, that Z is $R_{\{x_{\max}\}}$-étale, but not $R_{\{x_{\min}\}}$-étale. This underscores the requirement (iii) of Theorem 1.2, namely that Z should be $R_{\{x\}}$-étale for *all* $x \in X$.

We end this paper by giving the following result which extends Theorem 1.2 when the subset $Z \subset X$ satisfies a certain condition.

COROLLARY 5.2. *Let (X, T) be a Cantor minimal system and let $(R, \mathcal{T})$ be the associated étale equivalence relation. Let Z be a non-empty closed subset of X such that there exists $z_0 \in X$ with $[z_0]_T \cap Z$ contained in either $\{T^n z_0 \mid n \geq 1\}$ or $\{T^{-n} z_0 \mid n \geq 0\}$. (Recall that $[z_0]_T$ denotes the T-orbit $\{T^n z_0 \mid n \in \mathbb{Z}\}$ of z_0.) The following are equivalent*:

(i) *Z is T-regular*;
(ii) *Z is R-étale*;
(iii) *Z is $R_{\{x\}}$-étale for all $x \in X$*;
(iv) *there exists a simple Bratteli diagram (V, E), containing a subdiagram (W, F), and a map $h : X_{(V,E)} \to X$ such that $h \times h : \mathrm{AF}(V, E) \to R_{\{z_0\}}$ is an isomorphism and $h(X_{(W,F)}) = Z$.*

Proof. By Corollary 5.1 we have (ii)⇒(iv). We will prove (iv)⇒(ii). The rest follows by Theorem 1.2.

Let (V, E) and (W, F) be as in (iv). Now $\mathrm{AF}(W, F)$ is an AF-equivalence relation on $X_{(W,F)}$, and the topology coincides with the relative topology from $\mathrm{AF}(V, E)$. So $(h \times h)(\mathrm{AF}(W, F)) = R_{\{z_0\}}|_Z$ is an AF-equivalence relation, and hence étale, on $h(X_{(W,F)}) = Z$. This means that $R|_Z$ is étale, since $R|_Z = R_{\{z_0\}}|_Z$, and both $R|_Z$ and $R_{\{z_0\}}|_Z$ have relative topology from R. ■

Acknowledgements. This paper was partly written while the second author was employed at the Norwegian University of Science and Technology.

REFERENCES

[1] T. Giordano, H. Matui, I. F. Putnam and C. F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems*, preprint, 2007.

[2] —, —, —, —, *Orbit equivalence for Cantor minimal $\mathbb{Z}^2$-systems*, J. Amer. Math. Soc. 21 (2008), 863–892.

[3] T. Giordano, I. F. Putnam and C. F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. 469 (1995), 51–111.

[4] —, —, —, *Affable equivalence relations and orbit structure of Cantor dynamical systems*, Ergodic Theory Dynam. Systems 24 (2004), 441–475.

[5] R. H. Herman, I. F. Putnam and C. F. Skau, *Ordered Bratteli diagrams*, *dimension groups and topological dynamics*, Int. J. Math. 3 (1992), 827–864.

[6] I. F. Putnam, *Orbit equivalence of Cantor minimal systems*: *a survey and a new proof*, preprint, 2007.

[7] J. Renault, *A Groupoid Approach to C^*-algebras*, Lecture Notes in Math. 793, Springer, 1980.

[8] C. Skau, *Ordered K-theory and minimal symbolic dynamical systems*, Colloq. Math. 84/85 (2000), 203–227.

Department of Mathematical Sciences
Norwegian University of Science and Technology
N-7491 Trondheim, Norway
E-mail: heidida@math.ntnu.no

Education and Natural Sciences
Hedemark University College
N-2418 Elverum, Norway
E-mail: mats.molberg@hihm.no

Received 18 December 2007;
revised 4 December 2008 (4955)