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SELECTION PRINCIPLES AND UPPER SEMICONTINUOUS FUNCTIONS

BY

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Abstract. In connection with a conjecture of Scheepers, Bukovský introduced properties wQN* and SSP* and asked whether wQN* implies SSP*. We prove it in this paper. We also give characterizations of properties $S_1(\Gamma,\Omega)$ and $S_{fin}(\Gamma,\Omega)$ in terms of upper semicontinuous functions.

1. Introduction. In this paper all topological spaces are assumed to be infinite. We denote by $\mathbb{I}$ the closed unit interval $[0,1]$. The symbol $\mathbf{0}$ is the constant function with the value 0. For real-valued functions f_n $(n\in\omega)$ on a set X, the symbol $f_n\to\mathbf{0}$ means that the sequence $\{f_n\}_{n\in\omega}$ converges pointwise to $\mathbf{0}$ (i.e. for every $x\in X$ the sequence $\{f_n(x)\}_{n\in\omega}$ converges to 0). A real-valued function f on a space X is said to be *upper semicontinuous* [4] if for every real number r, the set $\{x\in X : f(x)<r\}$ is open in X.

DEFINITION 1.1 ([5]). A family $\{A_n\}_{n\in\omega}$ of subsets of a set X is a *γ-cover* of X if every point $x\in X$ is contained in A_n for all but finitely many $n\in\omega$ and $A_n\neq X$ for every $n\in\omega$. A space X has *property* $S_1(\Gamma,\Gamma)$ if for every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open γ-covers of X, there are $U_n\in\mathcal{U}_n$ $(n\in\omega)$ such that $\{U_n\}_{n\in\omega}$ is a γ-cover of X.

DEFINITION 1.2. A sequence $\{f_n\}_{n\in\omega}$ of real-valued functions on a set X *converges quasi-normally* to $\mathbf{0}$ [1] if there is a sequence $\{\varepsilon_n\}_{n\in\omega}$ of positive real numbers converging to 0 such that for each $x\in X$, $|f_n(x)|<\varepsilon_n$ for all but finitely many $n\in\omega$. A space X has *property* wQN [3] if whenever $\{f_n\}_{n\in\omega}$ is a sequence of real-valued continuous functions on X such that $f_n\to\mathbf{0}$, the sequence contains a subsequence which converges quasi-normally to $\mathbf{0}$. A space X has *property* SSP (*the sequence selection property*) [6] if whenever $\{f_{n,m}\}_{n,m\in\omega}$ is a family of real-valued continuous functions on X such that for each $n\in\omega$, $f_{n,m}\to\mathbf{0}$ $(m\to\infty)$, there is a function $\varphi\in\omega^\omega$ with $f_{n,\varphi(n)}\to\mathbf{0}$.

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Uniform convergence implies quasi-normal convergence, and quasi-normal convergence implies pointwise convergence. It is known that properties wQN and SSP are equivalent (for instance see [2, Theorem 1]). Scheepers [7] proved that property $\mathrm{S}_1(\Gamma,\Gamma)$ implies property wQN, and conjectured that for perfectly normal spaces, properties wQN and $\mathrm{S}_1(\Gamma,\Gamma)$ are equivalent. In connection with this conjecture, Bukovský introduced properties SSP* and wQN* below as modifications of SSP and wQN.

Definition 1.3 ([2]). A space X has *property* wQN* if whenever $\{f_n\}_{n\in\omega}$ is a sequence of upper semicontinuous functions from X into $\mathbb{I}$ such that $f_n \to \mathbf{0}$, the sequence contains a subsequence which converges quasi-normally to $\mathbf{0}$. A space X has *property* SSP* if whenever $\{f_{n,m}\}_{n,m\in\omega}$ is a family of upper semicontinuous functions from X into $\mathbb{I}$ such that for each $n\in\omega$, $f_{n,m}\to\mathbf{0}$ $(m\to\infty)$, there is a function $\varphi\in\omega^\omega$ with $f_{n,\varphi(n)}\to\mathbf{0}$.

Bukovský proved:

Theorem 1.4 ([2]).

(1) *Property* SSP* *implies property* wQN*,
(2) *Property* $\mathrm{S}_1(\Gamma,\Gamma)$ *is equivalent to property* SSP*.

But it was open whether wQN* implies SSP* [2, Problem 2]. In the next section we show that this is indeed the case. In the third section we give characterizations of properties $\mathrm{S}_1(\Gamma,\Omega)$ and $\mathrm{S}_{\mathrm{fin}}(\Gamma,\Omega)$ in terms of upper semicontinuous functions.

2. Properties $\mathrm{S}_1(\Gamma,\Gamma)$, SSP* and wQN*

Lemma 2.1. *Let $\{f_m\}_{m\in\omega}$ be a sequence of real-valued functions on a set X which converges quasi-normally to $\mathbf{0}$. Let $\{\delta_n\}_{n\in\omega}$ be a sequence of positive real numbers converging to* 0. *Then there is a subsequence $\{f_{m_n}\}_{n\in\omega}\subset\{f_m\}_{m\in\omega}$ such that for every $x\in X$, $|f_{m_n}(x)|<\delta_n$ for all but finitely many $n\in\omega$.*

Proof. Since $\{f_m\}_{m\in\omega}$ converges quasi-normally to $\mathbf{0}$, there is a sequence $\{\varepsilon_m\}_{m\in\omega}$ of positive real numbers converging to 0 such that for every $x\in X$, $|f_m(x)|<\varepsilon_m$ for all but finitely many $m\in\omega$. For each $n\in\omega$ take $m_n\in\omega$ with $\varepsilon_{m_n}<\delta_n$. Then for every $x\in X$, $|f_{m_n}(x)|<\varepsilon_{m_n}<\delta_n$ for all but finitely many $n\in\omega$. ∎

We denote by $\mathrm{USC}_p(X,\mathbb{I})$ the space of all upper semicontinuous functions from a space X into $\mathbb{I}$ with the topology of pointwise convergence.

Theorem 2.2. *Property* wQN* *implies property* $\mathrm{S}_1(\Gamma,\Gamma)$.

Proof. For each $n\in\omega$, let $\mathcal{U}_n=\{U_{n,m}: m\in\omega\}$ be an open γ-cover of X. For each $n,m\in\omega$, we put $V_{n,m}=U_{0,m}\cap\cdots\cap U_{n,m}$, and let $\mathcal{V}_n=$

$\{V_{n,m} : m \in \omega\}$. Each $\mathcal{V}_n$ is an open γ-cover of X. We define $f_m : X \to [0,1]$ as follows:

$$f_m(x) = \begin{cases} 1 & \text{if } x \in X \setminus V_{0,m}, \\ 1/(k+2) & \text{if } x \in V_{k,m} \setminus V_{k+1,m} \ (k \in \omega), \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_m \in \mathrm{USC}_p(X,\mathbb{I})$. Note that $f_m(x) < 1/(n+1)$ if and only if $x \in V_{n,m}$. Since each $\mathcal{V}_n$ is a γ-cover of X, $f_m \to \mathbf{0}$. By property wQN*, the sequence $\{f_m\}_{m\in\omega}$ has a subsequence converging quasi-normally to $\mathbf{0}$. Applying Lemma 2.1 to this quasi-normal subsequence and $\{\delta_n = 1/(n+1)\}_{n\in\omega}$, we obtain a subsequence $\{f_{m_n}\}_{n\in\omega} \subset \{f_m\}_{m\in\omega}$ such that for each $x \in X$, $f_{m_n}(x) < \delta_n = 1/(n+1)$ for all but finitely many $n \in \omega$. This shows that $\{V_{n,m_n}\}_{n\in\omega}$ (hence $\{U_{n,m_n}\}_{n\in\omega}$) is a γ-cover of X. ∎

Combining Theorems 2.2 and 1.4, we obtain the following (so Problems 1 and 3 in [2] coincide):

COROLLARY 2.3. *Properties* $\mathrm{S}_1(\Gamma,\Gamma)$, SSP* *and* wQN* *are all equivalent.*

3. Properties $\mathrm{S}_1(\Gamma,\Omega)$ and $\mathrm{S}_{\mathrm{fin}}(\Gamma,\Omega)$

DEFINITION 3.1 ([5]). A family $\mathcal{A}$ of subsets of a set X is an *ω-cover* of X if every finite subset of X is contained in some member of $\mathcal{A}$ and X is not a member of $\mathcal{A}$. A space X has *property* $\mathrm{S}_1(\Gamma,\Omega)$ (resp. $\mathrm{S}_{\mathrm{fin}}(\Gamma,\Omega)$) if for every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open γ-covers of X, there are $U_n \in \mathcal{U}_n$ (resp. finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$) ($n \in \omega$) such that $\{U_n\}_{n\in\omega}$ (resp. $\bigcup_{n\in\omega} \mathcal{V}_n$) is an ω-cover of X.

Obviously the following implications hold:

$$\mathrm{S}_1(\Gamma,\Gamma) \Rightarrow \mathrm{S}_1(\Gamma,\Omega) \Rightarrow \mathrm{S}_{\mathrm{fin}}(\Gamma,\Omega).$$

The following is easy to show, so we omit the proof.

LEMMA 3.2. *If $\mathcal{U}$ is an ω-cover of a set X, then every finite subset of X is contained in infinitely many members of $\mathcal{U}$.*

We denote by $[X]^{<\omega}$ the set of all finite subsets of a set X.

THEOREM 3.3. *The following properties of a space X are equivalent.*

(1) $\mathrm{S}_{\mathrm{fin}}(\Gamma,\Omega)$,
(2) *If* $\{f_{n,m}\}_{n,m\in\omega} \subset \mathrm{USC}_p(X,\mathbb{I})$ *and for each* $n \in \omega$, $f_{n,m} \to \mathbf{0}$ $(m \to \infty)$, *then there is* $\varphi \in \omega^\omega$ *with* $\mathbf{0} \in \overline{\{f_{n,m} : n \in \omega,\ m \le \varphi(n)\}}$ *in* $\mathrm{USC}_p(X,\mathbb{I})$,
(3) *If* $\{f_m\}_{m\in\omega} \subset \mathrm{USC}_p(X,\mathbb{I})$ *and* $f_m \to \mathbf{0}$, *then there is a sequence* $\{\varepsilon_m\}_{m\in\omega} \subset (0,1)$ *converging to* 0 *such that for every* $F \in [X]^{<\omega}$ *there is* $m \in \omega$ *with* $\max\{f_m(x) : x \in F\} < \varepsilon_m$.

Proof. (1)⇒(2). Assume $\{f_{n,m}\}_{n,m\in\omega} \subset \mathrm{USC}_p(X,\mathbb{I})$ and for each $n \in \omega$, $f_{n,m} \to \mathbf{0}$ $(m \to \infty)$. For each $n, m \in \omega$, let $U_{n,m} = \{x \in X : f_{n,m}(x) < 1/(n+1)\}$. Since each $f_{n,m}$ is upper semicontinuous, $U_{n,m}$ is open in X. Let $\mathcal{U}_n = \{U_{n,m} : m \in \omega\}$. If there are infinitely many $n \in \omega$ with $X \in \mathcal{U}_n$, then we can take a sequence $\{f_{n_j,m_j}\}_{j\in\omega}$ which converges uniformly to $\mathbf{0}$. Therefore we may assume $X \notin \mathcal{U}_n$ for every $n \in \omega$. Hence each $\mathcal{U}_n$ is an open γ-cover of X. Using property $\mathrm{S}_{\mathrm{fin}}(\Gamma,\Omega)$, we can take $\varphi \in \omega^\omega$ such that $\mathcal{U} = \{U_{n,m} : n \in \omega,\, m \le \varphi(n)\}$ is an ω-cover of X. Let $F \in [X]^{<\omega}$ and let $\varepsilon > 0$. By Lemma 3.2, F is contained in infinitely many members of $\mathcal{U}$, hence there are $n, m \in \omega$ such that $F \subset U_{n,m}$, $m \le \varphi(n)$ and $1/(n+1) < \varepsilon$. Then for every $x \in F$, $f_{n,m}(x) < 1/(n+1) < \varepsilon$. This shows $\mathbf{0} \in \overline{\{f_{n,m} : n \in \omega,\, m \le \varphi(n)\}}$.

(2)⇒(3). Assume that $\{f_m\}_{m\in\omega} \subset \mathrm{USC}_p(X,\mathbb{I})$ and $f_m \to \mathbf{0}$. For each $n, m \in \omega$, let $g_{n,m} = \min\{1, (n+1)f_m\}$. Then $g_{n,m} \in \mathrm{USC}_p(X,\mathbb{I})$ and $g_{n,m} \to \mathbf{0}$ $(m \to \infty)$. We take $\varphi \in \omega^\omega$ with $\mathbf{0} \in \overline{\{g_{n,m} : n \in \omega,\, m \le \varphi(n)\}}$. We may assume that φ is strictly increasing. We define a sequence $\{\varepsilon_m\}_{m\in\omega} \subset (0,1)$ as follows:

$$\varepsilon_m = \begin{cases} 1/2 & \text{if } m \le \varphi(0), \\ 1/(n+2) & \text{if } \varphi(n) < m \le \varphi(n+1)\ (n \in \omega). \end{cases}$$

Note that $\{\varepsilon_m\}_{m\in\omega}$ is decreasing and $\varepsilon_{\varphi(n)} = 1/(n+1)$ $(n \ge 1)$. Let $F \in [X]^{<\omega}$. Take $g_{n,m}$ such that $m \le \varphi(n)$ and $\max\{g_{n,m}(x) : x \in F\} < 1$. Then $\max\{f_m(x) : x \in F\} < 1/(n+1) = \varepsilon_{\varphi(n)} \le \varepsilon_m$.

(3)⇒(1). This can be proved by similar arguments to the proof of Theorem 2.2. For each $n \in \omega$, let $\mathcal{U}_n = \{U_{n,m} : m \in \omega\}$ be an open γ-cover of X. For each $n, m \in \omega$, we put $V_{n,m} = U_{0,m} \cap \cdots \cap U_{n,m}$ and let $\mathcal{V}_n = \{V_{n,m} : m \in \omega\}$. Each $\mathcal{V}_n$ is an open γ-cover of X. We define $f_m : X \to [0,1]$ as follows:

$$f_m(x) = \begin{cases} 1 & \text{if } x \in X \setminus V_{0,m}, \\ 1/(k+2) & \text{if } x \in V_{k,m} \setminus V_{k+1,m}\ (k \in \omega), \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_m \in \mathrm{USC}_p(X,\mathbb{I})$ and $f_m \to \mathbf{0}$. We take a sequence $\{\varepsilon_m\}_{m\in\omega} \subset (0,1)$ converging to 0 such that for every $F \in [X]^{<\omega}$ there is $m \in \omega$ with $\max\{f_m(x) : x \in F\} < \varepsilon_m$. Note that $1/(n+2) < \varepsilon_m \le 1/(n+1)$ implies $f_m^{-1}([0,\varepsilon_m)) = V_{n,m}$. For each $n \in \omega$, let

$$\mathcal{V}'_n = \{V_{n,m} : m \in \omega,\, 1/(n+2) < \varepsilon_m \le 1/(n+1)\}.$$

Since $\{\varepsilon_m\}_{m\in\omega}$ converges to 0, each $\mathcal{V}'_n$ is a finite subfamily of $\mathcal{V}_n$. We observe that $\bigcup_{n\in\omega} \mathcal{V}'_n$ is an ω-cover of X. Let $F \in [X]^{<\omega}$. Then there is

$m \in \omega$ with $\max\{f_m(x) : x \in F\} < \varepsilon_m$. Take $n \in \omega$ with $1/(n+2) < \varepsilon_m \le 1/(n+1)$. Then $F \subset V_{n,m} \in \mathcal{V}'_n$. Consequently, $\bigcup_{n\in\omega}\{U_{n,m} : m \in \omega, 1/(n+2) < \varepsilon_m \le 1/(n+1)\}$ is an ω-cover of X. ∎

THEOREM 3.4. *The following properties of a space X are equivalent.*

(1) $\mathrm{S}_1(\Gamma, \Omega)$.
(2) *If* $\{f_{n,m}\}_{n,m\in\omega} \subset \mathrm{USC}_p(X, \mathbb{I})$ *and for each* $n \in \omega$, $f_{n,m} \to \mathbf{0}$ $(m \to \infty)$, *then there is* $\varphi \in \omega^\omega$ *with* $\mathbf{0} \in \overline{\{f_{n,\varphi(n)} : n \in \omega\}}$ *in* $\mathrm{USC}_p(X, \mathbb{I})$.
(3) *If* $\{f_m\}_{m\in\omega} \subset \mathrm{USC}_p(X, \mathbb{I})$, $f_m \to \mathbf{0}$ *and* $\{\varepsilon_m\}_{m\in\omega} \subset (0, 1)$ *is a convergent sequence to* 0, *then there is* $\varphi \in \omega^\omega$ *such that for every* $F \in [X]^{<\omega}$ *there is* $m \in \omega$ *with* $\max\{f_{\varphi(m)}(x) : x \in F\} < \varepsilon_m$.

Proof. (1)⇒(2). Assume $\{f_{n,m}\}_{n,m\in\omega} \subset \mathrm{USC}_p(X, \mathbb{I})$ and for each $n \in \omega$, $f_{n,m} \to \mathbf{0}$ $(m \to \infty)$. For each $n, m \in \omega$, let $U_{n,m} = \{x \in X : f_{n,m}(x) < 1/(n+1)\}$. Since each $f_{n,m}$ is upper semicontinuous, $U_{n,m}$ is open in X. Let $\mathcal{U}_n = \{U_{n,m} : m \in \omega\}$. By the same argument as in the proof of Theorem 3.3, we may assume that each $\mathcal{U}_n$ is an open γ-cover of X. Using property $\mathrm{S}_1(\Gamma, \Omega)$, we take $\varphi \in \omega^\omega$ such that $\mathcal{U} = \{U_{n,\varphi(n)} : n \in \omega\}$ is an ω-cover of X. Let $F \in [X]^{<\omega}$ and let $\varepsilon > 0$. By Lemma 3.2, there is $n \in \omega$ such that $F \subset U_{n,\varphi(n)}$ and $1/(n+1) < \varepsilon$. This shows $\mathbf{0} \in \overline{\{f_{n,\varphi(n)} : n \in \omega\}}$.

(2)⇒(3). Assume $\{f_m\}_{m\in\omega} \subset \mathrm{USC}_p(X, \mathbb{I})$, $f_m \to \mathbf{0}$ and let $\{\varepsilon_m\}_{m\in\omega} \subset (0, 1)$ be a convergent sequence to 0. For each $n, m \in \omega$, let $g_{n,m} = \min\{1, (1/\varepsilon_n)f_m\}$. Then $g_{n,m} \in \mathrm{USC}_p(X, \mathbb{I})$ and $g_{n,m} \to \mathbf{0}$ $(m \to \infty)$. We take $\varphi \in \omega^\omega$ with $\mathbf{0} \in \overline{\{g_{n,\varphi(n)} : n \in \omega\}}$. Let $F \in [X]^{<\omega}$. Take $g_{m,\varphi(m)}$ with $\max\{g_{m,\varphi(m)}(x) : x \in F\} < 1$. Then $\max\{f_{\varphi(m)}(x) : x \in F\} < \varepsilon_m$.

(3)⇒(1). This can also be proved by similar arguments to the proof of Theorem 2.2. For each $n \in \omega$, let $\mathcal{U}_n = \{U_{n,m} : m \in \omega\}$ be an open γ-cover of X. For each $n, m \in \omega$, we put $V_{n,m} = U_{0,m} \cap \cdots \cap U_{n,m}$ and let $\mathcal{V}_n = \{V_{n,m} : m \in \omega\}$. Each $\mathcal{V}_n$ is an open γ-cover of X. We define $f_m : X \to [0, 1]$ as follows:

$$f_m(x) = \begin{cases} 1 & \text{if } x \in X \setminus V_{0,m}, \\ 1/(k+2) & \text{if } x \in V_{k,m} \setminus V_{k+1,m}\ (k \in \omega), \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_m \in \mathrm{USC}_p(X, \mathbb{I})$ and $f_m \to \mathbf{0}$. For the sequences $\{f_m\}_{m\in\omega}$ and $\{\varepsilon_0 = 1/2, \varepsilon_m = 1/(m+1)\}_{m\ge 1}$, there is $\varphi \in \omega^\omega$ such that for every $F \in [X]^{<\omega}$ there is $m \in \omega$ with $\max\{f_{\varphi(m)}(x) : x \in F\} < \varepsilon_m$. Note that the condition $\max\{f_{\varphi(m)}(x) : x \in F\} < \varepsilon_m$ implies $F \subset V_{m,\varphi(m)}$. Therefore $\{V_{n,\varphi(n)} : n \in \omega\}$ (hence $\{U_{n,\varphi(n)} : n \in \omega\}$) is an ω-cover of X. ∎

REFERENCES

[1] Z. Bukovská, *Thin sets in trigonometrical series and quasinormal convergence*, Math. Slovaca 40 (1990), 53–62.
[2] L. Bukovský, *On* wQN$_*$ *and* wQN* *spaces*, Topology Appl. 156 (2008), 24–27.
[3] L. Bukovský, I. Recław and M. Repický, *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, ibid. 41 (1991), 25–40.
[4] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.
[5] M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology Appl. 69 (1996), 31–62.
[6] —, *A sequential property of* $C_p(X)$ *and a covering property of Hurewicz*, Proc. Amer. Math. Soc. 125 (1997), 2789–2795.
[7] —, *Sequential convergence in* $C_p(X)$ *and a covering property*, East-West J. Math. 1 (1999), 207–214.

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