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DIVISORS, PARTITIONS AND SOME NEW q-SERIES IDENTITIES

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Abstract. We obtain new q-series identities that have interesting interpretations in terms of divisors and partitions. We present a proof of a theorem of Z. B. Wang, R. Fokkink, and W. Fokkink, which follows as a corollary to our main q-series identity, and offer a similar result.

1. Introduction. In the history of partition theory, divisor functions have played an important part in understanding different partition functions. Perhaps one of the more well-known results in partition theory relating divisors to the partition function $p(n)$, the number of unrestricted partitions of n, is the simple and elegant recurrence equation due to Euler (see [2, p. 108]), which can be paraphrased from the identity

$$\sum_{n=0}^{\infty} np(n)q^n = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}. \tag{1.1}$$

It follows directly that the right hand side of (1.1) is the product of the generating functions for $\sigma_1(n) = \sum_{d|n} d$ and $p(n)$, and consequently,

$$np(n) = \sum_{k=1}^{n} \sigma_1(k)p(n-k). \tag{1.2}$$

The q-series identities established in this paper have deep relations to partitions, divisors, and recurrence equations for partitions. In particular, we obtain a result that is of a similar nature to that of Euler's formula. We also obtain some results that are related to a type of "middle" divisor function that appears in [1, 5]. Lastly, we consider a theorem of Z. B. Wang, R. Fokkink, and W. Fokkink, as well as a new theorem that is similar in form.

For the combinatorial considerations (the generating function interpretations) convergence is not an issue, but to make sense analytically $|q|$ is taken to be smaller than 1.

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We use the standard notation [4]

$$(x)_n = (x;q)_n := \prod_{k=1}^{n}(1-xq^{k-1}), \tag{1.3}$$

$$(x)_\infty = (x;q)_\infty := \prod_{k=1}^{\infty}(1-xq^{k-1}). \tag{1.4}$$

2. Results. In establishing our main results, we will need two known general results for q-series. The first of these can be found in [1] and can be seen as an extension of Abel's lemma.

PROPOSITION 2.1. *Suppose that* $f(z) = \sum_{n=0}^{\infty}\alpha(n)z^n$ *is analytic for* $|z|<1$. *If* α *is a complex number for which*

$$\text{(1)} \quad \sum_{n=0}^{\infty}(\alpha-\alpha(n)) < \infty,$$

$$\text{(2)} \quad \lim_{n\to\infty} n(\alpha-\alpha(n)) = 0,$$

then

$$\lim_{z\to 1-}\frac{d}{dz}(1-z)f(z) = \sum_{n=0}^{\infty}(\alpha-\alpha(n)).$$

PROPOSITION 2.2.

$$\sum_{n=0}^{\infty}\frac{(\alpha q)_{2n}(\beta q)_n z^n}{(\alpha q)_n (q)_n} = \frac{(\beta z q)_\infty}{(z)_\infty}\sum_{n=0}^{\infty}\frac{(\beta q)_n (z)_n}{(q)_n(\beta z q)_{2n}}(-\alpha z)^n q^{n(3n+1)/2}. \tag{2.1}$$

This is an identity that can be found in Fine's book [4, eq. (25.96)].

THEOREM 2.3.

$$\begin{aligned}\sum_{n=0}^{\infty}\bigg(\frac{(\beta q)_\infty}{(q)_\infty} - \frac{(\alpha q)_{2n}(\beta q)_n}{(\alpha q)_n (q)_n}\bigg) &\\ = \frac{(\beta q)_\infty}{(q)_\infty}\bigg(&\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} - \beta\sum_{n=0}^{\infty}\frac{q^n}{1-\beta q^n} \\ &- \sum_{n=1}^{\infty}\frac{(\beta q)_n}{(1-q^n)(\beta q)_{2n}}(-\alpha)^n q^{n(3n+1)/2}\bigg).\end{aligned} \tag{2.2}$$

Proof. Applying Propositions 2.2 and 2.1, we have

$$\begin{aligned}
\lim_{z\to 1-} \frac{d}{dz}(1-z)\,\frac{(\beta z q)_\infty}{(z)_\infty}\sum_{n=0}^{\infty}\frac{(\beta q)_n (z)_n}{(q)_n(\beta z q)_{2n}}\,(-\alpha z)^n q^{n(3n+1)/2}& \\
= \frac{(\beta q)_\infty}{(q)_\infty}\Bigg(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} - \beta\sum_{n=0}^{\infty}\frac{q^n}{1-\beta q^n}& \\
+ \lim_{z\to 1-}\frac{d}{dz}\sum_{n=0}^{\infty}\frac{(z)_n(\beta q)_n}{(q)_n(\beta z q)_{2n}}(-\alpha z)^n q^{n(3n+1)/2}\Bigg)& \\
= \frac{(\beta q)_\infty}{(q)_\infty}\Bigg(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} - \beta\sum_{n=0}^{\infty}\frac{q^n}{1-\beta q^n}& \\
- \sum_{n=1}^{\infty}\frac{(\beta q)_n}{(1-q^n)(\beta q)_{2n}}(-\alpha)^n q^{n(3n+1)/2}\Bigg).& \quad\blacksquare
\end{aligned}$$

COROLLARY 2.4.

$$\begin{aligned}
\sum_{n=0}^{\infty}\Bigg(\frac{1}{(q)_\infty} - \frac{(\alpha q^{n+1})_n}{(q)_n}\Bigg)& \\
= \frac{1}{(q)_\infty}\Bigg(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} - \sum_{n=1}^{\infty}\frac{(-\alpha)^n q^{n(3n+1)/2}}{(1-q^n)}\Bigg).&
\end{aligned} \tag{2.3}$$

This follows by simply setting $\beta = 0$ in Theorem 2.3. It can be seen that (2.3) reveals some interesting results in the elementary theory of partitions for $\alpha = -1$.

THEOREM 2.5. *Let $p(n)$ denote the number of partitions of n. Moreover, let $j(N,n)$ denote the partitions of N where parts are less than or equal to $2n$, and parts greater than n and less than or equal to $2n$ are distinct. Lastly, let $l(n)$ denote the number of odd divisors of n that occur outside the interval $[\sqrt{2n/3}, \sqrt{6n}]$. Then*

$$(N+1)p(N) - \sum_{n=1}^{N} j(N,n) = \sum_{k=1}^{N} p(k)(d(N-k) - l(N-k)). \tag{2.4}$$

Proof. Recall that the generating function for $p(n)$ is given by

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n.$$

It can be seen that (see [3])

$$\frac{(-aq^{n+1})_n}{(q)_n}$$

is the generating function for the partitions where parts are less than or equal to $2n$, and parts greater than n and less than or equal to $2n$ are distinct. Moreover, the exponent of a is the number of parts greater than n. We now choose $\alpha = -1$ in (2.3) and observe that the coefficient of q^N on the left hand side is precisely

$$(N+1)p(N) - \sum_{n=0}^{N} j(N,n).$$

Now the generating function for the divisor function $d(n)$ is given by

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} d(n)q^n.$$

The generating function for $l(n)$ (defined in Theorem 2.5) is seen to be

$$\sum_{n=1}^{\infty} (q^{n(3n+1)/2} + q^{n(3n+3)/2} + q^{n(3n+5)/2} + \cdots) = \sum_{n=1}^{\infty} \frac{q^{n(3n+1)/2}}{(1-q^n)}.$$

It is clear to see that coefficient of q^N on the right hand side is

$$\sum_{n=0}^{N} p(n)(d(N-n) - l(N-n)).$$

This completes the proof. ■

Corollary 2.6.

$$\sum_{n=0}^{\infty} \big(1 - (1-\alpha q^{n+1})(1-\alpha q^{n+2}) \cdots (1-\alpha q^{2n})\big)$$
$$= -\sum_{n=1}^{\infty} \frac{(-\alpha)^n q^{n(3n+1)/2}}{(1-q^n)(1-q^{n+1}) \cdots (1-q^{2n})}.$$

Proof. This is the $\beta = 1$ case of (2.2). Combinatorially, both sides give the same generating function. In particular, a partition into distinct parts $\geq n+1$, and with all other parts $\leq 2n$, is easily seen to be

$$(1-\alpha q^{n+1})(1-\alpha q^{n+2}) \cdots (1-\alpha q^{2n}),$$

where α keeps track of the number of parts.

It is left to the reader to show the combinatorial equivalence of the two series in Corollary 2.6. ■

In the work of Z. B. Wang, R. Fokkink, and W. Fokkink (see [6]) it has been shown that $b_o(n) - b_e(n) = d(n)$ for all positive natural numbers n. Here $b_o(n)$ (resp. $b_e(n)$) denotes the sum of the smallest parts in all the partitions of n into an odd (resp. even) number of distinct parts. For example, the four partitions of 6 into distinct parts are 6, $5+1$, $4+2$, $3+2+1$. So

$b_o(6) = 6 + 1 = 7$ and $b_e(6) = 1 + 2 = 3$. On the other hand, the total number of divisors of 6 is $4 = 7 - 3$. We now state this as the following theorem and offer an alternative proof to the one given in [6].

THEOREM 2.7. *Let $b_o(n)$ (resp. $b_e(n)$) denote the sum of the smallest parts in all the partitions of n into an odd (resp. even) number of distinct parts. Moreover, let $d(n)$ denote the classical divisor function. If n is a positive integer, then*

$$b_o(n) - b_e(n) = d(n). \tag{2.5}$$

Proof. It turns out that this result is an immediate corollary of

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q)_\infty} - \frac{1}{(q)_n}\right) = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}, \tag{2.6}$$

which is the case $\alpha = 0$ of (2.3). To see this we multiply both sides by $(q)_\infty$ to obtain

$$\sum_{n=0}^{\infty}(1 - (q^{n+1})_\infty) = \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}. \tag{2.7}$$

The right hand side of (2.7) is clearly the generating function for $d(n)$. On the left hand side we see that $1-(q^{n+1})_\infty$ counts every partition into distinct parts each $\geq n+1$ with $+1$ if there are an odd number of parts and -1 if there are an even number of parts. We find that a given partition will be counted each time $n+1$ does not exceed the smallest part, i.e. it will be counted in

$$\sum_{n=0}^{\infty}(1 - (q^{n+1})_\infty)$$

exactly as many times as the size of its smallest part. Consequently,

$$\sum_{n=0}^{\infty}(1 - (q^{n+1})_\infty) = \sum_{n=1}^{\infty}(b_o(n) - b_e(n))q^n,$$

and this proves that

$$b_o(n) - b_e(n) = d(n). \blacksquare$$

Now we consider a partition theorem very similar in form to the theorem we have just proved.

THEOREM 2.8. *Let $U_o(n)$ (resp. $U_e(n)$) denote the sum of the smallest parts in all the partitions of n into an odd (resp. even) number of parts, where all parts less than twice the smallest minus one appear zero or two times, and parts greater than or equal to twice the smallest minus one are distinct. Then*

$$U_o(n) - U_e(n) = d(n) - l(n).$$

Proof. First, set $\alpha = -1$ in (2.3) and multiply both sides by $(q)_\infty$ to get

$$\sum_{n=0}^{\infty}(1-(-q^{n+1})_n(q^{n+1})_\infty) = \sum_{n=1}^{\infty}\frac{q^n}{1-q^n} - \sum_{n=1}^{\infty}\frac{q^{n(3n+1)/2}}{(1-q^n)}. \tag{2.8}$$

Now we can write the left hand side as

$$\sum_{n=0}^{\infty}(1-(1-q^{n+n+1+1})(1-q^{n+n+2+2})\cdots(1-q^{2n+2n})(1-q^{2n+1})\cdots).$$

Now the product is similar to the generating function in the previous proof, the only difference being that parts $\geq n+1$ and less than or equal to $2n$ appear zero or two times, parts $> 2n$ being distinct. The result now follows after recalling the generating functions for $d(n)$ and $l(n)$. ■

3. Conclusion. It is clear that we have not seen the full extent of the results obtainable from the extension of Abel's lemma. The results contained herein illustrate some of the variety of partition theorems obtainable from Proposition 2.1.

REFERENCES

[1] G. E. Andrews, J. Jiménez-Urroz, and K. Ono, *q-series identities and values of certain L-functions*, Duke Math. J. 108 (2001), 395–419.

[2] B. C. Berndt, *Ramanujan's Notebooks*, *Part IV*, Springer, New York, 1994.

[3] B. C. Berndt and A. J. Yee, *Combinatorial proofs of identities in Ramanujan's lost notebook associated with the Rogers–Fine identity and false theta functions*, Ann. Combin. 7 (2003), 409–423.

[4] N. J. Fine, *Basic Hypergeometric Series and Applications*, Math. Surveys Monogr. 27, Amer. Math. Soc., Providence, RI, 1988.

[5] J. Lovejoy, *Rank and conjugation for the Frobenius representation of an overpartition*, Ann. Combin. 9 (2005), 321–334.

[6] Z. B. Wang, R. Fokkink, and W. Fokkink, *A relation between partitions and the number of divisors*, Amer. Math. Monthly 102 (1995), 345–347.

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