COLLOQUIUM MATHEMATICUM

VOL. 117 2009 NO. 2

ADDENDUM TO "GENERALIZED RADICAL RINGS, UNKNOTTED BIQUANDLES, AND QUANTUM GROUPS"

(COLLOQ. MATH. 109 (2007), 85–100)

BY

WOLFGANG RUMP (Stuttgart)

Dedicated to B. V. M.

Semigroups of I-type arose in the study of Sklyanin algebras [11, 12, 5, 13], a class of algebras related to an elliptic curve which allow a presentation by a set of non-commuting generators $x_1, \dots, x_n$ with $\binom{n}{2}$ quadratic relations $x_i x_j = x_k x_l$. Such relations (with extra conditions) define a semigroup of I-type. The corresponding semigroup ring has nice homological properties (see [3, Theorem 1.4]). Semigroups of I-type and their quotient groups, also called *groups of I-type*, are investigated in [3, 4].

Cycle sets [6] were introduced to study set-theoretic solutions [2] of the quantum Yang–Baxter equation. A *cycle set* is a set X with a binary operation $\cdot$ such that the left multiplication $y \mapsto x \cdot y$ is bijective and the equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{1}$$

holds for all $x, y, z \in X$. The relation (1) occurred earlier in Bosbach's investigations on the positive cone of a lattice-ordered group [1]. Here, (1) arises for $x \cdot y := 1 \vee x^{-1} y$. In algebraic logic, the equation appears if $x \cdot y$ is regarded as an implication $x \to y$ (see [14]). For relationships between cycle sets and other mathematical structures, see [10, 7, 8, 9] and the literature cited there.

Theorem 1 of [9] states that groups of I-type are equivalent to cycle sets. However, the result was doubted by the reviewer (1) since our proof was not complete. In the present note, we prove the missing equation by a simple geometric argument, and show its equivalence to an important property of

2000 *Mathematics Subject Classification*: Primary 20M05, 20M10, 16T20; Secondary 81R50, 14A22, 14H52.

Key words and phrases: group of I-type, quantum Yang–Baxter equation, cycle set.

(1) MR2308827 (2008g:16066).

DOI: 10.4064/cm117-2-12

any group G_X of I-type, namely, that the natural map $G_X \to S(X)$ into the symmetric group on the generators of G_X is a group homomorphism.

Let X be a finite set. A *group of I-type* is given by a second, not necessarily commutative, group structure $G_X := (\mathbb{Z}^{(X)}, \circ)$ on the free abelian group $\mathbb{Z}^{(X)}$, such that the neutral elements of both groups coincide, and

$$\{x \circ a \mid x \in X\} = \{x + a \mid x \in X\} \tag{2}$$

for all $a \in \mathbb{Z}^{(X)}$. If we restrict $\mathbb{Z}^{(X)}$ to the free abelian monoid $\mathbb{N}^{(X)}$, the monoid $(\mathbb{N}^{(X)}, \circ)$ is called a *semigroup of I-type.*

Thus every element $a \in \mathbb{Z}^{(X)}$ gives rise to a permutation $\sigma(a)$ of X such that

$$\sigma(a)(x) \circ a = x + a \tag{3}$$

for all $x \in X$. So we have a map

$$\sigma \colon G_X \to S(X). \tag{4}$$

In particular, this gives a binary operation on X via

$$x \cdot y := \sigma(x)(y). \tag{5}$$

Now we can state our equivalence result.

THEOREM. *Every group $(\mathbb{Z}^{(X)}, \circ)$ of I-type defines a cycle set $(X, \cdot)$, and up to isomorphism, this correspondence is bijective.*

Proof. Let $(\mathbb{Z}^{(X)}, \circ)$ be a group of I-type. We regard $\mathbb{Z}^{(X)}$ as the set of vertices of an infinite quiver $Q(X)$ with set of arrows $X \times \mathbb{Z}^{(X)}$, such that the arrow $(x, a) \in X \times \mathbb{Z}^{(X)}$ starts at a and ends at $x + a$. The map (4) can thus be regarded as a colouring of the edges

$$c \colon X \times \mathbb{Z}^{(X)} \to X \tag{6}$$

with $c(x, a) := \sigma(a)(x)$. For any path

$$a \xrightarrow{e_0} a_1 \to \cdots \to a_n \xrightarrow{e_n} b$$

in $Q(X)$, this implies that

$$b = c(e_n) \circ \cdots \circ c(e_0) \circ a. \tag{7}$$

From (3) and (5), we get the equation

$$(x \cdot y) \circ x = (y \cdot x) \circ y = x + y \tag{8}$$

for $x, y \in X$. On the other hand, (2) implies that for different $x, y \in X$, the equation

$$p \circ x = q \circ y \tag{9}$$

has a unique solution $(p,q) \in X \times X$. By (8), this solution is $(p,q) = (x \cdot y, y \cdot x)$. For any $a \in \mathbb{Z}^{(X)}$, we thus have a mesh

$$
\begin{tikzcd}[column sep=small]
 & a \arrow[dl, "x"'] \arrow[dr, "y"] & \\
x \circ a \arrow[dr, "x\cdot y"'] & & y \circ a \arrow[dl, "y\cdot x"] \\
 & (x+y) \circ a &
\end{tikzcd}
\tag{10}
$$

in the quiver $Q(X)$. Now let $x, y, z \in X$ with $y \neq z$ be given. Then $x\cdot y \neq x\cdot z$, and (10) gives a mesh

$$
\begin{tikzcd}[column sep=small]
 & x \arrow[dl, "x\cdot y"'] \arrow[dr, "x\cdot z"] & \\
x + y \arrow[dr, "(x\cdot y)\cdot(x\cdot z)"'] & & x + z \arrow[dl, "(x\cdot z)\cdot(x\cdot y)"] \\
 & x+y+z &
\end{tikzcd}
$$

in $Q(X)$. Hence $x+y+z = ((x\cdot y)\cdot(x\cdot z))\circ(x\cdot y)\circ x = ((x\cdot y)\cdot(x\cdot z))\circ(x+y)$. By (3), we have $\sigma(x+y)(z) \circ (x+y) = z + (x+y)$. So we get

$$\sigma(x+y)(z) = (x \cdot y) \cdot (x \cdot z) = \sigma(x \cdot y)\sigma(x)(z) \tag{11}$$

for $z \neq y$, hence for all $z \in X$. By symmetry, this yields (1). So we have associated a cycle set to any group of I-type. The bijectivity of this correspondence was proved in [9]. ■

Let us briefly sketch how the converse is proved. For a given cycle set X, there is a natural extension to a cycle set structure on the free abelian group $\mathbb{Z}^{(X)}$ (see [6, Proposition 6]) such that (1) can be replaced by

$$(a+b) \cdot c = (a \cdot b) \cdot (a \cdot c) \tag{12}$$

for all $a, b, c \in \mathbb{Z}^{(X)}$. Furthermore, the extended cycle set satisfies

$$x \cdot (y+z) = (x \cdot y) + (x \cdot z) \tag{13}$$

for all $x, y, z \in X$. The operation $\cdot$ on $\mathbb{Z}^{(X)}$ defines a permutation $\sigma(a)$ on X via (5) for all $a \in \mathbb{Z}^{(X)}$ which relates $\mathbb{Z}^{(X)}$ to a group of I-type by (3).

Remark. In [9], the proof of (1) for a group of I-type makes use of (13) without further comment in a chain of equations which all follow from (8). This creates the impression that (8) might imply (13) too. However, the above proof strongly suggests that (13) cannot be proved without the finiteness assumption on X.

Note that for a group G_X of I-type, it is not even clear in advance whether the map (4) is a group homomorphism, i.e. whether

$$(x \circ y) \cdot z = x \cdot (y \cdot z) \tag{14}$$

for $x, y, z \in X$. Let us show that (13) and (14) are equivalent by mere use of (8).

First, we transform (14) into $((x \circ y) \cdot z) \circ x \circ y = (x \cdot (y \cdot z)) \circ x \circ y$, that is, $(x \circ y) + z = (x + y \cdot z) \circ y$. Replacing x by $y \cdot x$, we get

$$((y \cdot x) \circ y) + z = ((y \cdot x) + (y \cdot z)) \circ y.$$

The left-hand side of this equation is $y + x + z = (y \cdot (x + z)) \circ y$. Thus (14) becomes equivalent to $y \cdot (x + z) = (y \cdot x) + (y \cdot z)$, that is, (13).

REFERENCES

[1] B. Bosbach, *Concerning cone algebras*, Algebra Universalis 15 (1982), 58–66.

[2] V. G. Drinfeld, *On some unsolved problems in quantum group theory*, in: Quantum Groups (Leningrad, 1990), Lecture Notes in Math. 1510, Springer, Berlin, 1992, 1–8.

[3] T. Gateva-Ivanova and M. Van den Bergh, *Semigroups of I-type*, J. Algebra 206 (1998), 97–112.

[4] E. Jespers and J. Okniński, *Monoids and groups of I-type*, Algebras Represent. Theory 8 (2005), 709–729.

[5] A. V. Odesskiĭ and B. L. Feĭgin, *Sklyanin's elliptic algebras*, Funktsional. Anal. i Prilozhen. 23 (1989), no. 3, 45–54, 96 (in Russian); English transl.: Funct. Anal. Appl. 23 (1989), 207–214.

[6] W. Rump, *A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation*, Adv. Math. 193 (2005), 40–55.

[7] —, *Braces, radical rings, and the quantum Yang–Baxter equation*, J. Algebra 307 (2007), 153–170.

[8] —, *Classification of cyclic braces*, J. Pure Appl. Algebra 209 (2007), 671–685.

[9] —, *Generalized radical rings, unknotted biquandles, and quantum groups*, Colloq. Math. 109 (2007), 85–100.

[10] —, *L-algebras, self-similarity, and l-groups*, J. Algebra 320 (2008), 2328–2348.

[11] E. K. Sklyanin, *Some algebraic structures connected with the Yang–Baxter equation*, Funktsional. Anal. i Prilozhen. 16 (1982), no. 4, 27–34, 96 (in Russian); English transl.: Funct. Anal. Appl. 16 (1982), 263–270.

[12] —, *Some algebraic structures connected with the Yang–Baxter equation, II. Representations of a quantum algebra*, Funktsional. Anal. i Prilozhen. 17 (1983), no. 4, 34–48 (in Russian); English transl.: Funct. Anal. Appl. 17 (1983), 273–284.

[13] J. Tate and M. Van den Bergh, *Homological properties of Sklyanin algebras*, Invent. Math. 124 (1996), 619–647.

[14] T. Traczyk, *On the structure of BCK-algebras with $zx \cdot yx = zy \cdot xy$*, Math. Japon. 33 (1988), 319–324.

Institut für Algebra und Zahlentheorie
Universität Stuttgart
Pfaffenwaldring 57
D-70550 Stuttgart, Germany
E-mail: rump@mathematik.uni-stuttgart.de

Received 7 March 2009;
revised 8 May 2009 (5191)