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MANIFOLDS WITH A UNIQUE EMBEDDING

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Abstract. We show that if X, Y are smooth, compact k-dimensional submanifolds of $\mathbb{R}^n$ and $2k+2 \le n$, then each diffeomorphism $\phi : X \to Y$ can be extended to a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ which is tame (to be defined in this paper). Moreover, if X, Y are real analytic manifolds and the mapping ϕ is analytic, then we can choose Φ to be also analytic.

We extend this result to some interesting categories of closed (not necessarily compact) subsets of $\mathbb{R}^n$, namely, to the category of Nash submanifolds (with Nash, real-analytic and smooth morphisms) and to the category of closed semi-algebraic subsets of $\mathbb{R}^n$ (with morphisms being semi-algebraic continuous mappings). In each case we assume that X, Y are k-dimensional and ϕ is an isomorphism, and under the same dimension restriction $2k+2 \le n$ we assert that there exists an extension $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ which is an isomorphism and it is tame.

The same is true in the category of smooth algebraic subvarieties of $\mathbb{C}^n$, with morphisms being holomorphic mappings and with morphisms being polynomial mappings.

1. Introduction. A diffeomorphism is said to be a *triangle* diffeomorphism if it is of the form

$$\Phi : \mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n + p(x_1, \ldots, x_{n-1})) \in \mathbb{R}^n,$$

where $p(x_1, \ldots, x_{n-1})$ is a smooth function. A diffeomorphism F which can be obtained as a composition of triangle diffeomorphisms and linear automorphisms with determinant 1 is called *tame.* Of course, a tame diffeomorphism is diffeotopic to the identity and *it preserves the volume.*

Let X be a smooth manifold. We say that two embeddings $f, g : X \to \mathbb{R}^n$ are *equivalent* if there is a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $g = \Phi \circ f$. If additionally Φ is a tame diffeomorphism, we say that f, g are *tamely equivalent.* If any two embeddings of X into $\mathbb{R}^n$ are equivalent (resp. tamely equivalent) we say that X has a *unique* (resp. *tamely unique*) *embedding* into $\mathbb{R}^n$.

For example if $X = \mathbb{S}^1$ is a circle, then X has infinitely many non-equivalent embeddings into $\mathbb{R}^3$ (every knot gives a non-standard embedding). It is of interest to find sufficient conditions for a manifold X to have

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a unique or tamely unique embedding into $\mathbb{R}^n$. One can deduce from Whitney's paper [13] that if X is a compact k-dimensional smooth manifold, then any two smooth embeddings $f, g : X \to \mathbb{R}^n$, where $n \geq 2k+2$, are equivalent. In this note we improve this result and show that in fact in this case any two smooth embeddings $f, g : X \to \mathbb{R}^n$ are tamely equivalent. Moreover, we show that if X is *a compact real analytic manifold* and f, g are *real analytic embeddings*, then we can find Φ that is *a tame real analytic isomorphism* $\Phi : \mathbb{R}^n \to \mathbb{R}^n$.

Of course, the same question can be posed for a larger class of categories. In particular, in [6], [7], [9] and [12] this problem was solved for the category of smooth complex algebraic affine varieties (where morphisms are polynomial mappings). The second aim of this paper is to generalize (and simplify) these results to the case of some other interesting *pseudo-algebraic categories* (see Definitions 4.3 and 4.11). Examples of pseudo-algebraic categories are: the category of Nash (i.e., analytic and semi-algebraic) submanifolds of $\mathbb{R}^n$ with Nash (i.e., analytic and semi-algebraic) mappings as morphisms, the category of Nash submanifolds of $\mathbb{R}^n$ with smooth mappings as morphisms, the category of Nash submanifolds of $\mathbb{R}^n$ with real analytic mappings as morphisms, the category of smooth complex affine subvarieties of $\mathbb{C}^n$ with holomorphic (or polynomial, or smooth) mappings as morphisms.

In particular, we prove that if X, Y are Nash k-dimensional submanifolds of $\mathbb{R}^n$ (where $n \geq 2k+2$) and $\phi : X \to Y$ is a diffeomorphism (resp. Nash isomorphism, real-analytic isomorphism), then ϕ can be extended to a tame diffeomorphism (resp. Nash isomorphism, real-analytic isomorphism) $\Phi : \mathbb{R}^n \to \mathbb{R}^n$.

We also prove that if X, Y are k-dimensional smooth algebraic subvarieties of $\mathbb{C}^n$ (where $n \geq 2k+2$), and $\phi : X \to Y$ is a biholomorphism, then ϕ can be extended to a global tame biholomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$.

Finally, we show this theorem for the category of closed semi-algebraic sets with continuous semi-algebraic mappings as morphisms. More precisely, we show: if X, Y are k-dimensional closed semi-algebraic subsets of $\mathbb{R}^n$ (where $n \geq 2k+2$), and $\phi : X \to Y$ is a semi-algebraic homeomorphism, then ϕ can be extended to a global tame semi-algebraic homeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ (in particular, X and Y are homeotopic). This is a semi-algebraic counterpart of the classical theorem of Hermann Gluck on extension of homeomorphisms of polyhedrons (see [5]).

We also give examples of $k = n+1$-dimensional Nash manifolds $X_k \subset \mathbb{R}^{2n}$ (where n is any even number different from $2, 4, 8$) which have at least two different embeddings into $\mathbb{R}^{2n}$. This shows that our results cannot be much improved for large n. Note also that for $k = 1$ and $n = 3$ our result (about Nash manifolds) is optimal.

2. Preliminaries. We start with the following basic definition:

DEFINITION 2.1. Let X, Y be smooth manifolds and let $f : X \to Y$ be a smooth morphism. We say that the mapping f is an *embedding* if

- $f(X)$ is a closed submanifold of Y,
- the mapping $f : X \to f(X)$ is a diffeomorphism.

Let Y be a smooth manifold (all manifolds we consider here may be disconnected). We will denote by $\mathbf{C}(Y)$ the algebra of all smooth functions on Y. If $f : X \to Y$ is a smooth morphism of smooth manifolds, then we have the natural homomorphism $f^* : \mathbf{C}(Y) \ni h \mapsto h \circ f \in \mathbf{C}(X)$.

Let $X \subset Y$ be a closed submanifold. Using a partition of unity it is easy to see that every function on X is the restriction of some function on Y. Consequently, the mapping $i^* : \mathbf{C}(Y) \to \mathbf{C}(X)$ induced by the inclusion $i : X \to Y$ is an epimorphism. In fact, we have the following more general:

PROPOSITION 2.2. *Let X, Y be smooth manifolds and $f : X \to Y$ be a smooth map. The following conditions are equivalent*:

(1) *f is an embedding*,
(2) *the induced mapping $f^* : \mathbf{C}(Y) \to \mathbf{C}(X)$ is an epimorphism*,
(3) *f is proper*, *injective and $d_x f : T_x X \to T_{f(x)} Y$ is a monomorphism for every $x \in X$.*

Proof. (1)$\Rightarrow$(2). This follows from the remarks above.

(2)$\Rightarrow$(3). We can assume that X is embedded in some $\mathbb{R}^N$ (as a closed submanifold). Let $x_1, \ldots, x_N$ be coordinates on $\mathbb{R}^N$. By the assumption we can find smooth functions $H_i \in \mathbf{C}(Y)$ such that $x_i = H_i \circ f$ (on X). Put $H = (H_1, \ldots, H_N)$. We have identity $= H \circ f$. This easily implies that f is injective and proper. Moreover, after computing the derivatives of both sides we have

$$\text{identity} = d_{f(x)} H \circ d_x f,$$

which easily implies that $d_x f$ is a monomorphism.

(3)$\Rightarrow$(1). This is well known from differential geometry. ∎

3. Smooth and analytic compact case. In this section we will prove our first main result. To do this we need a series of lemmas:

LEMMA 3.1. *Let X be a submanifold of $\mathbb{R}^n$. Assume that the projection $\pi : X \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_l, 0, \ldots, 0) \in \mathbb{R}^l \times \{0\}$ is an embedding. Then there exists a tame diffeomorphism $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Pi|_X = \pi$.*

Proof. Let $X' := \pi(X)$; it is a closed submanifold of $\mathbb{R}^n$. Consider the mapping $\pi : X \to X' \subset \mathbb{R}^n$. It is an embedding, so $\pi^* : \mathbf{C}(\mathbb{R}^n) \to \mathbf{C}(X)$ is an epimorphism. In particular, for every $k > l$ there exists a function

$p_k \in \mathbf{C}(\mathbb{R}^n)$ such that $x_k = p_k(x_1, \ldots, x_l)$ (on X). Then the mapping

$\Pi(x_1, \ldots, x_n) = (x_1, \ldots, x_l, x_{l+1} - p_{l+1}(x_1, \ldots, x_l), \ldots, x_n - p_n(x_1, \ldots, x_l))$

is a tame diffeomorphism of $\mathbb{R}^n$ and $\Pi|_X = \pi$. ∎

The next lemma is a smooth variant of the Bertini Theorem:

LEMMA 3.2. *Let X be a smooth manifold. Let $f : X \to \mathbb{P}^m$ be a smooth morphism. Then there is a subset $E \subset \mathbb{P}^{m*}$ of measure* 0 *such that if H is a projective hyperplane and $H \notin E$, then $f^{-1}(H)$ is a smooth submanifold of X.*

Proof. First assume that $f : X \to \mathbb{R}^m$. Hence $f = (f_1, \ldots, f_m)$ and f_i are smooth functions. Consider the mapping

$$\Psi : X \times \mathbb{R}^m \ni (x, (\lambda_1, \ldots, \lambda_m)) \mapsto \Big(\sum_{i=1}^{n} \lambda_i f_i(x), (\lambda_1, \ldots, \lambda_m)\Big) \in \mathbb{R} \times \mathbb{R}^m.$$

Now our conclusion follows from the Sard Theorem (see [10]).

To prove the general case let $H_1, \ldots, H_{m+1} \subset \mathbb{P}^m$ be hyperplanes in general position. By the previous result the preimage of a general hyperplane is smooth in each open subset $U_i = X \setminus f^{-1}(H_i)$. Since the sets $\{U_i\}$ cover X the lemma follows. ∎

We also need the following:

LEMMA 3.3. *Let X be a compact submanifold of $\mathbb{R}^n$ of dimension k. If $n > 2k + 1$, then there exists a system of coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ such that the projection $\pi : X \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{2k+1}, 0, \ldots, 0) \in \mathbb{R}^{2k+1} \times \{0\}$ is an embedding.*

Proof. Let us denote by π_∞ the hyperplane at infinity of $\mathbb{R}^n$. Thus $\pi_\infty \cong \mathbb{P}^{n-1}$ is a real projective space of dimension $n-1 > 2k$. For a non-zero vector $v \in \mathbb{R}^n$ let $[v]$ denote the corresponding point in $\mathbb{P}^{n-1}$.

Let $\Delta = \{(x, y) \in X \times X : x = y\}$ and let TX denote the tangent bundle of X. Set $TX' = TX \setminus X \times \{0\}$. Consider the mappings

$$A : X \times X \setminus \Delta \ni (x, y) \mapsto [x - y] \in \pi_\infty$$

and

$$B : TX' \ni (x, v) \mapsto [v] \in \pi_\infty.$$

Since A, B are smooth and the manifolds $X \times X \setminus \Delta$ and TX are of dimension $2k$, the Sard Theorem (see [10]) implies $\pi_\infty \setminus (A(X \times X \setminus \Delta) \cup B(TX')) \neq \emptyset$. Let $P \in \pi_\infty \setminus (A(X \times X \setminus \Delta) \cup B(TX'))$ and let $H \subset \mathbb{R}^{n-1}$ be a hyperplane which does not contain the point P (at infinity). Thus the projection $S : X \ni x \mapsto \overline{Px} \cap H \in H \cong \mathbb{R}^{n-1}$ is an injective immersion. Since the manifold X is compact the mapping S is also proper. This means by Proposition 2.1 that S is an embedding. Now we can apply induction. ∎

LEMMA 3.4. *Let X be a compact manifold of dimension k. Assume that $X \subset \mathbb{R}^{2n}$, where $n \geq 2k+1$. Assume that the mappings*

$$\pi_1 : X \ni (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n) \in \mathbb{R}^n$$

and

$$\pi_2 : X \ni (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_n) \in \mathbb{R}^n$$

are embeddings. Then there exist linear mappings $S, T \in \mathrm{SL}(n)$ such that if we change the coordinates in $\mathbb{R}^n \times \{0\}$ to $(z_1, \ldots, z_n) = T(x_1, \ldots, x_n)$ and the coordinates in $\{0\} \times \mathbb{R}^n$ to $(w_1, \ldots, w_n) = S(y_1, \ldots, y_n)$, then all projections

$$q_r : X \ni (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (z_1, \ldots, z_r, w_{r+1}, \ldots, w_n) \in \mathbb{R}^n, \quad r = 0, \ldots, n,$$

are embeddings.

Proof. Let us denote by π_∞ the hyperplane at infinity of $\mathbb{R}^n \times \mathbb{R}^n$. Thus $\pi_\infty \cong \mathbb{P}^{2n-1}$ is a real projective space of dimension $2n-1$.

Again let $\Delta = \{(x, y) \in X \times X : x = y\}$ and TX the tangent bundle. Set $TX' = TX \setminus X \times \{0\}$. Consider again the mappings

$$A : X \times X \setminus \Delta \ni (x, y) \mapsto [x - y] \in \pi_\infty$$

and

$$B : TX' \ni (x, v) \mapsto [v] \in \pi_\infty.$$

Define $\Lambda := A(X \times X \setminus \Delta) \cup B(TX') \subset \pi_\infty$. Let $L = (L_1, \ldots, L_n) : \mathbb{R}^{2n} \to \mathbb{R}^n$ be a linear mapping. Set $S(L) := \{x \in \pi_\infty : L_i(x) = 0,\ i = 1, \ldots, n\}$. It is easy to see that $L|X$ is an injective immersion if and only if $\Lambda \cap S = \emptyset$.

Now we show that there are affine coordinates $(z_1, \ldots, z_n)$ in $\mathbb{R}^n \times \{0\}$ and affine coordinates $(w_1, \ldots, w_n)$ in $\{0\} \times \mathbb{R}^n$ such that all projections

$$q_i : X \ni (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (z_1, \ldots, z_{n-i}, w_{n-i+1}, \ldots, w_n) \in \mathbb{R}^n, \quad i = 0, \ldots, n,$$

are embeddings. On π_∞ we have coordinates $(x : y)$. Since $\pi_1|X$ is an embedding we have $\{(x : y) \in \pi_\infty : x_1 = 0, \ldots, x_n = 0\} \cap \Lambda = \emptyset$. Consequently, if we set $\psi : \pi_\infty \ni (x : y) \mapsto x \in \mathbb{P}^{n-1}$ (it is a rational mapping), then the mappings $g := \psi \circ A : X \times X \setminus \Delta \to \mathbb{P}^{n-1}$ and $k := \psi \circ B : TX' \to \mathbb{P}^{n-1}$ are well defined and smooth.

By Lemma 3.2 this means that if $H = \{x \in \mathbb{P}^{n-1} : \sum_{i=1}^n c_i x_i = 0\}$ is a generic hyperplane, then $g^{-1}(H)$ and $k^{-1}(H)$ are smooth submanifolds of $X \times X \setminus \Delta$ and TX', of dimension at most $2k-1$. Set $z_1 = \sum_{i=1}^n c_i x_i$.

Continuing in this fashion we see that we can choose n generic hyperplanes given by equations $z_i = \sum_{k=1}^n a_{i,k} x_k$, $i = 1, \ldots, n$, such that for every $1 \leq r \leq n$ the sets $A^{-1}(\{z_1 = 0, \ldots, z_r = 0\})$ and $B^{-1}(\{z_1 = 0, \ldots, z_r = 0\})$ are smooth submanifolds of $X \times X \setminus \Delta$ and TX' of dimension at most $2k - r$.

In particular, $\dim A^{-1}(\{z_1 = 0, \ldots, z_{n-1} = 0\}) \le 0$ and $\dim B^{-1}(\{z_1 = 0, \ldots, z_{n-1} = 0\}) \le 0$.

Now in the same way we can choose a generic hyperplane given by the equation $w_n = \sum_{k=1}^{n} b_{n,k} y_k$, $i = 1, \ldots, n$, such that $A^{-1}(\{z_1 = 0, \ldots, z_{n-1} = 0, w_n = 0\}) = \emptyset$ and $B^{-1}(\{z_1 = 0, \ldots, z_{n-1} = 0, w_n = 0\}) = \emptyset$ and additionally for every $0 \le r \le n-1$ we have $\dim A^{-1}(\{z_1 = 0, \ldots, z_r = 0, w_n = 0\}) \le 2k - r - 1$ and $\dim B^{-1}(\{z_1 = 0, \ldots, z_r = 0, w_n = 0\}) \le 2k - r - 1$. Further we can construct $w_{n-1} = \sum_{k=1}^{n} b_{n-1,k} y_k$, $i = 1, \ldots, n$, such that $A^{-1}(\{z_1 = 0, \ldots, z_{n-2} = 0, w_{n-1} = 0, w_n = 0,\}) = \emptyset$ and $B^{-1}(\{z_1 = 0, \ldots, z_{n-2} = 0, w_{n-1} = 0, w_n = 0\}) = \emptyset$ and additionally for every $0 \le r \le n-2$ we have $\dim A^{-1}(\{w_1 = 0, \ldots, w_r = 0,\ z_{n-1} = 0,\ z_n = 0\}) \le 2k - r - 2$ and $\dim B^{-1}(\{z_1 = 0, \ldots, z_r = 0,\ w_{n-1} = 0,\ w_n = 0\}) \le 2k - r - 2$. Continuing in this manner we find a system of coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_n)$ we are looking for: for all $0 \le r \le n$ we have

$$\Lambda \cap \{z_1 = 0, \ldots, z_r = 0,\ w_{r+1} = 0, \ldots, w_n = 0\} = \emptyset,$$

which implies that the mapping

$$q_r : X \ni (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (z_1, \ldots, z_r, w_{r+1}, \ldots, w_n) \in \mathbb{R}^n, \qquad r = 0, \ldots, n,$$

is an immersion. Since X is compact the mapping q_r is an embedding. Moreover, we can always assume (by the construction) that the transformations $T : (x_1, \ldots, x_n) \mapsto (z_1, \ldots, z_n)$ and $S : (y_1, \ldots, y_n) \mapsto (w_1, \ldots, w_n)$ are from $\mathrm{SL}(n)$. ∎

Now we are in a position to prove the first main result of this section:

THEOREM 3.5. *Let X be a compact smooth submanifold of $\mathbb{R}^n$ of dimension k. Let $f : X \to \mathbb{R}^n$ be an embedding. If $n \ge 2k+2$, then there exists a tame diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F|_X = f$.*

Proof. Apply Lemma 3.3 to X and $f(X)$. Then in virtue of Lemma 3.1 we can assume that there exist tame diffeomorphisms $A, B : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(X) \subset \mathbb{R}^{2k+1} \times \{0\}$ and $B(f(X)) \subset \{0\} \times \mathbb{R}^{2k+1}$ (if necessary we compose A and B with suitable affine transformations with determinant 1). Consider $f' = B \circ f \circ A^{-1}$; of course we can assume that $f = f'$. In particular, we can assume that $X \subset \mathbb{R}^{2k+1} \times \{0\}$ and $f(X) \subset \{0\} \times \mathbb{R}^{2k+1}$ and that $n = 2k+2$. Thus $f = (0, f_1, \ldots, f_{n-1})$.

Applying Lemma 3.4 to the set $X' = \mathrm{graph}(f) \subset \mathbb{R}^{2k+1} \times \mathbb{R}^{2k+1}$ we see that there are linear transformations $T, S \in \mathrm{SL}(n-1)$ such that if we put $(z_1, \ldots, z_{n-1}) = T(x_1, \ldots, x_{n-1})$ and $(f_1'(z), \ldots, f_{n-1}'(z)) = S(f_1 \circ T^{-1}(z), \ldots, f_{n-1} \circ T^{-1}(z))$ then all mappings

$$q_r' : X \ni (z_1, \ldots, z_{n-1}, 0) \mapsto (z_1, \ldots, z_r, f_{r+1}'(z), \ldots, f_{n-1}'(z)) \in \mathbb{R}^{n-1}$$

are embeddings (as compositions of the standard diffeomorphism $X \to \operatorname{graph}(f)$ with q_r; notation as in Lemma 3.4).

Now we construct a sequence of tame diffeomorphisms $A_{n-1}, A_{n-2}, \ldots, A_1, A_0$ and $B_{n-1}, B_{n-2}, \ldots, B_1, B_0$ such that for $z \in X$ we have

$$B_r \circ A_r \circ \cdots \circ B_{n-1} \circ A_{n-1}(x) = (z_1, \ldots, z_r, 0, f'_{r+1}(z), \ldots, f'_{n-1}(z)).$$

We proceed by induction. For $r = n-1$ it is enough to put $A_{n-1} = B_{n-1} =$ identity. Now assume that $1 \le r < n-1$ and we have constructed a sequence of tame diffeomorphisms $A_r, \ldots, A_{n-1}$ and $B_r, \ldots, B_{n-1}$ such that for $z \in X$ we have

$$B_r \circ A_r \circ \cdots \circ B_{n-1} \circ A_{n-1}(x) = (z_1, \ldots, z_r, 0, f'_{r+1}(z), \ldots, f'_{n-1}(z)).$$

We show how to construct A_{r-1} and B_{r-1}. Note that the mapping

$$q'_r : X \ni (z_1, \ldots, z_{n-1}, 0) \mapsto (z_1, \ldots, z_r, f'_{r+1}(z), \ldots, f'_{n-1}(z)) \in \mathbb{R}^{n-1}$$

is an embedding. Consequently, there exists a smooth function P_{r-1} such that

$$f'_r(z) = P_{r-1}(z_1, \ldots, z_r, f'_{r+1}(z), \ldots, f'_{n-1}(z)).$$

Consider the tame diffeomorphism A_{r-1} :

$$(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_r, t_{r+1} + P_{r-1}(t_1, \ldots, t_r, t_{r+2}, \ldots, t_n), t_{r+2}, \ldots, t_n).$$

Thus for $z \in X$ we have

$$A_{r-1}(z_1, \ldots, z_r, 0, f'_{r+1}(z), \ldots, f'_{n-1}(z)) = (z_1, \ldots, z_r, f'_r(z), \ldots, f'_{n-1}(z)).$$

Since the mapping $(z_1, \ldots, z_{r-1}, f'_r(z), \ldots, f'_{n-1}(z))$ restricted to X is an embedding, there exists a smooth function Q_{r-1} such that

$$z_r = Q_{r-1}(z_1, \ldots, z_{r-1}, f'_r(z), \ldots, f'_{n-1}(z)).$$

Consider the tame diffeomorphism

$$B_{r-1} : \mathbb{R}^n \ni (t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_r - Q_{r-1}(t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_n), t_{r+1}, \ldots, t_n) \in \mathbb{R}^n.$$

For $z \in X$ we have

$$B_{r-1} \circ A_{r-1}(z_1, \ldots, z_r, 0, f'_{r+1}(z), \ldots, f'_{n-1}(z)) = (z_1, \ldots, z_{r-1}, 0, f'_r(z), \ldots, f'_{n-1}(z)).$$

Finally, by induction we obtain a sequence of tame diffeomorphisms $A_{n-1}, A_{n-2}, \ldots, A_1, A_0$ and $B_{n-1}, B_{n-2}, \ldots, B_1, B_0$ such that for $z \in X$ we have

$$B_0 \circ A_0 \circ \cdots \circ B_{n-1} \circ A_{n-1}(x) = (0, f'_1(z), \ldots, f'_{n-1}(z)).$$

If we take $T_1(x_1, \ldots, x_n) = (T(x_1, \ldots, x_{n-1}), x_n)$ and $S_1(y_1, \ldots, y_n) = (y_1, S(y_2, \ldots, y_n))$, then

$$S_1^{-1} \circ B_0 \circ A_0 \circ \cdots \circ B_{n-1} \circ A_{n-1} \circ T_1(x) = (0, f_1(x), \ldots, f_{n-1}(x)).$$

Now it is enough to put $F = S_1^{-1} \circ B_0 \circ A_0 \circ \cdots \circ B_{n-1} \circ A_{n-1} \circ T_1$. ∎

Corollary 3.6. *With the preceding notation, there is a smooth family of tame diffeomorphisms* $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *such that* $F_0 =$ identity *and* $F_1|_X = f$.

Proof. Indeed, every triangle diffeomorphism

$$G = (x_1, \ldots, x_{n-1}, x_n + P_n(x_1, \ldots, x_{n-1}))$$

is diffeotopic to the identity by

$$t \mapsto G_t = (x_1, \ldots, x_{n-1}, x_n + tP_n(x_1, \ldots, x_{n-1})).$$

The same is true for any linear mapping with determinant 1, since such a mapping is a product of triangular linear mappings. ∎

Corollary 3.7. *Let X be a compact smooth manifold of dimension k. In $n \geq 2k + 2$, then X has a (tamely) unique embedding into $\mathbb{R}^n$.*

Now note that we can repeat this proof for real analytic submanifolds of $\mathbb{R}^n$ nearly word for word, with one exception: we need the fact that if $f : X \to \mathbb{R}$ is a real analytic function on a real analytic manifold X, then we can extend f to a real analytic function $F : \mathbb{R}^n \to \mathbb{R}$. This follows from a result of Cartan (see [3, p. 89]). In this way we have the following interesting:

Theorem 3.8. *Let $X \subset \mathbb{R}^n$ be a compact real analytic submanifold of dimension k. Let $f : X \to \mathbb{R}^n$ be a real analytic embedding. If $n \geq 2k + 2$, then f can be extended to a tame real analytic isomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$.*

Corollary 3.9. *With the preceding notation, there is an analytic family of tame analytic isomorphisms* $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *such that* $F_0 =$ identity *and* $F_1|_X = f$.

Corollary 3.10. *Let X be a compact real analytic manifold of dimension k. If $n \geq 2k + 2$, then X has a (tamely) unique analytic embedding into $\mathbb{R}^n$.*

Example 3.11. As shown by the example of a non-trivial knot $f : S^1 \to \mathbb{R}^3$ (note that we can take f real analytic) the assumption $n \geq 2k + 2$ in Theorems 3.5 and 3.8 is essential.

4. Real pseudo-algebraic categories. Now we apply our results to other categories of manifolds. In this section we will assume that the underlying field is the real numbers. We denote by $\mathbf{S}_0$ the category of all pairs $(X, \mathbb{R}^{n(X)})$, where $X \subset \mathbb{R}^{n(X)}$ is a smooth closed submanifold of $\mathbb{R}^{n(X)}$ and morphisms are smooth mappings. Let $\mathbf{S}$ be a subcategory of $\mathbf{S}_0$. Every object of $\mathbf{S}$ is a pair $(X, \mathbb{R}^{n(X)})$; we will identify it simply with X. In particular, we will identify $(\mathbb{R}^n, \mathbb{R}^n)$ with $\mathbb{R}^n$. We start with:

Definition 4.1. Let $\mathbf{S}$ be as above and let $X, Y \in \mathbf{S}$. We say that a mapping $f : X \to Y$ is an $\mathbf{S}$-*embedding* if

- $(f(X), \mathbb{R}^{n(Y)}) \in \mathbf{S}$,
- f as well as $f^{-1} : f(X) \to X \subset \mathbb{R}^{n(X)}$ are $\mathbf{S}$-morphisms.

Remark 4.2. In particular, an $\mathbf{S}$-embedding is always a smooth embedding (see Definition 2.1).

Definition 4.3. We say that $\mathbf{S}$ is a *fine category* if:

(1) $\mathbb{R}^n \in \mathbf{S}$ for every $n \in \mathbb{N}$,
(2) if $f : X \to \mathbb{R}$ is in $\mathbf{S}$, then f can be extended to a mapping $F : \mathbb{R}^{n(X)} \to \mathbb{R}$ which is also in $\mathbf{S}$,
(3) the linear mappings are in $\mathbf{S}$, and if $X \in \mathbf{S}$, then the restrictions to X of mappings from $\mathbf{S}$ are in $\mathbf{S}$,
(4) if $X \in \mathbf{S}$, then the set $\mathbf{CS}(X) = \{f : X \to \mathbb{R};\ f \in \mathbf{S}\}$ is an $\mathbb{R}$-algebra,
(5) if $X \in \mathbf{S}$ and $\pi : X \to \mathbb{R}^n$ is a linear projection which is a smooth embedding, then π is an $\mathbf{S}$-embedding,
(6) if $X \in \mathbf{S}$ and $f : X \to \mathbb{R}^n$ and $g : X \to \mathbb{R}^m$ are $\mathbf{S}$-morphisms, then $(f, g) : X \to \mathbb{R}^n \times \mathbb{R}^m$ is also an $\mathbf{S}$-morphism,
(7) if $f : X \to \mathbb{R}^n$ is in $\mathbf{S}$, then $(\mathrm{graph}(f), \mathbb{R}^{n(X)+k}) \in \mathbf{S}$.

The following definition will be crucial (see e.g. [2]):

Definition 4.4. We say that a submanifold $X \subset \mathbb{R}^n$ is a *Nash manifold* if X is a real analytic manifold and a closed semi-algebraic subset of $\mathbb{R}^n$. Moreover, if X, Y are Nash manifolds and $f : X \to Y$ is a mapping, then f is a Nash mapping if f is real analytic and semi-algebraic.

Example 4.5. Examples of fine categories are: the category $\mathbf{S}_0$ itself, the category $\mathbf{RA}$ of smooth real analytic submanifolds with real analytic mappings as morphisms (it satisfies (2) by [3]) and the category $\mathbf{NA}$ of Nash submanifolds with Nash mappings as morphisms (it satisfies (2) by [2, Corollary 8.9.13]).

A simple but important consequence of Definition 4.3 is:

Proposition 4.6. *Let $\mathbf{S}$ be a fine category and let $X \in \mathbf{S}$. If $f : X \to \mathbb{R}^n$ is an $\mathbf{S}$-embedding, then the induced mapping $f^* : \mathbf{CS}(\mathbb{R}^n) \to \mathbf{CS}(X)$ is an epimorphism.*

Proof. Indeed, let $Y = f(X)$. By definition $Y \in \mathbf{S}$ and the mapping

$$a : \mathbf{CS}(Y) \ni \alpha \mapsto \alpha \circ f \in \mathbf{CS}(X)$$

is an isomorphism. Now let $i : Y \to \mathbb{R}^n$ be the inclusion. Since every $\mathbf{S}$-function $\sigma : Y \to \mathbb{R}$ can be extended to a global $\mathbf{S}$-function $\Sigma : \mathbb{R}^n \to \mathbb{R}$, the mapping $i^* : \mathbf{CS}(\mathbb{R}^n) \to \mathbf{CS}(Y)$ is an epimorphism. But $f^* = a \circ i^*$. ∎

Now we have an (obvious) generalization of Lemma 3.1:

Lemma 4.7. *Let $\mathbf{S}$ be a fine category and let $(X, \mathbb{R}^n) \in \mathbf{S}$ be a submanifold. Assume that the projection $\pi : X \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_l, 0, \ldots, 0)$*

$\in \mathbb{R}^l \times \{0\}$ *is an embedding. Then there exists a tame* $\mathbf{S}$*-diffeomorphism* $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\Pi|_X = \pi$.

We will need the following:

DEFINITION 4.8. For a hyperplane $H \subset \mathbb{P}^n$ let us consider the Zariski open affine set $U_H = \mathbb{P}^n \setminus H$. We will say that U_H is a *standard open affine subset* of $\mathbb{P}^n$. Now let X be a semi-algebraic set and let $f : X \to \mathbb{P}^n$ be a mapping. We say that the mapping f is *projectively semi-algebraic* if for every standard affine set $U_H \subset \mathbb{P}^n$ the set $f^{-1}(U_H)$ is semi-algebraic and the mapping

$$f|_{f^{-1}(U_H)} : f^{-1}(U_H) \to U_H \cong \mathbb{R}^n$$

is semi-algebraic.

The next lemma is a semi-algebraic variant of Lemma 3.3:

LEMMA 4.9. *Let X be a semi-algebraic submanifold of $\mathbb{R}^n$ of dimension k. If $n > 2k+1$, then there exists a system of coordinates $(x_1, \dots, x_{2k+1}, x_{2k+2}, \dots, x_n)$ on $\mathbb{R}^n$ such that the projection $\pi : X \ni (x_1, \dots, x_{2k+1}, x_{2k+2}, \dots, x_n) \mapsto (x_1, \dots, x_{2k+1}, 0, \dots, 0) \in \mathbb{R}^{2k+1} \times \{0\}$ is an embedding.*

Proof. We follow closely the proof of Lemma 3.3, using the same notation. Since A, B are now semi-algebraic mappings and the manifolds $X \times X \setminus \Delta$ and TX' are of dimension $2k$, the set $\Lambda := A(X \times X \setminus \Delta) \cup B(TX)$ is a projectively semi-algebraic set (i.e., it is semi-algebraic in every standard affine open subset of $\mathbb{P}^{n-1}$) of dimension at most $2k$. Hence so is its closure Σ (for details see e.g. [1]). Consequently, $\pi_\infty \setminus \Sigma \neq \emptyset$.

Let $P \in \pi_\infty \setminus \Sigma$ and let $H \subset \mathbb{R}^{n-1}$ be a hyperplane which does not contain the point P (at infinity). Since $P \notin \Lambda$, the projection $S : X \ni x \mapsto \overline{Px} \cap H \in H \cong \mathbb{R}^{n-1}$ is an immersion. Moreover, since $P \notin \Sigma$, this projection is also proper, hence it is an embedding.

Now we can apply induction. ∎

LEMMA 4.10. *Let* $\mathbf{S}$ *be a fine category. Let* $(X, \mathbb{R}^{2n}) \in \mathbf{S}$, *where* $n \geq 2k+1$. *Assume that the mappings*

$$\pi_1 : X \ni (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (x_1, \dots, x_n) \in \mathbb{R}^n$$

and

$$\pi_2 : X \ni (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (y_1, \dots, y_n) \in \mathbb{R}^n$$

are (closed) embeddings and the submanifolds $\pi_1(X) = \Omega_1$ and $\pi_2(X) = \Omega_2$ are semi-algebraic. Then there exist linear mappings $S, Q \in \mathrm{SL}(n)$ such that if we change the coordinates in $\mathbb{R}^n \times \{0\}$ to $(z_1, \dots, z_n) = Q(x_1, \dots, x_n)$ and the coordinates in $\{0\} \times \mathbb{R}^n$ to $(w_1, \dots, w_n) = S(y_1, \dots, y_n)$, then all projections

$$q_r : X \ni (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (z_1, \ldots, z_r, w_{r+1}, \ldots, w_n) \in \mathbb{R}^n, \qquad r = 0, \ldots, n,$$

are $\mathbf{S}$*-embeddings.*

Proof. As in the proof of Lemma 3.4, we can show that if $(z_1, \ldots, z_n)$ in $\mathbb{R}^n \times \{0\}$ and $(w_1, \ldots, w_n)$ in $\{0\} \times \mathbb{R}^n$ are sufficiently generic affine coordinates, then all projections

$$q_i : X \ni (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (z_1, \ldots, z_{n-i}, w_{n-i+1}, \ldots, w_n) \in \mathbb{R}^n, \qquad i = 0, \ldots, n,$$

are immersions. The key point now is to prove that they are also proper. Let Ω_1' be the (euclidean) projective closure of Ω_1 in $\mathbb{P}^n$ and take $W_1 = \Omega_1' \setminus \Omega_1$. Define W_2 analogously. Of course W_1, W_2 are algebraic sets of dimension $k-1$.

We can choose coordinates $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ so that additionally

$$\dim W_1 \cap \{z_1 = 0, \ldots, z_t = 0\} \le k - 1 - t \quad \text{for } t = 1, \ldots, k,$$

and

$$\dim W_2 \cap \{w_n = 0, \ldots, w_{n-t} = 0\} \le k - t - 2 \quad \text{for } t = 0, 1, \ldots, k-1.$$

This implies that the mappings

$$T : \Omega_1 \ni (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_k) \in \mathbb{R}^k$$

and

$$R : X_2 \ni (w_1, \ldots, w_n) \mapsto (w_{n-k+1}, \ldots, w_n) \in \mathbb{R}^k$$

are proper. Consequently, the projection

$$P_1 = T \circ \pi_1 : X \ni (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (z_1, \ldots, z_k) \in \mathbb{R}^k$$

is proper. Similarly the projection

$$P_2 = R \circ \pi_1 : X \ni (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (w_{n-k+1}, \ldots, w_n) \in \mathbb{R}^k$$

is proper. Set

$$T_r : \mathbb{R}^n \ni (z_1, \ldots, z_r, w_{r+1}, \ldots, w_n) \mapsto (z_1, \ldots, z_r) \in \mathbb{R}^r$$

and

$$R_r : \mathbb{R}^n \ni (z_1, \ldots, z_r, w_{r+1}, \ldots, w_n) \mapsto (w_{r+1}, \ldots, w_n) \in \mathbb{R}^{n-r}.$$

It is easy to see that for every r either $T_r \circ q_r$ is proper (if $r \ge k$) or $R_r \circ q_r$ is proper (if $r < k$ and hence $r + 1 < n - k + 1$). In both cases this implies that q_r is proper. This finishes the proof. ∎

Definition 4.11. Let $\mathbf{S}$ be a fine category and let $\mathbf{S}' \subset \mathbf{S}$ be a subcategory. We say that $\mathbf{S}'$ is a *pseudo-algebraic subcategory in* $\mathbf{S}$ (or briefly a *pseudo-algebraic category*) if

- $\mathbb{R}^n \in \mathbf{S}'$ for every $n \in \mathbb{N}$,
- if $X \in \mathbf{S}'$, then X is a Nash manifold,
- if X, Y in $\mathbf{S}'$ then $\mathrm{Mor}_{\mathbf{S}}(X,Y) = \mathrm{Mor}_{\mathbf{S}'}(X,Y)$,

where $\mathrm{Mor}_{\mathbf{S}}(X,Y) = \{f \in \mathbf{S}; f : X \to Y\}$.

Now we can repeat word for word the proof of Theorem 3.5 to obtain:

THEOREM 4.12. *Let $\mathbf{S}'$ be a pseudo-algebraic category and let X, Y, $\in \mathbf{S}'$ be smooth manifolds of dimension k. Let $f : X \to Y$ be an $\mathbf{S}'$-diffeomorphism. If $n \geq 2k+2$, then f can be extended to a tame $\mathbf{S}'$-diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$. Moreover, there is a smooth family of tame $\mathbf{S}'$-diffeomorphisms $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ such that $F_0 = \text{identity}$ and $F_1|_X = f$.*

It is easy to check that the following categories are pseudo-algebraic: the category of Nash submanifolds of $\mathbb{R}^n$ with Nash mappings as morphisms (here $\mathbf{S}' = \mathbf{NA}$), the category of Nash submanifolds of $\mathbb{R}^n$ with real analytic mappings as morphisms (here $\mathbf{S}' \subset \mathbf{RA}$), and the category of Nash submanifolds of $\mathbb{R}^n$ with smooth mappings as morphisms (here $\mathbf{S}' \subset \mathbf{S}_0$). In particular, we have:

THEOREM 4.13. *Let $X, Y \subset \mathbb{R}^n$ be Nash manifolds of dimension k. Let $f : X \to Y$ be a diffeomorphism. If $n \geq 2k+2$, then f can be extended to a tame diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$. Moreover, there is a smooth family of tame diffeomorphisms $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ such that $F_0 = \text{identity}$ and $F_1|_X = f$.*

This gives the following nice application to complex algebraic varieties:

COROLLARY 4.14. *Let $X, Y \subset \mathbb{C}^n$ be smooth complex algebraic manifolds of complex dimension k. Let $f : X \to Y$ be a diffeomorphism. If $n \geq 2k+1$, then f can be extended to a tame diffeomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$. In particular, if two smooth algebraic complex curves $X, Y \subset \mathbb{C}^3$ are diffeomorphic, then they are embedded into $\mathbb{C}^3$ in the same way (up to a diffeomorphism).*

Proof. Indeed, we can treat X, Y as $2k$-dimensional real algebraic smooth submanifolds of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. By the assumption $2n \geq 2(2k+1) = 2(2k)+2$. ∎

We also have the analytic variant of Theorem 4.13:

THEOREM 4.15. *Let $X, Y \subset \mathbb{R}^n$ be Nash (not necessarily connected) submanifolds of dimension (not necessarily pure) k. Let $f : X \to Y$ be a Nash isomorphism (resp. real analytic isomorphism). If $n \geq 2k+2$, then f can be extended to a tame Nash isomorphism (resp. tame real analytic isomorphism) $F : \mathbb{R}^n \to \mathbb{R}^n$. Moreover, there is an analytic family of tame Nash isomorphisms (resp. tame real analytic isomorphisms) $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ such that $F_0 = \text{identity}$ and $F_1|_X = f$.*

EXAMPLE 4.16. Since some non-trivial knots $f : \mathbb{R} \to \mathbb{R}^3$ can be realized as polynomial embeddings (see e.g. [11]), we see that the assumption $n \geq 2k + 2$ in Theorems 4.13 and 4.15 is optimal.

5. Complex pseudo-algebraic categories. Now assume that the underlying field is the complex field. We let $\mathbf{S}_0$ denote the category of all pairs $(X, \mathbb{C}^{n(X)})$, where $X \subset \mathbb{C}^{n(X)}$ is a smooth closed submanifold of $\mathbb{C}^{n(X)}$ and morphisms are smooth mappings. Every object of $\mathbf{S}_0$ is a pair $(X, \mathbb{C}^{n(X)})$; we will identify it simply with X. In particular, we will identify $(\mathbb{C}^n, \mathbb{C}^n)$ with $\mathbb{C}^n$. Let $\mathbf{S}$ be a subcategory of $\mathbf{S}_0$. We can easily extend Definition 4.3 to:

DEFINITION 5.1. We say that $\mathbf{S}$ is a *fine category* if:

(1) $\mathbb{C}^n \in \mathbf{S}$ for every $n \in \mathbb{N}$,
(2) if $f : X \to \mathbb{C}$ is in $\mathbf{S}$, then f can be extended to a mapping $F : \mathbb{C}^{n(X)} \to \mathbb{C}$ which is also in $\mathbf{S}$,
(3) the $\mathbb{C}$-linear mappings are in $\mathbf{S}$, and if $X \in \mathbf{S}$, then the restrictions to X of mappings from $\mathbf{S}$ are in $\mathbf{S}$,
(4) if $X \in \mathbf{S}$, then the set $\mathbf{CS}(X) = \{f : X \to \mathbb{C};\ f \in \mathbf{S}\}$ is a $\mathbb{C}$-algebra,
(5) if $X \in \mathbf{S}$ and $\pi : X \to \mathbb{C}^n$ is a linear projection which is a smooth embedding, then π is an $\mathbf{S}$-embedding,
(6) if $X \in \mathbf{S}$ and $f : X \to \mathbb{C}^n$ and $g : X \to \mathbb{C}^m$ are $\mathbf{S}$-morphisms, then $(f, g) : X \to \mathbb{C}^n \times \mathbb{C}^m$ is also an $\mathbf{S}$-morphism,
(7) if $f : X \to \mathbb{C}^n$ is in $\mathbf{S}$, then $(\operatorname{graph}(f), \mathbb{C}^{n(X)+k}) \in \mathbf{S}$.

From now on our models of fine categories will be the category $\mathbf{St}$ of smooth Stein submanifolds $X \subset \mathbb{C}^{n(X)}$ with holomorphic mappings as morphisms and the category $\mathbf{Pl}$ of smooth algebraic submanifolds with polynomial mappings as morphisms. Note that again the fact that the category $\mathbf{St}$ has property (2) is a non-trivial fact, which follows from Cartan's Theorem B (for details see [3]). We can also define a complex pseudo-algebraic category:

DEFINITION 5.2. Let $\mathbf{S}$ be a (complex) fine category and let $\mathbf{S}' \subset \mathbf{S}$ be a subcategory. We say that $\mathbf{S}'$ is a (complex) *pseudo-algebraic subcategory in* $\mathbf{S}$ (or briefly a *pseudo-algebraic category*) if

- $\mathbb{C}^n \in \mathbf{S}'$ for every $n \in \mathbb{N}$,
- if $X \in \mathbf{S}'$, then X is a complex algebraic manifold,
- if X, Y in $\mathbf{S}'$ then $\operatorname{Mor}_{\mathbf{S}}(X, Y) = \operatorname{Mor}_{\mathbf{S}'}(X, Y)$,

where $\operatorname{Mor}_{\mathbf{S}}(X, Y) = \{f \in \mathbf{S}; f : X \to Y\}$.

Now we can repeat word for word the results of the previous section to obtain:

THEOREM 5.3. *Let* $\mathbf{S}'$ *be a pseudo-algebraic category and let* $(X,\mathbb{C}^n)$, $(Y,\mathbb{C}^n)\in\mathbf{S}'$ *be smooth complex manifolds of dimension* k. *Let* $f:X\to Y$ *be an* $\mathbf{S}'$-*diffeomorphism. If* $n\ge 2k+2$, *then* f *can be extended to a tame* $\mathbf{S}'$-*diffeomorphism* $F:\mathbb{C}^n\to\mathbb{C}^n$. *Moreover*, *there is a smooth family of tame* $\mathbf{S}'$-*diffeomorphisms* $F_t:\mathbb{C}^n\times\mathbb{C}\to\mathbb{C}^n$ *such that* $F_0=\text{identity}$ *and* $F_1|_X=f$.

It is easy to check that the category of smooth complex algebraic submanifolds with holomorphic mappings as morphisms is a pseudo-algebraic category (here $\mathbf{S}'\subset\mathbf{S}=\mathbf{St}$). Similarly the category of smooth complex algebraic submanifolds with polynomial mappings as morphisms is a pseudo-algebraic category (here $\mathbf{S}'=\mathbf{S}=\mathbf{Pl}$). In particular, we have:

THEOREM 5.4. *Let* $X,Y\subset\mathbb{C}^n$ *be smooth complex algebraic submanifolds of dimension* k. *Let* $f:X\to Y$ *be a biholomorphism* (*resp. polynomial isomorphism*). *If* $n\ge 2k+2$, *then* f *can be extended to a tame biholomorphism* (*resp. a tame polynomial automorphism*) $F:\mathbb{C}^n\to\mathbb{C}^n$. *Moreover*, *there is a smooth family of tame biholomorphisms* (*resp. tame polynomial isomorphisms*) $F_t:\mathbb{C}^n\times\mathbb{C}\to\mathbb{C}^n$ *such that* $F_0=\text{identity}$ *and* $F_1|_X=f$.

EXAMPLE 5.5 (see [8]). Let $n\ge 4$ be an even number and consider the variety $S_{2n-1}=\{(x,y)\in\mathbb{C}^{2n}:\sum_{i=1}^n x_iy_i=1\}$. Then the embeddings

$$\iota:S_{2n-1}\times\mathbb{C}^2\ni((x,y),(s,t))\mapsto((x,y),s,t,0,\dots,0)\in\mathbb{C}^{2n}\times\mathbb{C}^n$$

and $\phi:S_{2n-1}\times\mathbb{C}^2\to\mathbb{C}^{2n}\times\mathbb{C}^n$ given by

$$((x,y),(s,t))\mapsto$$
$$((x,y),y_1s+x_2t,y_2s-x_1t,y_3s+x_4t,y_4s-x_3t,\dots,y_{n-1}s+x_nt,y_ns-x_{n-1}t)$$

are non-equivalent, i.e., there does not exist a biholomorphism

$$\Phi:\mathbb{C}^{2n}\times\mathbb{C}^n\to\mathbb{C}^{2n}\times\mathbb{C}^n$$

such that $\Phi\circ\iota=\phi$.

EXAMPLE 5.6. For every $n\ge 2$ there is a (closed) holomorphic embedding $f:\mathbb{C}\times\{0,\dots,0\}\to\mathbb{C}^n$ which cannot be extended to a biholomorphism $F:\mathbb{C}^n\to\mathbb{C}^n$ (for details see [4]). Of course, the reason is that the smooth Stein curve $Y=f(\mathbb{C})$ is far from being algebraic.

EXAMPLE 5.7. Let $X,Y\subset\mathbb{C}$ be finite sets of points with $\#X=\#Y\ge 3$. Since every biholomorphism of $\mathbb{C}$ is a $\mathbb{C}$-linear mapping, a general bijection $f:X\to Y$ cannot be extended to a global biholomorphism $F:\mathbb{C}\to\mathbb{C}$. This means that at least for $k=0$ the assumption $n\ge 2k+2$ of Theorem 5.4 is optimal.

6. Semi-algebraic category. To end this paper we consider a category which is not smooth. Let $\mathbf{SE}$ be the category of closed semi-algebraic subsets of $\mathbb{R}^n$, i.e., objects of this category are pairs $(X,\mathbb{R}^{n(X)})$ and $X\subset\mathbb{R}^{n(X)}$ is a closed semi-algebraic subset of $\mathbb{R}^{n(X)}$. The morphisms in **SE** are continuous semi-algebraic mappings. This is a fine category in some sense, since:

(1) $\mathbb{R}^n \in \mathbf{SE}$ for every $n \in \mathbb{N}$,
(2) if $f : X \to \mathbb{R}$ is in **SE**, then f can be extended to a mapping $F : \mathbb{R}^{n(X)} \to \mathbb{R}$ which is also in **SE** (this is a semi-algebraic version of the Tietze Extension Theorem, see e.g. [2, Proposition 2.6.9]),
(3) the linear mappings are in **SE**, and if $X \in \mathbf{SE}$, then the restrictions to X of mappings from **SE** are in **SE**,
(4) if $X \in \mathbf{SE}$, then the set $\mathbf{CSE}(X) = \{f : X \to \mathbb{R};\ f \in \mathbf{SE}\}$ is an $\mathbb{R}$-algebra,
(5) if $X \in \mathbf{SE}$ and $\pi : X \to \mathbb{R}^n$ is a projection which is a topological embedding, then f is an **SE**-embedding,
(6) if $X \in \mathbf{SE}$ and $f : X \to \mathbb{R}^n$ and $g : X \to \mathbb{R}^m$ are **SE**-morphisms, then $(f, g) : X \to \mathbb{R}^n \times \mathbb{R}^m$ is also an **SE**-morphism.
(7) if $f : X \to \mathbb{R}^n$ is in **SE**, then $(\operatorname{graph}(f), \mathbb{R}^{n(X)+k}) \in \mathbf{SE}$.

By Proposition 4.6 we have:

LEMMA 6.1. *If X is a semi-algebraic set and $f : X \to \mathbb{R}^n$ is an* **SE**-*embedding, then the mapping*

$$f^* : \mathbf{CSE}(\mathbb{R}^n) \ni h \mapsto h \circ f \in \mathbf{CSE}(X)$$

is an epimorphism.

Moreover, using basic properties of semi-algebraic sets it is not difficult to prove topological counterparts of Lemmas 4.9, 4.7 and 4.10; the main idea is the same, we have to use the lemma below.

LEMMA 6.2. *Let W be a semi-algebraic subset of $\mathbb{P}(\mathbb{R}^n)$. Let $(x_1, \ldots, x_k)$ be a system of linear homogeneous polynomials on $\mathbb{P}(\mathbb{R}^n)$ with $V(x_1, \ldots, x_k) \cap W = \emptyset$. Then for generic $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ we have*

$$\dim W \cap V\Big(\sum_{i=1}^{k} \lambda_i x_i\Big) \le \dim W - 1.$$

Proof. Let $W = \bigcup_{i=1}^{r} W_i$ be the decomposition of W into irreducible components (in the semi-algebraic sense, see Proposition 2.9.10 of [2]). For simplicity we may suppose that $\dim W_i = \dim W$ for every $i = 1, \ldots, s$.

Let L_i be the linear subspace of $\mathbb{P}^n(\mathbb{R})$ spanned by W_i, $i = 1, \ldots, s$. A hyperplane H satisfies $\dim W \cap H = \dim W$ if and only if it contains some L_i. If $\mathbb{P}(\lambda)$ parametrizes all hyperplanes of the type $\sum_{i=1}^{k} \lambda_i x_i = 0$, then those that contain some L_i form a linear subspace Λ_i of $\mathbb{P}(\lambda)$. By our assumption we have $\Lambda_i \ne \mathbb{P}(\lambda)$ for every i (since otherwise $W_i \subset V(x_1, \ldots, x_k) \cap W$). Hence the union $\bigcup_{i=1}^{s} \Lambda_i$ is a proper subset of $\mathbb{P}(\lambda)$ and the proof is finished. ■

For example we give a proof of:

Lemma 6.3. *Let X be a closed semi-algebraic subset of $\mathbb{R}^n$ of dimension k. If $n > 2k+1$, then there exists a system of coordinates $(x_1, \ldots, x_{2k+1}, x_{2k+2}, \ldots, x_n)$ such that the projection*

$$\pi : X \ni (x_1, \ldots, x_{2k+1}, x_{2k+2}, \ldots, x_n) \mapsto (x_1, \ldots, x_{2k+1}, 0, \ldots, 0) \in \mathbb{R}^{2k+1} \times \{0\}$$

is a topological embedding.

Proof. Again we follow the proof of Lemma 3.3. Since

$$A : X \times X \setminus \Delta \ni (x, y) \mapsto [x - y] \in \pi_\infty$$

is a semi-algebraic mapping and the semi-algebraic set $X \times X \setminus \Delta$ is of dimension $2k$, the set $\Lambda := A(X \times X \setminus \Delta)$ is (projectively) semi-algebraic of dimension at most $2k$, and hence so is its closure Σ (for details see e.g., [1]). Consequently, $\pi_\infty \setminus \Sigma \neq \emptyset$. Now we can finish as in Lemma 4.9. ∎

Lemma 6.4. *Let $(X, \mathbb{R}^n) \in$ **SE** be a closed subset of dimension k. Assume that the projection $\pi : X \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_l, 0, \ldots, 0) \in \mathbb{R}^l \times \{0\}$ is a topological embedding. Then there exists a tame* **SE***-homeomorphism $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Pi|_X = \pi$.* ∎

Lemma 6.5. *Let $(X, \mathbb{R}^{2n}) \in$ **SE**, where $n \geq 2k+1$. Assume that the mappings*

$$\pi_1 : X \ni (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n) \in \mathbb{R}^n$$

and

$$\pi_2 : X \ni (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_n) \in \mathbb{R}^n$$

are (*closed*) *embeddings. Then there exist linear mappings $S, T \in \mathrm{SL}(n)$ such that if we change the coordinates in $\mathbb{R}^n \times \{0\}$ to $(z_1, \ldots, z_n) = T(x_1, \ldots, x_n)$ and the coordinates in $\{0\} \times \mathbb{R}^n$ to $(w_1, \ldots, w_n) = S(y_1, \ldots, y_n)$, then all projections*

$$q_r : X \ni (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (z_1, \ldots, z_r, w_{r+1}, \ldots, w_n) \in \mathbb{R}^n, \quad r = 0, \ldots, n,$$

are closed (*topological*) *embeddings.* ∎

Now we can repeat nearly word for word the proof of Theorem 3.5 to obtain:

Theorem 6.6. *Let $X, Y \subset \mathbb{R}^n$ be closed semi-algebraic subsets of dimension k. Let $f : X \to Y$ be a semi-algebraic homeomorphism. If $n \geq 2k+2$, then f can be extended to a tame semi-algebraic homeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$.*

Since every triangle homeomorphism

$$G = (x_1, \ldots, x_{n-1}, x_n + P_n(x_1, \ldots, x_{n-1}))$$

is homeotopic to the identity by

$$t \mapsto G_t = (x_1, \dots, x_{n-1}, x_n + tP_n(x_1, \dots, x_{n-1}))$$

and the same is true for linear mappings with determinant 1, we have:

COROLLARY 6.7. *Let $X, Y \subset \mathbb{R}^n$ be closed semi-algebraic subsets of dimension k. Let $f : X \to Y$ be a semi-algebraic homeomorphism. If $n \geq 2k+2$, then X, Y are semi-algebraically homeotopic, i.e., there is a continuous semi-algebraic family $t \mapsto G_t$ of semi-algebraic homeomorphisms $G_t : \mathbb{R}^n \to \mathbb{R}^n$ such that $G_0 = \text{identity}$ and $G_1|_X = f$.*

REMARK 6.8. Example 4.16 shows that the assumption $n \geq 2k+2$ in Theorem 6.6 is essential.

REMARK 6.9. The results of this section are semi-algebraic counterparts of the classical topological results of Gluck on extension of homeomorphisms of compact polyhedra (see [5]).

7. Examples. If X is a k-dimensional Nash submanifold of $\mathbb{R}^n$ and $n > 2k+1$ then X has a unique Nash embedding into $\mathbb{R}^n$. We know that for $k = 1$ and $n = 3$ this result is optimal. It is of interest whether it is also optimal for large k and n.

We give examples of Nash manifolds $X_{n+1} \subset \mathbb{R}^{2n}$ (where n is any even number different from $2, 4, 8$) which have at least two different Nash embeddings into $\mathbb{R}^{2n}$. This means that our results cannot be much improved for large n.

THEOREM 7.1. *Let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ be a sphere where n is an even number different from $2, 4, 8$. The embeddings*

$$\iota : \mathbb{S}^{n-1} \times \mathbb{R}^2 \ni (x, (s,t)) \mapsto (x, s, t, 0, \dots, 0) \in \mathbb{R}^n \times \mathbb{R}^n$$

and $\phi : \mathbb{S}^{n-1} \times \mathbb{R}^2 \to \mathbb{R}^n \times \mathbb{R}^n$ given by

$$(x, (s,t)) \mapsto$$
$$(x, x_1 s + x_2 t, x_2 s - x_1 t, x_3 s + x_4 t, x_4 s - x_3 t, \dots, x_{n-1} s + x_n t, x_n s - x_{n-1} t)$$

are non-equivalent, i.e., there does not exist a diffeomorphism

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$$

such that $\Phi \circ \iota = \phi$.

Proof. It is well known that for even $n \neq 2, 4, 8$ the tangent bundle $\mathbb{A} = T\mathbb{S}^{n-1}$ is not trivial. However, since the normal bundle $N(\mathbb{S}^{n-1})$ is trivial, the bundle $\mathbb{A}$ is stably trivial. In fact, if $\mathbb{E}_r$ denotes a trivial bundle of rank r on the sphere, then $\mathbb{A} \oplus \mathbb{E}_1 = \mathbb{E}_n$.

Let $x_1, \dots, x_n$ be standard coordinates in $\mathbb{R}^n$. Let $\mathbb{E}' \subset \mathbb{E}_n$ be the subbundle of rank 1 generated by the vector $\mathbf{a} = (x_1, \dots, x_n)$ and $\mathbb{E}''$ be a subbundle generated by the vector $\mathbf{b} = (x_2, -x_1, x_4, -x_3, \dots, x_n, -x_{n-1})$.

It is easy to see that

$$F : \mathbb{E}_n \ni (v_1, \dots, v_n) \mapsto \sum_{i=1}^{n} x_i v_i \in \mathbb{E}_1$$

and

$$G : \mathbb{E}_n \ni (v_1, \dots, v_n) \mapsto x_2 v_1 - x_1 v_2 + \dots + x_n v_{n-1} - x_{n-1} v_n \in \mathbb{E}_1$$

are morphisms of vector bundles. Moreover, $F(\mathbf{a}) = \mathbf{1}$ and $G(\mathbf{b}) = \mathbf{1}$. This means that $\mathbb{E}'$ and $\mathbb{E}''$ are prime summands in $\mathbb{E}_n$.

Since $\ker F = \mathbb{A}$, we have $\mathbb{E}' \oplus \mathbb{A} = \mathbb{E}_n$. Moreover, since $F(\mathbf{b}) = \mathbf{0}$ we have $\mathbb{E}'' \subset \mathbb{A}$. In particular, this means that there exists a subbundle $\mathbb{D} \subset \mathbb{E}_n$ such that $\mathbb{A} = \mathbb{E}'' \oplus \mathbb{D}$. By the construction we have $\mathbb{D} \oplus \langle \mathbf{a}, \mathbf{b} \rangle = \mathbb{E}_n$, where $\langle \mathbf{a}, \mathbf{b} \rangle$ denote the subbundle generated by the vectors $\mathbf{a}$ and $\mathbf{b}$ (check that it is really a subbundle!).

Now consider the embedding

$$\phi : \mathbb{S}^{n-1} \times \mathbb{R}^2 \ni (x, (s,t)) \mapsto (x, x_1 s + x_2 t, x_2 s - x_1 t, \dots, x_{n-1} s + x_n t, x_n s - x_{n-1} t) \in \mathbb{R}^n \times \mathbb{R}^n.$$

By direct computations we see that the normal bundle $N(\phi(\mathbb{S}^{n-1} \times \mathbb{R}^2))$ restricted to the submanifold $\mathbb{S}^{n-1} \times \{0\}$ is equal to

$$N(\mathbb{S}^{n-1}) \oplus (\mathbb{E}_n / \langle \mathbf{a}, \mathbf{b} \rangle) = \mathbb{E}_1 \oplus \mathbb{D} = \mathbb{A} = T\mathbb{S}^{n-1}.$$

This means that this normal bundle is not trivial along $\mathbb{S}^{n-1} \times \{0\}$.

However, it is easy to see that the normal bundle $N(\iota(\mathbb{S}^{n-1} \times \mathbb{R}^2))$ restricted to the submanifold $\mathbb{S}^{n-1} \times \{0\}$ is trivial. Since ϕ and ι coincide along $\mathbb{S}^{n-1} \times \{0\}$, this implies that there is no diffeomorphism

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$$

such that $\Phi \circ \iota = \phi$. ∎

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