COLLOQUIUM MATHEMATICUM

VOL. 95 2003 NO. 2

AN EXTENSION THEOREM FOR A MATKOWSKI–SUTÔ PROBLEM

BY

ZOLTÁN DARÓCZY (Debrecen), GABRIELLA HAJDU (Debrecen)
and CHE TAT NG (Waterloo, ON)

Abstract. Let I be an interval, $0 < \lambda < 1$ be a fixed constant and $A(x,y) = \lambda x + (1-\lambda)y$, $x, y \in I$, be the weighted arithmetic mean on I. A pair of strict means M and N is complementary with respect to A if $A(M(x,y), N(x,y)) = A(x,y)$ for all $x, y \in I$. For such a pair we give results on the functional equation $f(M(x,y)) = f(N(x,y))$. The equation is motivated by and applied to the Matkowski–Sutô problem on complementary weighted quasi-arithmetic means M and N.

1. Introduction. We call a convex subset I of $\mathbb{R}$ an *interval*. An interval is *proper* when it has more than one element. We shall assume that I is proper. A function $M : I^2 \to I$ is said to be a *mean* on I if it satisfies the following conditions:

(M1) $\min\{x,y\} \le M(x,y) \le \max\{x,y\}$ for all $x, y \in I, x \ne y$;
(M2) M is continuous on I^2.

A mean is called *strict* if the inequalities in (M1) are strict. If M is a mean on I, then $M(x,x) = x$ for all $x \in I$. Let $\mathrm{CM}(I)$ denote the class of all continuous and strictly monotonic real functions defined on I. Let $0 < \lambda < 1$ be a fixed number. A function $M : I^2 \to I$ is called a *weighted quasi-arithmetic mean* on I (see [1]) if there exists $\varphi \in \mathrm{CM}(I)$ such that

$$M(x,y) = \varphi^{-1}(\lambda\varphi(x) + (1-\lambda)\varphi(y)) =: A_\varphi(x,y;\lambda)$$

for all $x, y \in I$. In this case, $\varphi \in \mathrm{CM}(I)$ is called the *generating function* of the weighted quasi-arithmetic mean with *weight* λ. Weighted quasi-arithmetic means are strict.

2000 *Mathematics Subject Classification*: 39B22, 39B12, 26A18.

Key words and phrases: functional equation, weighted quasi-arithmetic mean.

This research has been supported by the Hungarian National Research Science Foundation (OTKA) Grant T-030082, the High Educational Research and Development Fund (FKFP) Grant 0310/1997 and by NSERC of Canada Grant OGP 0008212.

If $\varphi, \chi \in \mathrm{CM}(I)$ then $A_\varphi(x,y;\lambda) = A_\chi(x,y;\lambda)$ for all $x, y \in I$ if and only if there exist real constants $\alpha \neq 0$ and β such that

$$\varphi(x) = \alpha\chi(x) + \beta \quad \text{for all } x \in I.$$

If $\varphi, \chi \in CM(I)$ and the above equation holds for some constants $\alpha \neq 0$ and β on a subset $J \subset I$ then we say that φ is *equivalent* to χ on J; and, in this case, we write $\varphi \sim \chi$ on J. For fixed J, it is easy to verify that $\sim$ is indeed an equivalence relation on $\mathrm{CM}(I)$, i.e., it is reflexive, symmetric and transitive. When $\varphi(x) = x$ for all $x \in I$, or when φ is equivalent to the identity map id on I, $A_\varphi(x,y;\lambda)$ is simply denoted by $A(x,y;\lambda)$ and is the well known weighted arithmetic mean

$$A(x,y;\lambda) := \lambda x + (1-\lambda)y \quad (x, y \in I).$$

Let M be a strict mean on I and let $0 < \lambda \leq 1/2$. Then the function defined by

$$\widehat{M}_\lambda(x,y) := \frac{\lambda}{1-\lambda}\,x + y - \frac{\lambda}{1-\lambda}\,M(x,y) \quad (x, y \in I)$$

is also a strict mean on I and for each $x, y \in I$, $M(x,y) = \lambda x + (1-\lambda)y$ if and only if $\widehat{M}_\lambda(x,y) = \lambda x + (1-\lambda)y$. The pair M, $\widehat{M}_\lambda$ satisfies

$$\lambda M(x,y) + (1-\lambda)\widehat{M}_\lambda(x,y) = A(x,y;\lambda) \tag{1}$$

for all $x, y \in I$. In this sense, $\widehat{M}_\lambda$ is complementary to M with respect to the weighted arithmetic mean.

The Matkowski–Sutô problem for weighted quasi-arithmetic means is the following: When will two complementary means M and $\widehat{M}$ be weighted quasi-arithmetic means with the same weight λ on I? In more detail, this means finding those functions $\varphi, \psi \in \mathrm{CM}(I)$ which satisfy

$$\lambda\varphi^{-1}(\lambda\varphi(x) + (1-\lambda)\varphi(y)) + (1-\lambda)\psi^{-1}(\lambda\psi(x) + (1-\lambda)\psi(y)) = \lambda x + (1-\lambda)y \tag{2}$$

for all $x, y \in I$.

The case $\lambda = 1/2$ is the original Matkowski–Sutô problem (see [7], [8], [2], [4]), which has recently been solved in [5] completely. The case $\lambda \neq 1/2$ has been solved in [6] under the assumptions that I is open and the generating functions are continuously differentiable on I with nonvanishing derivatives. Under this assumption, the conclusion is that $\varphi \sim \mathrm{id}$ and $\psi \sim \mathrm{id}$ on I. Conversely, it is easy to verify that when $\varphi \sim \mathrm{id}$ and $\psi \sim \mathrm{id}$ on I, (2) is satisfied. It is natural to ask if the differentiability assumption in the forward statement can be reduced.

Without loss of generality we can suppose that $\lambda \leq 1/2$, otherwise we change the roles of φ and ψ, and of x and y. So in what follows $\lambda \leq 1/2$ is assumed.

We ask the following local versus global question. Suppose that $\varphi, \psi \in \mathrm{CM}(I)$ satisfy (2) on I and there exists a proper interval $J \subset I$ such that $\varphi \sim \mathrm{id}$ and $\psi \sim \mathrm{id}$ on J. Is it true then that $\varphi \sim \mathrm{id}$ and $\psi \sim \mathrm{id}$ on I? In this paper we give an affirmative answer. With this result, the differentiability conditions on I used in [6] can be relaxed to their holding on some open subinterval of I. In Section 2 we solve an equivariance functional equation which is later applied in Section 3 to give the main result.

2. An equivariance equation on complementary means. Let M be a strict mean on I and let $0 < \lambda \leq 1/2$. A function $f : I \to \mathbb{R}$ is called (M, λ)-*associate* if it has the following property:

(MA) If $x, y \in I$ satisfy $M(x, y) = \lambda x + (1-\lambda)y$ and $f(x) = f(\lambda x + (1-\lambda)y)$ then $f(y) = f(x)$.

One can easily check that if f is (M, λ)-associate then it is also $(\widehat{M}_\lambda, \lambda)$-associate.

In this section we solve the equivariance functional equation

$$f(M(x, y)) = f(\widehat{M}_\lambda(x, y)) \quad (x, y \in I),$$

where $0 < \lambda \leq 1/2$ is fixed.

Theorem 1. *Let M be a strict mean on I, $0 < \lambda \leq 1/2$, and let $f : I \to \mathbb{R}$ be a function satisfying the functional equation*

$$f(M(x, y)) = f\left(\frac{\lambda}{1-\lambda}\, x + y - \frac{\lambda}{1-\lambda}\, M(x, y)\right) \tag{3}$$

for all $x, y \in I$. Then

(a) *For all $x, y \in I$ where $M(x, y) \neq A(x, y; \lambda)$, f is locally constant at $A(x, y; \lambda)$.*

(b) *If f is continuous and (M, λ)-associate then either*

(i) *f is constant on I, or*

(ii) *f is injective on I and $M(x, y) = A(x, y; \lambda)$ for all $x, y \in I$.*

Proof. Denote by I_{xy} the closed interval joining $M(x, y)$ and $\widehat{M}_\lambda(x, y)$ and recall that $A(x, y; \lambda) := \lambda x + (1-\lambda)y$ is the weighted arithmetic mean on I. We also recall that $\lambda M(x, y) + (1-\lambda)\widehat{M}_\lambda(x, y) = A(x, y; \lambda)$.

Claim 1. *For all $x_0, y_0 \in I$ two cases are possible*:

(I) *If $M(x_0, y_0) \leq \widehat{M}_\lambda(x_0, y_0)$ then*

$$f(A(x_0, y_0; \lambda) - s) = f\left(A(x_0, y_0; \lambda) + \frac{\lambda}{1-\lambda}\, s\right)$$

for all $0 \leq s \leq A(x_0, y_0; \lambda) - M(x_0, y_0)$.

(II) *If* $\widehat{M}_\lambda(x_0,y_0) < M(x_0,y_0)$ *then*

$$f(A(x_0,y_0;\lambda)-s) = f\left(A(x_0,y_0;\lambda) + \frac{1-\lambda}{\lambda}\,s\right)$$

for all $0 \le s \le A(x_0,y_0;\lambda) - \widehat{M}_\lambda(x_0,y_0)$.

Proof. The assertion is trivial when $I_{x_0y_0}$ is a singleton. Suppose $I_{x_0y_0}$ is proper. There are two cases: either $x_0 < y_0$ or $y_0 < x_0$. First let $x_0 < y_0$.

Consider $x_t := x_0 + t, y_t := y_0 - \frac{\lambda}{1-\lambda}t$ for $0 \le t \le A(x_0,y_0;\lambda) - x_0$. We note that for all $t \in [0, A(x_0,y_0;\lambda) - x_0]$ we have $\lambda x_t + (1-\lambda)y_t = A(x_0,y_0;\lambda)$, and consequently $\lambda M(x_t,y_t) + (1-\lambda)\widehat{M}_\lambda(x_t,y_t) = A(x_0,y_0;\lambda)$.

Now suppose $M(x_0,y_0) < \widehat{M}_\lambda(x_0,y_0)$. This immediately implies $M(x_0,y_0) < A(x_0,y_0;\lambda)$. The function $t \mapsto M(x_t,y_t)$ is continuous and takes the values $M(x_0,y_0)$ and $A(x_0,y_0;\lambda)$. By the Intermediate Value Theorem, for each $0 \le s \le A(x_0,y_0;\lambda) - M(x_0,y_0)$, there exists $t \in [0, A(x_0,y_0;\lambda) - x_0]$ such that $M(x_t,y_t) = A(x_0,y_0;\lambda) - s$ and $\widehat{M}_\lambda(x_t,y_t) = A(x_0,y_0;\lambda) + \frac{\lambda}{1-\lambda}s$. Thus by (3),

$$f(A(x_0,y_0;\lambda)-s) = f\left(A(x_0,y_0;\lambda) + \frac{\lambda}{1-\lambda}\,s\right).$$

A similar argument proves that if $\widehat{M}_\lambda(x_0,y_0) < M(x_0,y_0)$ then for each $0 \le s \le A(x_0,y_0;\lambda) - \widehat{M}_\lambda(x_0,y_0)$, there exists $t \in [0, A(x_0,y_0;\lambda) - x_0]$ such that $\widehat{M}_\lambda(x_t,y_t) = A(x_0,y_0;\lambda) - s$ and $M(x_t,y_t) = A(x_0,y_0;\lambda) + \frac{1-\lambda}{\lambda}s$. Then again by (3),

$$f(A(x_0,y_0;\lambda)-s) = f\left(A(x_0,y_0;\lambda) + \frac{1-\lambda}{\lambda}\,s\right).$$

If $y_0 < x_0$ then let $x_t := x_0 - \frac{1-\lambda}{\lambda}t, y_t := y_0 + t$ for $0 \le t \le A(x_0,y_0;\lambda) - y_0$.

The rest of the proof goes as above. ∎

Claim 2. *Suppose* $I_{x_0y_0}$ *is proper. Then* f *is locally constant at the point* $A(x_0,y_0;\lambda)$; *i.e., there exists a neighbourhood of* $A(x_0,y_0;\lambda)$ *on which* f *is constant.*

Proof. We only examine case (I), when $M(x_0,y_0) < \widehat{M}_\lambda(x_0,y_0)$. Let $x_0 < y_0$, say. For some sufficiently small $\delta > 0$, we have $\big[A(x_0,y_0;\lambda) - \delta, A(x_0,y_0;\lambda) + \frac{\lambda}{1-\lambda}\delta\big] \subset I_{xy_0}$ for all $x \in [x_0, x_0+\delta]$.

Now for all $x \in [x_0, x_0+\delta]$, I_{xy_0} is proper, and by Claim 1,

$$f(A(x,y_0;\lambda)-s) = f\left(A(x,y_0;\lambda) + \frac{\lambda}{1-\lambda}\,s\right)$$

whenever both arguments are in $\left[A(x_0, y_0; \lambda) - \delta, A(x_0, y_0; \lambda) + \frac{\lambda}{1-\lambda}\delta\right]$. The point $A(x, y_0; \lambda)$ being arbitrary in $[A(x_0, y_0; \lambda), A(x_0, y_0; \lambda) + \lambda\delta]$, this gives the constancy of f on $\left[A(x_0, y_0; \lambda) - \delta, A(x_0, y_0; \lambda) + \frac{\lambda}{1-\lambda}\delta\right]$.

The other cases, when $y_0 < x_0$ and (II) holds, can be proved similarly. ∎

The above proves (a) of Theorem 1. To prove (b), in what follows we assume that f is continuous and (M, λ)-associate.

CLAIM 3. *Suppose there exist $x_0 < y_0$ such that $I_{x_0 y_0}$ is proper. Then f is constant on I.*

Proof. Let $J \subset I$ be the *maximal* interval containing $A(x_0, y_0; \lambda)$ on which f is constant, i.e.,

$$J := \{x \in I \mid f(y) = c \text{ for all } y \text{ in the closed interval joining } x \text{ and } A(x_0, y_0; \lambda)\},$$

where $c := f(A(x_0, y_0; \lambda))$. By the continuity of f, J is closed relative to I; and by Claim 2, it is a proper interval neighbourhood of $A(x_0, y_0; \lambda)$. We shall argue that $J = I$; thus f is constant on I.

Suppose that $\beta := \sup J$ is an interior point of I. Then there exists $\varepsilon > 0$ such that $\beta - \varepsilon \in J$ and $\beta + \frac{\lambda}{1-\lambda}\varepsilon \in I$. Now for each $y \in \left]\beta, \beta + \frac{\lambda}{1-\lambda}\varepsilon\right]$ there exists a unique $x \in [\beta - \varepsilon, \beta[$ such that $A(x, y; \lambda) = \beta$. If the interval I_{xy} were proper then f would be constant in a neighbourhood of β by Claim 2 and so J would not be maximal. Therefore I_{xy} is a singleton, that is, $M(x, y) = \widehat{M}_\lambda(x, y)$. So

$$M(x, y) = A(x, y; \lambda) = \beta.$$

Because x and β belong to J,

$$f(x) = f(\beta) = c,$$

and since f is (M, λ)-associate, we get $f(y) = c$. As $y \in \left]\beta, \beta + \frac{\lambda}{1-\lambda}\varepsilon\right]$ is arbitrary, this implies that $\beta + \frac{\lambda}{1-\lambda}\varepsilon$ is in J, contradicting the assumption that $\beta = \sup J$. Thus $\sup J = \sup I$. One can similarly prove that $\inf J = \inf I$. Since J is closed in I, we have $J = I$. ∎

CLAIM 4. *If f is nonconstant on I, then $M(x, y) = A(x, y; \lambda)$ for all $x, y \in I$.*

Proof. By Claim 3, I_{xy} is a singleton for all $x, y \in I$, that is, $M(x, y) = \widehat{M}_\lambda(x, y)$. As $A(x, y; \lambda) = \lambda M(x, y) + (1 - \lambda)\widehat{M}_\lambda(x, y)$, we get $A(x, y; \lambda) = M(x, y)$. ∎

CLAIM 5. *If f is nonconstant, then it is injective on I.*

Proof. By Claim 4, the *continuous* and *nonconstant* function $f : I \to \mathbb{R}$ satisfies the condition (MA):

$$f(y) = f(x) \text{ whenever } f(x) = f(A(x,y;\lambda)), \quad x, y \in I. \tag{4}$$

(I) CASE 1. Suppose $\lambda = 1/2$. This has been dealt with in [3], where the proof of injectivity of f on all closed $[a,b] \subset I$ is given. So f is injective on I.

(II) CASE 2. Suppose $0 < \lambda < 1/2$. Let $\varrho := \frac{\lambda}{1-\lambda}$. Since $0 < \lambda < 1/2$, we have $0 < \varrho < 1$. We rewrite (4) in the form

$$f(u) = f(v) \quad \text{implies} \quad f(u + \varrho(u-v)) = f(u) = f(v), \qquad u, v, u + \varrho(u-v) \in I. \tag{5}$$

Suppose to the contrary that f is not injective. Then there exist $x_1 < x_2$ in the interior of I which are as close as we wish so that

$$f(x_1) = f(x_2).$$

Let them be chosen close enough that $x_2 + \varrho(x_2 - x_1)$ stays in I. We shall now argue that

$$f \text{ is constant on } [x_1, x_2]. \tag{6}$$

If this were not true, then there would exist a proper connected component interval $]x_3, x_4[$ of the nonempty open set $\{t \in]x_1, x_2[\mid f(t) \neq f(x_1)\}$ for which

$$f(x_1) = f(x_2) = f(x_3) = f(x_4), \text{ but } f(t) \neq f(x_1) \text{ for all } t \in]x_3, x_4[. \tag{7}$$

Since $x_2 + \varrho(x_2 - x_1) \in I$, this implies $x_5 := x_4 + \varrho(x_4 - x_3) \in I$. Applying (5) once we get $f(x_5) = f(x_4) = f(x_3)$. Let $x_6 := x_4 + \varrho(x_4 - x_5)$. Then $x_6 = x_4 + \varrho(-\varrho(x_4 - x_3)) = x_4 - \varrho^2(x_4 - x_3)$ where $0 < \varrho^2 < 1$; we have $x_6 \in]x_3, x_4[$. Applying (5) once more we get $f(x_6) = f(x_4) = f(x_5)$, i.e. $f(x_6) = f(x_1)$ while $x_6 \in]x_3, x_4[$. This is a contradiction to (7). This proves (6).

Let K be the maximal interval containing $[x_1, x_2]$ on which f is constant. Then K is proper, and is closed relative to I. It is easy to see that K must be equal to I. For otherwise, say $k := \sup K$ is an interior point in I; then by (5), f will remain constant on $[k, k+\varrho(k-x_1)] \cap I$ and K will not be maximal. This contradiction shows that $\sup K = \sup I$. Similarly, $\inf K = \inf I$ holds. K being closed in I, this gives $K = I$. Thus f is constant on I, and this is a contradiction. ■

This completes the proof of Theorem 1. ■

3. The extension theorem

LEMMA 1. *Let* $\varphi, \psi \in \mathrm{CM}(I)$ *satisfy* (2) *on* I *and let* $J \subset I$ *be a proper subinterval on which* $\varphi \sim \mathrm{id}$ *and* $\psi \sim \mathrm{id}$*. Then there exist* $\widetilde{\varphi}, \widetilde{\psi} \in \mathrm{CM}(I)$ *satisfying* (2) *such that* $\varphi \sim \widetilde{\varphi}$ *and* $\psi \sim \widetilde{\psi}$ *on* I *and*

$$\widetilde{\varphi}(x) = x, \quad \widetilde{\psi}(x) = x \quad \textit{for all } x \in J.$$

Proof. There exist constants $\alpha_i \neq 0$ and β_i $(i = 1, 2)$ such that

$$\alpha_1\varphi(x) + \beta_1 = x, \qquad \alpha_2\psi(x) + \beta_2 = x$$

for all $x \in J$. Then $\widetilde{\varphi} := \alpha_1\varphi + \beta_1$ and $\widetilde{\psi}(x) := \alpha_2\psi + \beta_2$ have the asserted properties. ■

THEOREM 2. *Let* $\varphi, \psi \in \mathrm{CM}(I)$ *satisfy* (2) *for all* $x, y \in I$ *and let* J *be a proper subinterval of* I *such that* $\varphi \sim \mathrm{id}$ *and* $\psi \sim \mathrm{id}$ *on* J*. Then* $\varphi \sim \mathrm{id}$ *and* $\psi \sim \mathrm{id}$ *on* I*.*

Proof. According to Lemma 1, we can suppose that

$$\varphi(x) = x, \quad \psi(x) = x \quad (x \in J)$$

and we need to show that $\varphi = \psi = \mathrm{id}$ on the full interval I. Let $K \subset I$ be the maximal interval containing J such that

$$\varphi(x) = x, \quad \psi(x) = x \quad (x \in K). \tag{8}$$

We are going to show that $K = I$. By the continuity of φ and ψ, K is closed in I. Suppose to the contrary that $K \neq I$; then either $\inf K$ or $\sup K$ is an interior point of I. Say, $a := \inf K$ is an interior point of I.

Choose another element $b \in K$ which is above a, i.e. $a < b$. Then $]a, b[$ is an open neighbourhood of $A_\varphi(a, b; \lambda)$ and $A_\psi(a, b; \lambda)$ because the two means are strict. By the continuity of $A_\varphi(\cdot, b; \lambda)$ and $A_\psi(\cdot, b; \lambda)$, and the fact that a is an interior point of I, there exists $\delta > 0$ such that $[a - \delta, a] \subset I$ and $A_\varphi(x, b; \lambda)$ and $A_\psi(x, b; \lambda)$ are both in $]a, b[$ for all $x \in [a - \delta, a]$.

Let $x \in [a - \delta, a]$. Then from (2) and (8) we have

$$\lambda(\lambda\varphi(x) + (1 - \lambda)b) + (1 - \lambda)(\lambda\psi(x) + (1 - \lambda)b) = \lambda x + (1 - \lambda)b,$$

which implies $\lambda\varphi(x) + (1-\lambda)\psi(x) = x$. The latter also holds true for $x \in [a, b]$ where $\varphi(x) = \psi(x) = x$ and so we have

$$\lambda\varphi(x) + (1 - \lambda)\psi(x) = x \quad \text{for all } x \in [a - \delta, b]. \tag{9}$$

That is,

$$\psi(x) = -\frac{\lambda}{1 - \lambda}\,\varphi(x) + \frac{x}{1 - \lambda}.$$

Since

$$\lambda A_\varphi(x,y;\lambda)+(1-\lambda)A_\psi(x,y;\lambda)=\lambda x+(1-\lambda)y,$$
$$\varphi(A_\varphi(x,y;\lambda))=\lambda\varphi(x)+(1-\lambda)\varphi(y),$$
$$\psi(A_\psi(x,y;\lambda))=\lambda\psi(x)+(1-\lambda)\psi(y),$$

equation (9) yields

$$\varphi(A_\psi(x,y;\lambda))-A_\psi(x,y;\lambda)=\varphi(A_\varphi(x,y;\lambda))-A_\varphi(x,y;\lambda)$$

for all $x,y\in[a-\delta,b]$. Now let $f(t):=\varphi(t)-t$. Then

$$f(A_\psi(x,y;\lambda))=f(A_\varphi(x,y;\lambda))\quad\text{for all } x\in[a-\delta,b]. \tag{10}$$

We show that f is $(A_\varphi(x,y;\lambda),\lambda)$-associate. Let $x,y\in[a-\delta,a]$ be such that $A_\varphi(x,y;\lambda)=\lambda x+(1-\lambda)y$ and $f(x)=f(\lambda x+(1-\lambda)y)$. Then

$$\lambda\varphi(x)+(1-\lambda)\varphi(y)=\varphi(\lambda x+(1-\lambda)y)$$

and

$$\varphi(x)-x=\varphi(\lambda x+(1-\lambda)y)-(\lambda x+(1-\lambda)y).$$

These equations imply

$$\varphi(y)-y=\varphi(x)-x,$$

that is, f is $(A_\varphi(x,y;\lambda),\lambda)$-associate.

By Theorem 1, either f is constant or $A_\varphi(x,y;\lambda)=A(x,y;\lambda)$ for all $x,y\in[a-\delta,b]$. In both cases $\varphi(x)=\alpha x+\beta$ for all $x\in[a-\delta,b]$ follows for some $\alpha\neq 0$ and β. Comparing this with $\varphi(x)=x$ for all $x\in[a,b]$, we get $\alpha=0$ and $\beta=0$. This in turn implies $\varphi(x)=x$ for all $x\in[a-\delta,b]$. Putting this in (8) we also have $\psi(x)=x$ for all $x\in[a-\delta,b]$. Thus $[a-\delta,b]\cup K$ is an interval larger than K on which (8) holds, and this is a contradiction to the maximality of K. Similarly, $\sup K$ cannot be an interior point of I. This proves that $K=I$. ∎

The results of [6] and Theorem 2 yield the following corollary.

Corollary 1. *Suppose $\lambda\neq 1/2$. Let $\varphi,\psi\in \mathrm{CM}(I)$ satisfy* (2) *for all $x,y\in I$ and let K be a proper open subinterval of I such that φ and ψ are continuously differentiable on K. Then $\varphi\sim\mathrm{id}$ and $\psi\sim\mathrm{id}$ on I.*

Proof. Let $H:=\{x\mid x\in K,\ \varphi'(x)=0\}$, which is a closed set in K. Then $H\neq K$, because $\varphi\in\mathrm{CM}(I)$. Therefore there exists a proper open interval $K_1\subset K$ such that $\varphi'(x)\neq 0$ if $x\in K_1$. Similarly, let $H_1:=\{x\mid x\in K_1,\ \psi'(x)=0\}$. Then there exists a proper open interval $K_2\subset K_1$ such that $\psi'(x)\neq 0$ if $x\in K_2$. Thus $\varphi'(x)\neq 0$ and $\psi'(x)\neq 0$ if $x\in K_2$. By [6], $\varphi\sim\mathrm{id}$ and $\psi\sim\mathrm{id}$ on K_2. Now Theorem 2 implies $\varphi\sim\mathrm{id}$ and $\psi\sim\mathrm{id}$ on I. ∎

REFERENCES

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York, 1966.

[2] Z. Daróczy, Gy. Maksa and Zs. Páles, *Extension theorems for the Matkowski–Sutô problem*, Demonstratio Math. 33 (2000), 547–556.

[3] Z. Daróczy and C. T. Ng, *A functional equation on complementary means*, Acta Sci. Math. (Szeged) 66 (2000), 603–611.

[4] Z. Daróczy and Zs. Páles, *On means that are both quasi-arithmetic and conjugate arithmetic*, Acta Math. Hungar. 90 (2001), 271–282.

[5] —, —, *Gauss-composition of means and the solution of the Matkowski–Sutô problem*, Publ. Math. Debrecen 61 (2002), 157–218.

[6] —, —, *The Matkowski–Sutô problem for weighted quasi-arithmetic means*, submitted.

[7] J. Matkowski, *Invariant and complementary quasi-arithmetic means*, Aequationes Math. 57 (1999), 87–107.

[8] O. Sutô, *Studies on some functional equations I–II*, Tôhoku Math. J. 6 (1914), 1–15 and 82–101.

Institute of Mathematics and Informatics
Lajos Kossuth University
H-4010 Debrecen, Pf. 12
Hungary
E-mail: daroczy@math.klte.hu
hajdug@math.klte.hu

Department of Pure Mathematics
University of Waterloo
Waterloo, ON, Canada N2L 3G1
E-mail: ctng@math.uwaterloo.ca

Received 19 February 2002 (4171)