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ON THE EXISTENCE OF PSEUDOSYMMETRIC KÄHLER MANIFOLDS

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Abstract. It is proved that there exists a non-semisymmetric pseudosymmetric Kähler manifold of dimension 4.

1. Preliminaries. Let (M,g) be a Riemannian manifold. Let $\mathfrak{X}(M)$ be the Lie algebra of smooth vector fields on M. If $U \in \mathfrak{X}(M)$, then $U^\flat$ denotes the 1-form defined by $U^\flat(X) = g(U,X)$ for any $X \in \mathfrak{X}(M)$. For $U,V \in \mathfrak{X}(M)$, we consider the curvature transformation $R(U,V)$ and an auxiliary transformation $R^\circ(U,V)$ defined by

$$R(U,V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]}, \qquad R^\circ(U,V) = U \otimes V^\flat - V \otimes U^\flat,$$

where ∇ is the Levi-Civita connection of (M,g). We extend $R(U,V)$ and $R^\circ(U,V)$ to derivations of the tensor algebra on M, assuming that they commute with contractions and $R(U,V)f = 0$, $R^\circ(U,V)f = 0$ for every smooth function f on M. Consequently, for the curvature $(0,4)$-tensor field (which is also traditionally denoted by R), $R(U,V)R$ and $R^\circ(U,V)R$ are $(0,4)$-tensor fields such that for any $W_1,\dots,W_4 \in \mathfrak{X}(M)$,

$$(R(U,V)R)(W_1,\dots,W_4) = -\sum_{s=1}^{4} R(W_1,\dots,R(U,V)W_s,\dots,W_4),$$

$$(R^\circ(U,V)R)(W_1,\dots,W_4) = -\sum_{s=1}^{4} R(W_1,\dots,R^\circ(U,V)W_s,\dots,W_4).$$

A Riemannian manifold is called *semisymmetric* if $R(U,V)R = 0$ for every $U,V \in \mathfrak{X}(M)$ (see [6], [1]).

A Riemannian manifold is said to be *pseudosymmetric* (see [4]) if there exists a function f on M such that for every $U,V \in \mathfrak{X}(M)$,

$$(1) \qquad (R(U,V) - fR^\circ(U,V))R = 0.$$

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References concerning examples of Riemannian pseudosymmetric manifolds which are not semisymmetric can be found in [4].

By a *Kähler manifold* (M, J, g) is meant a differentiable manifold M of real dimension $2n$, which is endowed with a Kähler structure (J, g). That is, J is an almost complex structure (i.e., a $(1,1)$-tensor field such that $J^2 = -\mathrm{Id}$), g is a Riemannian metric compatible with J ($g(JX, JY) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$) and $\nabla J = 0$; equivalently, J comes from a complex structure on M and the fundamental 2-form Φ ($\Phi(X, Y) = g(X, JY)$, $X, Y \in \mathfrak{X}(M)$) is closed.

By constructing an appropriate example, we show the following:

PROPOSITION. *There exists a non-semisymmetric pseudosymmetric* 4-*dimensional Kähler manifold.*

This contradicts Theorem 4 of J. Deprez, R. Deszcz and L. Verstraelen [3], and Propositions 3.2 and 3.3 of F. Defever, R. Deszcz and L. Verstraelen [2] (these two propositions concern a slightly wider class of manifolds).

Precisely, it is claimed in [3] and [2] that the class of pseudosymmetric Kähler manifolds is non-essential in the sense that they are necessarily semisymmetric in any dimension $N = 2n$. Unfortunately, the proof of Theorem 4 of [3] fails in dimension $N = 4$ (formula (6.3) needs a correction). This theorem remains true in any dimension $N \geq 6$. The proof of Proposition 3.2 of [2] also fails in dimension 4 because its final part requires the dimension to be ≥ 6. Consequently, also Proposition 3.3 of [2] cannot be true in dimension 4.

2. Example. Let (x, y, z, t) denote the Cartesian coordinates in $\mathbb{R}^4$. Let (θ^i) be the frame of differential 1-forms on $\mathbb{R}^4$ given by

$$\theta^1 = e^t dx, \quad \theta^2 = e^t dy, \quad \theta^3 = e^{-t}(-2x\,dy + dz), \quad \theta^4 = e^{3t} dt,$$

and let (e_i) be the dual frame of vector fields,

$$e_1 = e^{-t}\frac{\partial}{\partial x}, \quad e_2 = e^{-t}\left(\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}\right), \quad e_3 = e^t\,\frac{\partial}{\partial z}, \quad e_4 = e^{-3t}\,\frac{\partial}{\partial t}.$$

On $\mathbb{R}^4$, define an almost complex structure J and a Riemannian metric g by

$$J = e_2 \otimes \theta^1 - e_1 \otimes \theta^2 + e_4 \otimes \theta^3 - e_3 \otimes \theta^4, \quad g = \sum_i \theta^i \otimes \theta^i.$$

The frame (e_i) is orthonormal with respect g, $Je_1 = e_2$, $Je_2 = -e_1$, $Je_3 = e_4$, $Je_4 = -e_3$ and g is compatible with J. Let N_J be the Nijenhuis torsion tensor of J,

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] + J^2[X, Y],$$

$[\cdot, \cdot]$ being the Lie bracket of vector fields. By direct calculations, one checks that $N_J(e_1, e_3) = 0$. Consequently, by the antisymmetry of N_J and the general identity $N_J(JX,Y) = -JN_J(X,Y)$, we obtain $N_J(e_i,e_j) = 0$ for all $1 \le i,j \le 4$. Thus, $N_J = 0$ and consequently J comes from a complex structure on $\mathbb{R}^4$. Moreover, in our example, the fundamental form Φ has the shape

$$\Phi = -2(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) = 2e^{2t}(-dx \wedge dy - dz \wedge dt + 2x dy \wedge dt),$$

so it is closed ($d\Phi = 0$). Thus, the pair (J,g) is a Kähler structure on $\mathbb{R}^4$.

Let (ω_i^j) be the skew-symmetric matrix of Pfaff forms representing the Levi-Civita connection of g (see e.g. [5], [7]),

$$d\theta^j = -\sum_s \omega_s^j \wedge \theta^s, \qquad \omega_i^j + \omega_j^i = 0.$$

The non-zero forms ω_i^j are

$$\omega_1^2 = -\omega_3^4 = -e^{-3t}\theta^3, \qquad \omega_1^3 = \omega_2^4 = -e^{-3t}\theta^2, \qquad \omega_1^4 = -\omega_2^3 = -e^{-3t}\theta^1.$$

Let (Ω_i^j) be the skew-symmetric matrix of differential 2-forms corresponding to the curvature of ∇ (ibidem),

$$\Omega_i^j = d\omega_i^j - \sum_s \omega_i^s \wedge \omega_s^j, \qquad \Omega_i^j + \Omega_j^i = 0.$$

The non-zero forms Ω_i^j are

$$\begin{aligned}
\Omega_1^2 &= -\Omega_3^4 = 4e^{-6t}(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4),\\
\Omega_1^3 &= \Omega_2^4 = -2e^{-6t}(\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4),\\
\Omega_1^4 &= -\Omega_2^3 = -2e^{-6t}(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3).
\end{aligned}$$

Let R_{hijk} be the components of the curvature tensor with respect to (e_i), $R(e_h,e_i)e_j = \sum_s R_{hijs}e_s$. They are related to the curvature forms by

$$\Omega_j^k(e_h,e_i) = \frac{1}{2}R_{hijk}, \quad \text{or} \quad \Omega_j^k = \sum_{h<i} R_{hijk}\theta^h \wedge \theta^i.$$

Therefore, the non-zero components R_{hijk} are

$$\begin{aligned}
&R_{1212} = -R_{1234} = R_{3434} = 4e^{-6t},\\
&R_{1313} = R_{1324} = R_{1414} = R_{2323} = R_{2424} = -R_{1423} = -2e^{-6t}.
\end{aligned}$$

We now show that our metric g satisfies (1) with $f(t) = 4e^{-6t}$. To do this, consider the auxiliary transformation $Q(U,V)$, defined for any $U,V \in \mathfrak{X}(\mathbb{R}^4)$ by

$$Q(U,V) = R(U,V) - fR^\circ(U,V),$$

f being as above. Let Q_{hijk} be the components of Q with respect to (e_i), $Q(e_h,e_i)e_j = \sum_s Q_{hijs}e_s$, and note that (1) is fulfilled if and only if

$$(2)\quad (Q(e_p,e_q)R)(e_h,e_i,e_j,e_k) \\ = -\sum_s (Q_{pqhs}R_{sijk} + Q_{pqis}R_{hsjk} + Q_{pqjs}R_{hisk} + Q_{pqks}R_{hijs}) = 0.$$

To shorten verification of (2), define the skew-symmetric matrix $(\widetilde{\Omega}^k_j)$ of differential 2-forms by

$$\widetilde{\Omega}^k_j = \sum_{h<i} Q_{hijk}\theta^h \wedge \theta^i, \qquad \widetilde{\Omega}^k_j + \widetilde{\Omega}^j_k = 0.$$

Since

$$Q_{hijk} = R_{hijk} - f(\delta_{ij}\delta_{hk} - \delta_{hj}\delta_{ik}),$$

we have $\widetilde{\Omega}^k_j = \Omega^k_j + f\theta^j \wedge \theta^k$, and consequently the non-zero forms $\widetilde{\Omega}^k_j$ are

$$\begin{aligned} \widetilde{\Omega}^2_1 &= 4e^{-6t}(2\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4), \\ \widetilde{\Omega}^4_3 &= 4e^{-6t}(-\theta^1 \wedge \theta^2 + 2\theta^3 \wedge \theta^4), \\ \widetilde{\Omega}^3_1 &= -\widetilde{\Omega}^4_2 = 2e^{-6t}(\theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4), \\ \widetilde{\Omega}^4_1 &= \widetilde{\Omega}^3_2 = 2e^{-6t}(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3). \end{aligned}$$

Now to prove (2) it is necessary and sufficient to check the following:

$$\sum_s (\widetilde{\Omega}^s_h R_{sijk} + \widetilde{\Omega}^s_i R_{hsjk} + \widetilde{\Omega}^s_j R_{hisk} + \widetilde{\Omega}^s_k R_{hijs}) = 0.$$

This is done by direct calculations for certain indices only because of the (anti-)symmetry properties of the curvature tensor of a Kähler manifold.

Thus, the Kähler manifold $(\mathbb{R}^4, J, g)$ is pseudosymmetric. Moreover it is non-semisymmetric, since e.g.

$$(R(e_1,e_3)R)(e_1,e_2,e_1,e_4) = -24e^{-12t} \neq 0.$$

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