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AFFINE AND CONVEX FUNCTIONS WITH RESPECT TO THE LOGARITHMIC MEAN

BY

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Abstract. The class of all functions $f : (0, \infty) \to (0, \infty)$ which are continuous at least at one point and affine with respect to the logarithmic mean is determined. Some related results concerning the functions convex with respect to the logarithmic mean are presented.

1. Introduction. The *logarithmic mean* $L : (0, \infty)^2 \to (0, \infty)$ defined by

$$L(x, y) := \begin{cases} \dfrac{x-y}{\log x - \log y}, & x \neq y, \\ x, & x = y, \end{cases}$$

has numerous applications in physics (cf. for instance [14]).

The aim of this paper is to determine the class of functions $f : (0, \infty) \to (0, \infty)$ which are *affine* with respect to this mean (briefly, L-*affine*), which means that

$$f(L(x, y)) = L(f(x), f(y)), \qquad x, y > 0.$$

Since the logarithmic mean essentially differs from all the classical means—it is not *quasi-arithmetic* (cf. [4])—the problem appears to be rather difficult.

In Section 2 we recall the notions of a *mean* M, M-*affine* and M-*convex* functions. Some properties of the logarithmic mean (homogeneity, superadditivity, etc.) are presented in Section 3. The main result asserting that *a function* $f : (0, \infty) \to (0, \infty)$ *continuous at least at one point is* L-*affine if*, *and only if*, *either* f *is constant or* $f(x) = f(1)x$ *for all* $x > 0$, is proved in Section 4. Thus the family of all L-affine functions is the minimal one as the constant functions and linear functions are M-affine for every positively homogeneous mean. In particular, the functions of the form $f(x) = ax + b$ $(x > 0)$ with $a, b > 0$ are not L-affine. At the end of Section 4 some open questions are presented. In Section 5 we apply the main result to determine all (continuous at a point) functions affine with respect

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to the *quasi-logarithmic mean*. Some examples, remarks and results related to convex, *geometrically convex* (i.e. convex with respect to the geometric mean) and *L-convex* functions, as well as some open questions are presented in Section 6.

2. Functions affine with respect to a mean. Let $J \subset \mathbb{R}$ be an interval. A function $M : J^2 \to \mathbb{R}$ is said to be a *mean* on J if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \qquad x, y \in J;$$

moreover, if for all $x, y \in J$, $x \neq y$, these inequalities are strict, then M is called a *strict* mean; and if $M(x, y) = M(y, x)$ for all $x, y \in I$, then M is called *symmetric*.

If $M : J^2 \to \mathbb{R}$ is a mean then M is *reflexive*, that is,

$$M(x, x) = x, \qquad x \in J,$$

which implies that $M(I^2) = I$ for every interval $I \subset J$, and $M|_{I\times I}$ is a mean on I. This property permits us to generalize the classical notion of affine functions in the following way:

DEFINITION 1. Let $J \subset \mathbb{R}$ be an interval, $M : J^2 \to J$ a mean on J, and $I \subset J$ an interval. A function $f : I \to J$ is said to be *affine with respect to* M on I (briefly, *M-affine* on I) if

$$f(M(x, y)) = M(f(x), f(y)), \qquad x, y \in I.$$

A function $f : I \to J$ satisfying the inequality

$$f(M(x, y)) \leq M(f(x), f(y)), \qquad x, y \in I,$$

is called *M-convex* on I, and one satisfying the reverse inequality, *M-concave* on I (cf. [11], [12]).

Note that taking in these definitions $M = A$ where A denotes the arithmetic mean, $A(x, y) = (x + y)/2$, we obtain the classical Jensen affine and Jensen convex functions.

REMARK 1. Suppose that $M : (0, \infty)^2 \to (0, \infty)$ is a homogeneous mean, i.e.

$$M(tx, ty) = tM(x, y), \qquad t, x, y > 0.$$

Then

(i) for every interval $I \subseteq (0, \infty)$ any function $f : I \to (0, \infty)$ which is constant or of the form $f(x) = f(1)x$ $(x \in I)$ is M-affine;

(ii) if $c \in (0, \infty)$ and $f : I \to (0, \infty)$ is M-affine, then so is cf.

REMARK 2. Suppose that $M : J^2 \to J$ is a mean and $I_f, I_g \subseteq J$ are intervals. If $g : I_g \to I_f$ and $f : I_f \to J$ are M-affine, then, clearly, the composition $f \circ g$ is also M-affine.

3. Some properties of the logarithmic mean. We denote by log the real natural logarithmic function defined on $(0,\infty)$. The logarithmic mean $L:(0,\infty)^2\to(0,\infty)$ defined by (cf. [2, p. 345])

$$L(x,y):=\begin{cases}\dfrac{x-y}{\log x-\log y}, & x\neq y,\\ x, & x=y,\end{cases}$$

has the following properties, easy to verify:

PROPERTY 1. *L is symmetric.*

PROPERTY 2. *L is homogeneous.*

PROPERTY 3. *L is continuous and increasing with respect to each variable; moreover, for all $x,y>0$, $L(x,y)=x\Leftrightarrow y=x$, and*

$$\lim_{y\to 0}L(x,y)=0,\qquad \lim_{y\to\infty}L(x,y)=\infty,\qquad x>0.$$

PROPERTY 4. *The function $L_0:[0,\infty)^2\to[0,\infty)$ defined by*

$$L_0(x,y):=\begin{cases}L(x,y), & x,y>0,\\ 0, & xy=0,\end{cases}$$

is a continuous homogeneous mean on $[0,\infty)^2$.

PROPERTY 5. *For all $x,y\geq 0$,*

$$\sqrt{xy}\leq L_0(x,y)\leq\frac{x+y}{2},\qquad x,y\geq 0;$$

moreover these inequalities are strict for all $x,y>0$, $x\neq y$.

REMARK 3. The logarithmic mean is not quasi-arithmetic (cf. J. Aczél [1, p. 82], also [4]), i.e. there is no continuous and strictly monotonic function $g:(0,\infty)\to\mathbb{R}$ such that

$$L(x,y)=g^{-1}\left(\frac{g(x)+g(y)}{2}\right),\qquad x,y>0.$$

For the proof of the next property we need the following (cf. [6]–[8], [10])

LEMMA 1. *A function $h:(0,\infty)\to\mathbb{R}$ is strictly convex (respectively strictly concave) iff the function $F:(0,\infty)^2\to\mathbb{R}$ defined by*

$$F(x,y):=yh\left(\frac{x}{y}\right),\qquad x,y>0,$$

is subadditive on $(0,\infty)^2$, i.e.

$$(y_1+y_2)h\left(\frac{x_1+x_2}{y_1+y_2}\right)\leq y_1h\left(\frac{x_1}{y_1}\right)+y_2h\left(\frac{x_2}{y_2}\right),\qquad x_1,x_2,y_1,y_2>0,$$

(*respectively the reverse inequality holds*), *and this inequality becomes an equality if*, *and only if*, *there is a constant* $k > 0$ *such that*

$$y_i = kx_i, \qquad i = 1, 2.$$

PROPERTY 6. *The logarithmic mean is superadditive*, *i.e.*

$$L(x_1 + x_2, y_1 + y_2) \geq L(x_1, y_1) + L(x_2, y_2), \qquad x_1, x_2, y_1, y_2 > 0,$$

and this inequality becomes an equality iff there is a constant $k > 0$ *such that*

$$y_i = kx_i, \qquad i = 1, 2.$$

Proof. Define $h : (0, \infty) \to (0, \infty)$ by $h(t) := L(t, 1)$. Then

$$h(t) = \begin{cases} \dfrac{t-1}{\log t}, & t \neq 1,\, t > 0, \\ 1, & t = 1. \end{cases}$$

Calculating the second derivative of h we get

$$t^2 (\log t)^3 h''(t) = 2(t-1) - (t+1)\log t, \quad t > 0,\ t \neq 1; \qquad h''(1) = -\frac{1}{6}.$$

From Property 5 we have

$$\frac{t-1}{\log t} \leq \frac{t+1}{2}, \qquad t > 0,\ t \neq 1.$$

It follows that $h''(t) < 0$ for all $t > 0$, and consequently, h is strictly concave in $(0, \infty)$. Now an application of Lemma 1 completes the proof. ∎

PROPERTY 7 (cf. [15]). *The logarithmic mean is strictly concave on* $(0, \infty)^2$.

Proof. Making use of the homogeneity of L, by Lemma 1 we have

$$L\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \frac{1}{2} L(x_1 + x_2, y_1 + y_2) \geq \frac{1}{2} \left(L(x_1, y_1) + L(x_2, y_2)\right)$$

for all $x_1, x_2, y_1, y_2 > 0$, which shows that L is Jensen concave. The continuity of L implies that L is concave (cf. [5]). The strict concavity of L results from the "equality" part of Lemma 1. ∎

PROPERTY 8 (cf. [13]). *For all* $x, y > 0$,

$$L(x, y) = \int_0^1 x^t y^{1-t}\, dt.$$

PROPOSITION 1. *For a one-to-one* $f : (0, \infty) \to \mathbb{R}$ *define a function* $F : (0, \infty)^2 \to \mathbb{R}$ *by*

$$F(x, y) := \begin{cases} \dfrac{x - y}{f(x) - f(y)}, & x \neq y, \\ x, & x = y. \end{cases}$$

Then

(i) *F is a mean iff $f = \log + c$ for some $c \in \mathbb{R}$; moreover $F = L$;*

(ii) *F is homogeneous iff the function g defined by $g := f - f(1)$ satisfies the equation*

$$g(rx) = g(r) + g(x), \qquad r, x > 0.$$

If moreover the graph of f is not dense in $(0, \infty) \times \mathbb{R}$ or f is measurable, then $f = c \log + f(1)$ for some constant $c \in \mathbb{R}$.

Proof. The first part is an immediate consequence of the definition of a mean. To show the second, observe that g satisfies the Cauchy functional equation and apply the well known theory (e.g. [1], [4]). ■

4. L-affine functions. Let $I \subseteq (0, \infty)$. Since the logarithmic mean L is homogeneous, in view of Remark 1, the functions $f : I \to (0, \infty)$ that are constant or satisfy $f(x) = f(1)x$ for $x \in I$, are L-affine, i.e.

$$f(L(x, y)) = L(f(x), f(y)), \qquad x, y \in I.$$

Now we prove the following

THEOREM 1. *Let $I \subseteq (0, \infty)$ be an interval, and suppose that $f, g : I \to (0, \infty)$ are L-affine. Then the function $f + g$ is L-affine if, and only if, there is a $c > 0$ such that $g = cf$.*

Proof. By the assumptions we have

$$f(L(x, y)) = L(f(x), f(y)), \qquad g(L(x, y)) = L(g(x), g(y)), \qquad x, y \in I.$$

The function $f + g$ is L-affine iff

$$f(L(x, y)) + g(L(x, y)) = L(f(x) + g(x), f(y) + g(y)), \qquad x, y \in I,$$

i.e., iff

$$L(f(x), f(y)) + L(g(x), g(y)) = L(f(x) + g(x), f(y) + g(y)), \qquad x, y \in I.$$

In view of the equality part of Lemma 1, for every $x, y \in I$ there exists a $k(x, y) > 0$ such that

$$f(y) = k(x, y) f(x), \qquad g(y) = k(x, y) g(x).$$

Hence

$$\frac{g(x)}{f(x)} = \frac{g(y)}{f(y)}, \qquad x, y \in I.$$

It follows that $g = cf$ for some $c > 0$. ■

REMARK 4. Fix $a, b > 0$. The functions $f(x) = ax$, $g(x) = b$, $x > 0$, are L-affine but, in view of the above proposition, the function $(f + g)(x) = ax + b$, $x > 0$, is not.

LEMMA 2. *Let $I \subseteq \mathbb{R}$ be an interval. If $f : I \to (0, \infty)$ is L-affine and continuous at least at one point, then f is continuous everywhere.*

Proof. Suppose that f is L-affine in I and continuous at a point $z \in I$. Since $L(z,\cdot)$ is an increasing homeomorphism of $(0,\infty)$ (cf. Property 3), the set $L(z,I) = \{L(z,y) \mid y \in I\}$ is an open interval. As L is a mean, we have $z \in L(z,I)$, and $L(z,I) \subseteq I$. Fix $x \in L(z,I)$, $x \neq z$. Then there is exactly one $y \in I$ such that $x = L(z,y)$. Take a sequence (x_n), $x_n \in L(z,I)$ $(n \in \mathbb{N})$, such that $\lim_{n\to\infty} x_n = x$. By Property 3, for every $n \in \mathbb{N}$ there is a unique $z_n \in (0,\infty)$ such that $x_n = L(z_n, y)$. Since $L(\cdot, y)$ is a homeomorphism of $(0,\infty)$, we have $\lim_{n\to\infty} z_n = z$, and as $L(z,I)$ is open and $z, x \in L(z,I)$, there is an n_0 such that $z_n, x_n \in L(z,I)$ for $n \geq n_0$. Now, as f is L-affine, we have

$$f(x_n) = f(L(z_n,y)) = L(f(z_n), f(y)), \qquad n \geq n_0.$$

Letting here $n \to \infty$ and making use of the continuity of f at the point z and L-affinity of f, we get

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} f(L(z_n,y)) = \lim_{n\to\infty} L(f(z_n), f(y)) = L(f(z), f(y)) \\ &= f(L(z,y)) = f(x),\end{aligned}$$

which proves that f is continuous at x. Thus f is continuous on $L(z,I)$. (The idea of this proof is taken from [9].)

Now let $(a,b) \subseteq I$ be a maximal open interval of continuity of f. Suppose that $b < \sup I$. Choose $c \in (b, \sup I)$ and $u \in (a,b)$ such that $L(u,c) = b$. For every sequence (b_n) with $b_n \in I$ $(n \in \mathbb{N})$ and $\lim_{n\to\infty} b_n = b$, there exists a sequence $u_n > 0$ $(n \in \mathbb{N})$ such that $b_n = L(u_n, c)$. The properties of L imply that $\lim_{n\to\infty} u_n = u$. Thus, for sufficiently large n we have $u_n \in (a,b)$, and

$$\begin{aligned}\lim_{n\to\infty} f(b_n) &= \lim_{n\to\infty} f(L(u_n,c)) = \lim_{n\to\infty} L(f(u_n), f(c)) = L(f(u), f(c)) \\ &= f(L(u,c)) = f(b),\end{aligned}$$

which proves that f is continuous at b. By the previous part of the proof, f is continuous on the open interval $L(b,I)$. Since $b \in L(b,I)$, the function f is continuous on the open interval $(a,b) \cup L(b,I)$, which is strictly larger than (a,b). This contradicts the maximality of (a,b) and proves that $b = \sup I$. In a similar way one can show that $\inf I = a$. ∎

LEMMA 3. *Let $I \subseteq (0,\infty)$ be an interval. If $f : I \to (0,\infty)$ is L-affine and continuous at least at one point*, *then either f is strictly monotonic and continuous everywhere*, *or f is a constant function.*

Proof. By Lemma 2 the function f is continuous on I. Suppose that f is not strictly monotonic. Then there are $a, b \in I$ with $a < b$ such that $f(a) = f(b)$. Put $C := \{x \in [a,b] \mid f(x) = f(a)\}$. First we show that $C = [a,b]$. Suppose that $[a,b] \setminus C \neq \emptyset$. By the continuity of f the set C is closed. It follows that there is a nonempty open maximal interval $(c,d) \subset [a,b] \setminus C$. Of course we have $c, d \in C$, i.e. $f(c) = f(d) = f(a)$, and $c < d$. Hence, as f

is L-affine,

$$f(L(c,d)) = L(f(c), f(d)) = L(f(a), f(a)) = f(a),$$

which means that $L(c,d) \in C$. This is a contradiction because $c < L(c,d)) < d$.

Let $(\alpha, \beta) \subset I$ be a nonempty maximal open interval such that $f(x) = a$ for all $x \in (\alpha, \beta)$ and suppose that $\beta < \sup I$. Put $\gamma := (\alpha + \beta)/2$. Since

$$L(\gamma, \beta) < L(\beta, \beta) = \beta,$$

the continuity of L implies that there is a $u \in (\beta, \sup I)$ such that

$$L(\gamma, u) < \beta.$$

Since $\gamma < L(\gamma, u)$, we have $L(\gamma, u) \in (\alpha, \beta)$. Hence, as f is L-affine,

$$f(a) = f(L(\gamma, u)) = L(f(\gamma), f(u)) = L(f(a), f(u)).$$

Since $L(f(a), f(a)) = f(a)$ and the function $L(f(a), \cdot)$ is strictly increasing, we get $f(u) = f(a)$. According to the first part of the proof, $f(x) = f(a)$ for all $x \in (\alpha, u)$, which contradicts the maximality of (α, β). Thus $\beta = \sup I$. In a similar way one can show that $\alpha = \inf I$. ■

In what follows we assume that $I = (0, \infty)$.

LEMMA 4. *If $f : (0,\infty) \to (0,\infty)$ is L-affine and at least one of the limits*

$$f(0+) := \lim_{x\to 0+} f(x), \qquad f(+\infty) := \lim_{x\to\infty} f(x)$$

exists, is positive and finite, then f is constant.

Proof. Suppose that $c := f(0+)$ exists and $0 < c < \infty$. Letting $y \to \infty$ in the relation

$$f(L(x,y)) = L(f(x), f(y)), \qquad x, y > 0,$$

and making use of Property 3, we get

$$c = L(f(x), c), \qquad x > 0.$$

Since the function $L(f(x), \cdot)$ is strictly increasing and $L(c,c) = c$, we infer that $f(x) = c$ for all $x > 0$.

If $f(+\infty)$ exists, is positive and finite, we can apply a similar argument. ■

Now we shall prove the main result of this paper.

THEOREM 2. *Let $f : (0,\infty) \to (0,\infty)$ be continuous at least at one point. Then f is L-affine if, and only if, either f is constant or $f(x) = f(1)x$ for all $x > 0$.*

Proof. Suppose that f is L-affine. By Lemma 3, f is either constant, or strictly monotonic and continuous in $(0, \infty)$. If f is constant there is nothing to prove. Suppose that f is strictly monotonic. Then $f(0+)$ exists and

$$0 \le f(0+) \le \infty.$$

The case $0 < f(0+) < \infty$ cannot occur because, in view of Lemma 4, the function f would be constant. Thus either $f(0+) = 0$ or $f(0+) = \infty$.

Consider the first case: $f(0+) = 0$. By Property 4, setting additionally

$$L(x,0) = L(0,x) := 0, \qquad x \geq 0,$$

we extend L to a continuous homogeneous mean on $[0,\infty)^2$. If we set $f(0) := 0$, it is easy to see that the function f is L-affine in $[0,\infty)$, i.e.

$$f(L(x,y)) = L(f(x), f(y)), \qquad x, y \geq 0. \tag{1}$$

Take an arbitrary $a > 0$ and define $g : [0,\infty) \to [0,\infty)$ by

$$g(x) := a\,\frac{f(x)}{f(a)}, \qquad x \geq 0. \tag{2}$$

Since the logarithmic mean is positively homogeneous and $g(x) > 0$ for all $x > 0$, from (2) we infer that g is L-affine in $[0,\infty)$, i.e.

$$g(L(x,y)) = L(g(x), g(y)), \qquad x, y \geq 0, \tag{3}$$

and we have

$$g(0) = 0, \qquad g(a) = a. \tag{4}$$

We shall show that

$$g(x) = x, \qquad x \in [0,a]. \tag{5}$$

Put

$$C := \{x \in [0,a] \mid g(x) = x\},$$

and suppose that $[0,a] \setminus C$ is nonempty. By the continuity of g the set C is closed. Therefore there is a maximal nonempty interval $(c,d) \subset [0,a] \setminus C$. Then, of course,

$$g(c) = c, \qquad g(d) = d,$$

and, by (3),

$$g(L(c,d)) = L(g(c), g(d)) = L(c,d),$$

which means that $L(c,d) \in C$. On the other hand, since $c < d$, we have $L(c,d) \in (c,d)$ and, consequently, $L(c,d) \notin C$. This contradiction proves that (5) holds true.

Now (5) and (2) imply that

$$\frac{f(x)}{x} = \frac{f(a)}{a}, \qquad x \in [0,a].$$

Since a is chosen arbitrarily, we have $f(x) = f(1)x$ for all $x > 0$.

Now suppose that $f(0+) = \infty$. By Lemma 4 we have

$$f(+\infty) := \lim_{x\to\infty} f(x) = 0$$

(in the opposite case f would be a constant function). Consequently, f must be a decreasing bijection of $(0,\infty)$. Clearly, there is a unique $a>0$ such that

$$f(a)=a.$$

Since the composition of two L-affine functions is L-affine (cf. Remark 2), the function $f^2:=f\circ f$ is L-affine. Moreover, f^2 is strictly increasing, and $f^2(a)=a$. According to the previous part of the proof we have

$$f^2(x)=x,\qquad x>0.$$

Hence the substitution $y:=f(x)$ in (1) gives

$$f(L(x,f(x)))=L(f(x),f^2(x))=L(f(x),x),\qquad x>0,$$

which, by the symmetry of L, can be written in the form

$$f(L(x,f(x)))=L(x,f(x)),\qquad x>0.$$

Since a is the only fixed point of f in $(0,\infty)$, it follows that

$$L(x,f(x))=a,\qquad x>0.$$

Hence, by the definition of L, we have

$$x-f(x)=a\log x-a\log f(x),\qquad x>0. \tag{6}$$

For every $c_0>0$, $c_0\neq 1$, the function $g:=c_0f$ is also an L-affine decreasing bijection of $(0,\infty)$ (cf. Remark 1.2). It follows that there is a unique $b>0$, $b\neq a$, such that $g(b)=b$. Therefore we have $g^2(x)=x$ for all $x>0$, and in the same way as in the case of the function f, we obtain

$$L(x,g(x))=b,\qquad x>0,$$

i.e.

$$x-g(x)=b\log x-b\log g(x),\qquad x>0. \tag{7}$$

Multiplying (6) and (7) by b and a, respectively, gives

$$bx-bf(x)=ab\log x-ab\log f(x),\qquad x>0,$$
$$ax-ag(x)=ab\log x-ab\log g(x),\qquad x>0.$$

Subtracting these equations we obtain

$$(b-a)x=bf(x)-ag(x)-ab\log f(x)+ab\log g(x),\qquad x>0. \tag{8}$$

Since the function $g\circ f$ is L-affine and strictly increasing, according to the already proved part of the theorem, there is a constant $c>0$ such that

$$g(f(x))=cx,\qquad x>0.$$

Hence, replacing x by $f(x)$ in (8) and making use of the relation $f^2(x)=x$ for $x>0$, we obtain

$$(b-a)f(x)=(b-ac)x+ab\log c,\qquad x>0,$$

i.e.

$$f(x) = Ax + b, \qquad x > 0,$$

where

$$A := \frac{b - ac}{b - a} \quad \text{and} \quad B := \frac{b - ac}{b - a} \log c.$$

This contradicts the assumption that $f(0+) = +\infty$, and proves the "only if" part of our theorem. Since the converse implication is obvious, the proof is complete. ∎

The following problems are open:

PROBLEM 1. Let $I \subset (0, \infty)$ be a proper subinterval of $(0, \infty)$. Suppose that $h : I \to (0, \infty)$ is L-affine on I. Does there exist an L-affine function $f : (0, \infty) \to (0, \infty)$ such that $f|_I = h$?

REMARK 5. The uniqueness of the extension function f in Problem 1 is obvious.

PROBLEM 2. Determine all continuous L-affine functions defined on an interval $I \subset (0, \infty)$.

REMARK 6. If the answer to Problem 1 is affirmative, then, of course, Theorem 2 gives the solution of Problem 2.

PROBLEM 3. Does there exist a discontinuous L-affine function?

PROBLEM 4. Determine all L-affine functions.

5. Functions affine with respect to quasi-logarithmic means. Let $I \subset \mathbb{R}$ be an interval, and $\varphi : I \to (0, \infty)$ be a continuous and strictly monotonic function. It is easy to verify that the function $L_{[\varphi]} : I^2 \to I$ defined by

$$L_{[\varphi]}(x, y) := \varphi^{-1}(L(\varphi(x), \varphi(y))), \qquad x, y \in I,$$

is a mean on I. By analogy to the *quasi-arithmetic mean*, we call $L_{[\varphi]}$ a *quasi-logarithmic mean*, and φ its *generator*.

Applying Theorem 2 we prove the following

THEOREM 3. *Let $I \subset \mathbb{R}$ be an open interval and suppose that $\varphi, \psi : I \to (0, \infty)$ are continuous, strictly monotonic and onto. Then $L_{[\psi]} = L_{[\varphi]}$ if, and only if, $\psi = c\varphi$ for some $c > 0$.*

Proof. Suppose that $L_{[\psi]} = L_{[\varphi]}$, that is,

$$\psi^{-1} L(\psi(x), \psi(y))) = \varphi^{-1} L(\varphi(x), \varphi(y)), \qquad x, y \in I.$$

Putting $f := \psi \circ \varphi^{-1}$ and setting here $x := \varphi^{-1}(s)$, $y := \varphi^{-1}(t)$ we obtain

$$f(L(s, t)) = L(f(s), f(t)), \qquad s, t > 0,$$

which means that f is L-affine on $(0,\infty)$. Since f is not constant, by Theorem 2, $f(t)=f(1)t$ for all $t>0$. From the definition of f we get $\psi = c\varphi$ where $c := f(1)$. The converse implication is obvious. ■

The next result establishes the form of $L_{[\varphi]}$-affine functions which are continuous at a point.

THEOREM 4. *Let $I \subset \mathbb{R}$ be an open interval and $\varphi : I \to (0,\infty)$ continuous, strictly monotonic and onto. Then a function $g : I \to (0,\infty)$ continuous at least at one point is $L_{[\varphi]}$-affine if, and only if, either g is constant or*

$$g(x) = \varphi^{-1}(a\varphi(x)), \qquad x \in I,$$

where $a := \varphi \circ g \circ \varphi^{-1}(1)$.

Proof. By the definition, a function g is $L_{[\varphi]}$-affine iff

$$g(\varphi^{-1}L(\varphi(x),\varphi(y))) = \varphi^{-1}L(\varphi(g(x),\varphi(g(y))), \qquad x,y \in I.$$

Setting $f := \varphi \circ g \circ \varphi^{-1}$ we can write this equation in the equivalent form

$$f(L(s,t)) = L(f(s),f(t)), \qquad s,t>0,$$

which means that f is L-affine. Now the result is a consequence of Theorem 2. ■

EXAMPLE 1. Taking $\varphi := \exp$ we obtain

$$L_{\exp}(x,y) = \begin{cases} \log\left(\dfrac{e^x - e^y}{x-y}\right), & x \neq y, \\ x, & x = y. \end{cases}$$

Applying Theorem 4 we infer that a function $g : \mathbb{R} \to (0,\infty)$ is $L_{\exp}$-affine iff either g is constant or

$$g(x) = f(0) + x, \qquad x \in \mathbb{R}.$$

6. Some remarks on L-convex functions. In [11] some general criterions of convexity with respect to a positively homogeneous mean and some examples are given. In particular it is shown that, for each $a>1$, the exponential function $f(x) = a^x$, $x>0$, is L-convex, and, for each $a \in (0,1)$, this function is neither L-convex nor L-concave.

REMARK 7. To compare the classical convexity and L-convexity, recall that a function F defined on an open interval I is convex iff for each point $x_0 \in I$ there is at least one A-affine function f,

$$f(x) = m(x-x_0) + F(x_0), \qquad x \in \mathbb{R},$$

supporting F, i.e. such that $f(x) \le F(x)$ for all $x \in I$. Since, in view of Theorem 3 in [11], the function $f := \exp -1$ is L-convex, Theorem 2 shows that the L-convex functions do not reveal similar properties.

Property 8 allows us to give a new proof of the following

THEOREM 5 (cf. [11]). *Let $I \subseteq (0,\infty)$ be an open interval. If $f, g : I \to (0,\infty)$ are L-convex then so is $f+g$.*

Proof. Fix $t \in (0,1)$ and $x, y \in I$. The function $h : (0,\infty) \to (0,\infty)$, $h(u) := u^t$, is concave. Applying Lemma 1 with

$$x_1 := f(x), \quad y_1 := f(y), \quad x_2 := g(x), \quad y_2 := g(y),$$

we obtain the inequality

$$f(x)^t f(y)^{1-t} + g(x)^t g(y)^{1-t} \le (f(x)+g(x))^t (f(y)+g(y))^{1-t}.$$

Making use in turn of: the L-convexity of f and g, Property 8, the above inequality, and again Property 8, we obtain

$$\begin{aligned}
&(f+g)(L(x,y)) = f(L(x,y)) + g(L(x,y)) \le L(f(x),f(y)) + L(g(x),g(y)) \\
&= \int_0^1 f(x)^t f(y)^{1-t}\,dt + \int_0^1 g(x)^t g(y)^{1-t}\,dt = \int_0^1 [f(x)^t f(y)^{1-t} + g(x)^t g(y)^{1-t}]\,dt \\
&\le \int_0^1 [f(x)+g(x)]^t [f(y)+g(y)]^{1-t}\,dt = L((f+g)(x),(f+g)(y)). \ \blacksquare
\end{aligned}$$

Hence, applying Theorem 2, we obtain

COROLLARY 1. *If $a, b > 0$ then $f : (0,\infty) \to (0,\infty)$, $f(x) = ax+b$, is L-convex, but not L-concave.*

Let $G : (0,\infty)^2 \to (0,\infty)$, $G(x,y) := \sqrt{xy}$, denote the geometric mean. The functions which are G-convex (resp. G-concave, G-affine) are called *geometrically convex* (resp. *geometrically concave*, *geometrically affine*). In some papers "geometrically" is replaced by "multiplicatively".

THEOREM 6. *Let $I \subseteq (0,\infty)$ be an open interval. If $f : I \to (0,\infty)$ is convex (resp. concave), and geometrically convex (resp. concave), i.e.*

$$f(\sqrt{xy}) \le \sqrt{f(x)f(y)} \quad (\textit{resp. } f(\sqrt{xy}) \ge \sqrt{f(x)f(y)}), \quad x, y > 0,$$

then f is L-convex (resp. L-concave).

Proof. Take arbitrary $x, y > 0$. Applying in turn: Property 8, the integral Jensen inequality for (arithmetically) convex functions (cf. for instance [4, p. 181]), the geometrical convexity of f, and finally Property 8, we obtain

$$\begin{aligned}
f(L(x,y)) &= f\Big(\int_0^1 x^t y^{1-t}\,dt\Big) \le \int_0^1 f(x^t y^{1-t})\,dt \le \int_0^1 f(x)^t f(y)^{1-t}\,dt) \\
&= L(f(x),f(y)). \ \blacksquare
\end{aligned}$$

EXAMPLE 2. *The tangent function is L-convex in $(0,\pi/2)$.*

Proof. The function $f := \arctan$ is concave on $(0,\infty)$. Note that f is geometrically concave iff $\log\circ f\circ\exp$ is concave. Thus to prove that f is geometrically concave it is sufficient to show that the function $g:(0,\infty)\to\mathbb{R}$ defined by

$$g(x) := \frac{f'(x)}{f(x)}\,x, \qquad x>0;$$

is decreasing on $(0,\infty)$. We have

$$g(x) = \frac{x}{(1+x^2)\arctan x}, \qquad x>0,$$

and, by simple calculations,

$$g'(x)[(1+x^2)\arctan x]^2 = (1-x^2)\arctan x - x, \qquad x>0.$$

Hence $g'(x)<0$ for all $x>1$. For $x\in(0,1)$ we have

$$\arctan x = \sum_{k=0}^{\infty}(-1)^k\,\frac{x^{2k+1}}{2k+1} \quad\text{and}\quad \frac{x}{1-x^2} = \sum_{k=0}^{\infty} x^{2k+1}.$$

It follows that

$$\frac{x}{1-x^2} - \arctan x > 0, \qquad x\in(0,1),$$

or, equivalently,

$$(1-x^2)\arctan x - x < 0, \qquad x\in(0,1),$$

which shows that $g'(x)<0$ for all $x\in(0,1)$. Thus $f=\arctan$ is geometrically concave on $(0,\infty)$. By Theorem 6 the function f is L-concave and, consequently, its inverse $f^{-1}=\tan$ is L-convex in $(0,\infty)$. ∎

EXAMPLE 3. It is easy to verify that the functions $f,g:(0,\infty)\to(0,\infty)$ given by

$$f(x) = \begin{cases} 1, & 0<x\le 1,\\ x, & x>1,\end{cases} \qquad\qquad g(x) = \begin{cases} x, & 0<x\le 1,\\ 1, & x>1,\end{cases}$$

are, respectively, L-convex and L-concave. Simple calculations show that for all $a,b>0$, the functions $f_{a,b}$ and $g_{a,b}$,

$$f_{a,b}(x) := af(bx), \qquad g_{a,b}(x) := ag(bx), \qquad x>0,$$

are, respectively, L-convex and L-concave. For an arbitrary sequence (b_n) with $b_n>0$ $(n\in\mathbb{N})$ one can choose (a_n) with $a_n>0$ $(n\in\mathbb{N})$ such that the series

$$h(x) := \sum_{k=0}^{\infty} a_n f(b_n x), \qquad x>0,$$

converges. Applying Theorem 4 (cf. [11, Corollary 1]) one can show that the function h is L-convex. It is easy to see that choosing the sequence (b_n) properly, one can get h which is not differentiable on a dense set of points.

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