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ON GROUPS OF ESSENTIAL VALUES OF TOPOLOGICAL CYLINDER COCYCLES OVER MINIMAL ROTATIONS

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Abstract. A theory of essential values of cocycles over minimal rotations with values in locally compact Abelian groups, especially $\mathbb{R}^m$, is developed. Criteria for such a cocycle to be conservative are given. The group of essential values of a cocycle is described.

Introduction. The purpose of this paper is to describe the groups of essential values of continuous cocycles (over minimal rotations) taking values in locally compact Abelian groups whose dual is connected. Recall that in the measure-theoretic context the notion of essential values over ergodic actions has been introduced by Klaus Schmidt ([7]). In topological dynamics a parallel theory has been developed by Atkinson [1], although only for extensions by $\mathbb{R}^m$. An adaptation of Schmidt's concepts was published in [4]. In that paper it was suggested that a full description of all groups of essential values over minimal rotations was possible, and indeed, in [4] it has been shown that the only possible groups of essential values for cocycles taking values in $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$. Here we go further and study the case of cocycles taking values in locally compact Abelian groups without compact subgroups. By the classification of LCA groups ([6, Theorem 25]), such a group is of the form $\mathbb{R}^m \oplus D$, where D is discrete and torsion-free. Our main result shows that a group of essential values is then contained in $\mathbb{R}^m$ and moreover it must be a linear subspace of $\mathbb{R}^m$. We will also prove that an $\mathbb{R}^m$-extension of a minimal rotation is conservative iff the cocycle has zero mean (with respect to the Haar measure), and that topological non-ergodicity of a conservative $\mathbb{R}^m$-extension leads to a functional equation. Both these results are essential improvements of those of Atkinson [1].

In this paper we also propose a notion of regularity of a topological cocycle. Namely, we say that a cocycle φ is regular if it is cohomologous to a cocycle taking values in the group $E(\varphi)$ of essential values of φ. In this

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case infinity is not an essential value of the quotient map $\widetilde{\varphi} : X \to G/E(\varphi)$, but the converse does not hold in general. Due to our analysis of possible groups of essential values we show that a cocycle (over a minimal rotation) is regular iff the corresponding extension is conservative.

We emphasize that our analysis of cocycle group extensions essentially exploits the fact that we study cocycles over minimal rotations. It has already been noticed in [4] that the group of essential values may be $\mathbb{Z}$ for some minimal extensions by $\mathbb{R}$, but in this case the base cannot be a rotation.

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1. Preliminaries. Let T be a homeomorphism of a locally compact topological Hausdorff space X. The pair (X,T) will be called a *locally compact flow* or simply a *flow*. If X is a compact Hausdorff space then (X,T) will be called a *compact flow*. For non-empty sets $U, V \subset X$ the *dwelling set* $D(U,V) \subset \mathbb{Z}$ is defined by

$$D(U,V) = \{n \in \mathbb{Z} : T^n U \cap V \neq \emptyset\}.$$

Several notions in topological dynamics can be defined through properties of dwelling sets. A point $x \in X$ is *almost periodic* if for each non-empty open neighbourhood $U \ni x$ the dwelling set $D(x,U)$ is relatively dense. An $x \in X$ is called a *recurrent point* if for any open neighbourhood U of x the dwelling set $D(x,U)$ is both upper and lower unbounded. An $x \in X$ is called a *wandering point* if there exists an open neighbourhood U of x such that $D(U,U) = \{0\}$. If X is a complete metric space then the set consisting of all recurrent and wandering points is residual ([3, Theorem 7.24]). By definition, T is *conservative* if for any non-empty open set $U \subset X$, $D(U,U) \setminus \{0\} \neq \emptyset$. Clearly, T is conservative iff no point in X is wandering. If (X,T) is topologically transitive and X is a perfect space then T is conservative. Conservative flows are also called *regionally recurrent* ([3]) or *non-wandering* ([8]). Finally, T is *topologically ergodic* if for any non-empty open sets $U, V \subset X$, $D(U,V) \neq \emptyset$. We say that T is *uniformly rigid* (or briefly *rigid*) if there exists an unbounded sequence $(n_t)_{t\ge1}$ of integers such that $T^{n_t} \to \mathrm{Id}$ uniformly; we then call $(n_t)_{t\ge1}$ a *rigidity time* for T ([2]). The simplest uniformly rigid homeomorphisms are rotations on monothetic groups. If T is such a rotation and $T^{n_t}x \to x$ for some x, then $(n_t)_{t\ge1}$ is a rigidity time for T.

Let $(G,+,0)$ be a locally compact Abelian group. A continuous function $\Psi : \mathbb{Z} \times X \to G$ satisfying $\Psi(n+m,x) = \Psi(n,T^m x) + \Psi(m,x)$ will be called a

$\mathbb{Z}$-*cocycle* or simply a *cocycle*. If for some continuous $f : X \to G$ a cocycle Ψ satisfies $\Psi(n,x) = f(T^n x) - f(x)$, then Ψ will be called a *coboundary*. For a continuous map $\varphi : X \to G$ one can define a $\mathbb{Z}$-cocycle $\varphi^{(n)}$ by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(T^{n-1}x) + \varphi(T^{n-2}x) + \ldots + \varphi(Tx) + \varphi(x), & n \geq 1, \\ 0, & n = 0, \\ -\varphi(T^n x) - \varphi(T^{n+1}x) - \ldots - \varphi(T^{-1}x), & n \leq -1. \end{cases}$$

Then the cocycle condition $\varphi^{(n+k)}(x) = \varphi^{(n)}(T^k x) + \varphi^{(k)}(x)$ is fulfilled. Thus a continuous map φ defines a $\mathbb{Z}$-cocycle $\varphi^{(n)}$. Conversely, each $\mathbb{Z}$-cocycle $\Psi : \mathbb{Z} \times X \to G$ is of the form $\Psi(n,x) = \varphi^{(n)}(x)$, where $\varphi(x) = \Psi(1,x)$. Therefore we will call every continuous function $\varphi : X \to G$ a $\mathbb{Z}$-*cocycle*; and φ will be called a *coboundary* if $\varphi(x) = \xi(Tx) - \xi(x)$ for some continuous function ξ. For such a cocycle φ define $T_\varphi : X \times G \to X \times G$ by setting

$$T_\varphi(x,g) = (Tx, \varphi(x) + g).$$

The flow $(X \times G, T_\varphi)$ is called a *cocycle extension* of (X,T). Clearly

$$T_\varphi^n(x,g) = (T^n x, \varphi^{(n)}(x) + g).$$

We say that the cocycle φ is *ergodic* if T_φ is topologically ergodic.

In what follows we denote by G_∞ the Aleksandrov compactification of G: $G_\infty = G \cup \{\infty\}$.

2. Essential values of a cocycle

DEFINITION 2.1. Let (X,T) be a flow and $\varphi : X \to G$ a cocycle. We say that $v \in G_\infty$ is an *essential value* of φ if for each non-empty open $U \subset X$ and each neighbourhood V of v there exists $N \in \mathbb{Z}$ such that

$$U \cap T^{-N}U \cap \{x \in X : \varphi^{(N)}(x) \in V\} \neq \emptyset. \tag{1}$$

The set of all essential values of φ will be denoted by $E_\infty(\varphi)$. Moreover, set $E(\varphi) = E_\infty(\varphi) \cap G$.

The set $E(\varphi)$ turns out to be a closed subgroup of G (see [4, Proposition 3.1]). From [4, Proposition 3.1] we also find that $E_\infty(\varphi) = E_\infty(\psi)$ for cohomologous cocycles φ and ψ. In [4, Proposition 3.2] the following characterization of topological ergodicity in the language of essential values is given.

FACT 2.2. *Assume that (X,T) is a compact topologically ergodic flow, G a locally compact group, and $\varphi : X \to G$ a continuous map. Then $(X \times G, T_\varphi)$ is topologically ergodic if and only if $E(\varphi) = G$.*

It follows from [4, Proposition 3.4] that if $E_\infty(\varphi) = \{0\}$, then φ is a coboundary. Conversely, if $\varphi = f \circ T - f$ for some continuous f, then taking an open neighbourhood $V \subset G$ of zero and an open $U \subset X$ such that

$x', x'' \in U$ implies $f(x') - f(x'') \in V$ we see that whenever $U \cap T^{-n}U \neq \emptyset$, then $f(T^n x) - f(x) \in V$ for each $x \in U \cap T^{-n}U$. Therefore $E_\infty(\varphi) = \{0\}$. Thus we have the following fact.

FACT 2.3. *Let (X,T) be a compact flow, G a locally compact Abelian group, and $\varphi : X \to G$ a continuous map. Then $E_\infty(\varphi) = \{0\}$ iff φ is a coboundary.*

We will also need [4, Proposition 3.3].

FACT 2.4. *Let (X,T) be topologically ergodic and $\varphi : X \to G$ a cocycle. Assume that $K \subset G$ is compact and $K \cap E(\varphi) = \emptyset$. Then for each non-empty open $U \subset X$ there exists a non-empty open set $V \subset U$ satisfying*

$$\bigcup_{n\in\mathbb{Z}} (V \cap T^{-n}V \cap \{x \in X : \varphi^{(n)}(x) \in K\}) = \emptyset.$$

The following proposition is a topological version of a similar statement from [5].

PROPOSITION 2.5. *Let (X,T) be a flow. Assume that G, H are locally compact Abelian groups and let $\pi : G \to H$ be a continuous group homomorphism. If $\varphi : X \to G$ is a continuous map, then*

$$\overline{\pi(E(\varphi))} \subset E(\pi \circ \varphi).$$

Directly from the definition of an essential value we have the following.

PROPOSITION 2.6. *Assume that (X,T) is a topological flow, G a locally compact Abelian group, $\varphi : X \to G$ a continuous map, and $H \subset E(\varphi)$ a closed subgroup. Let $\varphi_H : X \to G/H$, $\varphi_H(x) = \varphi(x) + H$. Then $E(\varphi_H) = E(\varphi)/H$.*

DEFINITION 2.7. Let (X,T) be a flow, G a locally compact Abelian group, and $\varphi : X \to G$ a continuous map. We say that the cocycle φ is *regular* if there exists a continuous map $f : X \to G$ such that all values of the cocycle $\psi = \varphi + f \circ T - f$ are in $E(\varphi)$.

From Proposition 2.6 we have the following corollary:

COROLLARY 2.8. *Assume that (X,T) is a flow, G a locally compact Abelian group, and $\varphi : X \to G$ a continuous cocycle. Let $\widetilde{\varphi} : X \to G/E(\varphi)$ be given by $\widetilde{\varphi}(x) = \varphi(x) + E(\varphi)$. Then $E(\widetilde{\varphi}) = \{0\}$. If additionally φ is regular, then also $E_\infty(\widetilde{\varphi}) = \{0\}$.*

One can easily prove the following result.

THEOREM 2.9. *Let (X,T) be a topologically ergodic flow, G a locally compact Abelian group, and $\varphi : X \to G$ a regular cocycle. Then $(X \times G, T_\varphi)$ is a disjoint union of topologically ergodic subflows, each isomorphic to $(X \times E(\varphi), T_\psi)$, where $\psi : X \to E(\varphi)$, $\psi = \varphi + f \circ T - f$.*

REMARK 1. If G is a locally compact Abelian group, then there exists a closed-open subgroup H of G which is a direct sum of a compact group and $\mathbb{R}^m$ (see for instance [6, Theorem 25]). If additionally G has no compact subgroups, then $H=\mathbb{R}^m$. Because $\mathbb{R}^m$ is a divisible group, G is a direct sum of $\mathbb{R}^m$ and $G/\mathbb{R}^m$. Note that the latter group is always discrete.

LEMMA 2.10. *Let (X,T) be a compact flow and $\varphi : X \to \mathbb{R}^m$ a cocycle. If for some unbounded sequence $(n_t)_{t\geq 1}$ of integers, the sequence $(\varphi^{(n_t)})_{n\geq 1}$ converges uniformly to a constant $a \in \mathbb{R}^m$, then $a=0$.*

The following lemma will be essential to the proof of the important Lemma 3.4.

LEMMA 2.11. *Let (X,T) be a compact flow and G a locally compact Abelian group with no non-trivial compact subgroup. Assume that $\varphi : X \to G$ is continuous. If $\varphi^{(n_t)} \to g \in G$ uniformly for some unbounded sequence $(n_t)_{t\geq 1}$ of integers, then $g=0$.*

Proof. Let $G=\mathbb{R}^m \oplus D$, where D is a discrete group without compact subgroups. Put $\varphi=\varphi_1+\varphi_2$, where $\varphi_1 : X \to \mathbb{R}^m$, $\varphi_2 : X \to D$, and $g=g_1+g_2$, where $g_1 \in \mathbb{R}^m$, $g_2 \in D$. By Lemma 2.10, $g_1=0$. For the converse suppose $g_2 \neq 0$. Since $\varphi_2^{(n_t)} \to g_2$ uniformly and D is discrete, $\varphi_2^{(n_t)} \equiv g_2$ for t large enough. Fix such a t. For $k \geq 1$ we can find integers $s=s_k$, $r=r_k$ such that $n_{t+k}=sn_t+r$. Then applying s times the cocycle equality $\psi^{(m+n)}=\psi^{(m)}\circ T^n+\psi^{(n)}$ we get $g_2=\varphi_2^{(n_{t+k})}=\varphi_2^{(sn_t+r)}=sg_2+\varphi_2^{(r)}$. Since D has no compact subgroups, $sg_2 \to \infty$, which gives a contradiction. ∎

LEMMA 2.12. *Let (X,T) be a compact flow, G a locally compact Abelian group, and $G_0=K\oplus\mathbb{R}^m$ an open subgroup of G with K compact. Assume that $\varphi : X \to G$ is a continuous map. If $\varphi^{(n_t)} \to g \in G_0$ uniformly for some increasing (or decreasing) sequence $(n_t)_{t\geq 1}$ of integers, then $g \in K$.*

Proof. Since G_0 is open in G, there exists a positive integer m such that $\varphi^{(m)}(X) \subset G_0$. By hypothesis, $\varphi^{(mn_t)} \to mg$ uniformly as $t \to \infty$. By Lemma 2.11, $mg \in K$, hence $g \in K$. ∎

LEMMA 2.13. *Let (X,T) be a compact flow, G a locally compact Abelian group, $G_0=K\oplus\mathbb{R}^m$ an open subgroup of G with K compact, and $\varphi : X \to G$ a continuous map. Assume that $(n_t)_{t\geq 1}$ is an increasing (or decreasing) sequence of integers such that $\varphi^{(n_t)} \to g \in G$ uniformly. Then $rg \in K$ for some non-zero integer r.*

Proof. Let $\overline{\varphi} : X \to G/G_0$, $\overline{\varphi}(x)=\varphi(x)+G_0$. Then $\overline{\varphi}$ takes values in a finitely generated subgroup G' of G/G_0 and G' has a finite maximal compact subgroup. Therefore we can use Lemmas 2.11 and 2.12 to get the result. ∎

Definition 2.14. If A, B are topological spaces and $\pi : A \to B$ a continuous map with $\pi(A) = B$, then a continuous map $s : B \to A$ is called a *continuous selector* for π if $\pi(s(y)) = y$ for all $y \in B$.

Lemma 2.15. *Let (X,T) be a flow, G a locally compact Abelian group, and $\varphi : X \to G$ a continuous map. Let $\widetilde{\varphi} : X \to G/E(\varphi)$, $\widetilde{\varphi}(x) = \varphi(x) + E(\varphi)$. If $E_\infty(\widetilde{\varphi}) = \{0\}$ and there exists a continuous selector for the natural quotient map $G \to G/E(\varphi)$, then φ is regular.*

Proof. Let s be a selector for $G \to G/E(\varphi)$. Since $E_\infty(\varphi) = \{0\}$, $\widetilde{\varphi}$ is a coboundary (Fact 2.3), that is, $\widetilde{\varphi} = \widetilde{f} \circ T - \widetilde{f}$, where $\widetilde{f} : X \to G/E(\varphi)$. Define $f : X \to G$ by $f(x) = s(\widetilde{f}(x))$. Then $\varphi(x) - f(Tx) + f(x) = \varphi(x) - s(\widetilde{f}(Tx)) + s(\widetilde{f}(x)) \in E(\varphi)$ and φ is regular. ∎

Remark 2. In the following cases the natural quotient maps $G \to G/H$, where G, $H \subset G$ are topological Abelian groups, admit continuous selectors:

(a) G/H is a discrete group;
(b) $H = \mathbb{R}^m$ for some integer $m \geq 0$.

3. The groups of essential values for extensions of minimal rotations. In this section we will concentrate on the following situation: $T : X \to X$ is a minimal rotation, X a compact metric monothetic group, and G a locally compact Abelian group.

Lemma 3.1. *Let T be a minimal rotation on a compact metric monothetic group X, D a discrete Abelian group, and $\varphi : X \to D$ a continuous map. If $(n_t)_{t\geq 1}$ is a rigidity time for T, then each $\varphi^{(n_t)}$ is a constant function for t large enough.*

Proof. We have $T^{n_t} \to \mathrm{Id}$ uniformly. Since φ is continuous and D is discrete, there exists a t_0 such that $\varphi(T^{n_t}x) = \varphi(x)$ for all $t \geq t_0$ and each $x \in X$. Fix $x_0 \in X$ and $t \geq t_0$. Then $\varphi^{(n_t)}(T^{i+1}x_0) - \varphi^{(n_t)}(T^i x_0) = \varphi(T^{n_t+i}x_0) - \varphi(T^i x_0) = 0$ for all $i \in \mathbb{Z}$, so $\varphi^{(n_t)}(T^i x_0) = \varphi^{(n_t)}(T^{i+1}x_0)$, $i \in \mathbb{Z}$. It follows from the minimality of T that $\varphi^{(n_t)} = \mathrm{const} = d_t \in D$. ∎

The following characterization of essential values can be found in [4].

Fact 3.2. *Let T be a minimal rotation on a compact metric monothetic group X, G a locally compact Abelian group, and $\varphi : X \to G$ a continuous map. Assume that $0 \neq g \in G_\infty$. Then $g \in E_\infty(\varphi)$ iff there exists a rigidity time $(n_t)_{t\geq 1}$ and a sequence $(x_t)_{t\geq 1}$ in X such that $\varphi^{(n_t)}(x_t) \to g$.*

Using Proposition 2.6 and Fact 3.2 one can easily prove the following proposition.

Proposition 3.3. *Let T be a minimal rotation on a compact metric monothetic group X, G a locally compact Abelian group, and $\varphi : X \to G$*

a continuous map. Let $H \subset G$ be a closed subgroup such that $H/E(\varphi) \cap H$ is compact. Put $\varphi_H : X \to G/H$, $\varphi_H(x) = \varphi(x) + H$. Then $E(\varphi_H)$ is naturally isomorphic to $E(\varphi)/E(\varphi) \cap H$. Moreover $\infty \in E_\infty(\varphi_H)$ if and only if $\infty \in E_\infty(\varphi_{E(\varphi)\cap H})$.

LEMMA 3.4. *Let T be a minimal rotation on a compact metric monothetic group X, G a locally compact Abelian group with no non-trivial compact subgroup, and $\varphi : X \to G$ a continuous map. If $E(\varphi) \neq \{0\}$, then no point in $E(\varphi)$ is isolated.*

Proof. Assume that $0 \neq g \in G$ is an isolated element of $E(\varphi)$. By Fact 3.2, $g = \lim_{t\to\infty} \varphi^{(n_t)}(x_t)$, where $(n_t)_{t\geq 1}$ is a rigidity time for T. It follows from Lemma 2.11 that $\varphi^{(n_t)} \not\to g$ uniformly. Since g is isolated, there exists an open set $0 \in V \subset G$ such that $\overline{V}$ is compact and $E(\varphi) \cap [g + (V + V)] = \{g\}$. Moreover we may assume that

$$\forall_{t\geq 1} \exists_{z\in X} \quad \varphi^{(n_t)}(z) \notin g + (V + V). \tag{2}$$

Find an open symmetric set V_1, $0 \in V_1 \subset G$, such that $V_1 + V_1 \subset V$ and put

$$K = g + (\overline{V} \setminus V_1).$$

Clearly K is a compact set with $K \cap E(\varphi) = \emptyset$. By Fact 2.4 there exists an open non-empty set $U \subset X$ such that

$$U \cap T^{-n}U \cap \{x \in X : \varphi^{(n)} \in K\} = \emptyset, \quad n \in \mathbb{Z}. \tag{3}$$

Let d be an invariant metric on X. Fix $x_0 \in U$ and $\delta > 0$ such that the ball with centre at x_0 and radius 2δ is included in U, and the condition $d(x, x') < \delta \Rightarrow \varphi(x) - \varphi(x') \in V_1$ is valid. Let B be the ball with centre x_0 and radius $\delta/2$. Then B is included in U together with its $\frac{3}{2}\delta$-neighbourhood. Let M be a positive integer such that

$$\forall_{x\in X} \exists_{0\leq i\leq M-1} \quad T^i x \in B. \tag{4}$$

Such an M exists because T is minimal.

Let $W \subset G$ be an open symmetric neighbourhood of zero such that $M \cdot W \subset V_1$. Fix a t satisfying

$$\varphi(T^{n_t}x) - \varphi(x) \in W, \quad x \in X, \tag{5}$$

$$d(T^{n_t}x, x) < \delta, \quad x \in X, \tag{6}$$

$$\varphi^{(n_t)}(x_t) \in g + W. \tag{7}$$

Let z be given by (2) for the fixed t. Since the positive part of the orbit of x_t is dense in X ($\overline{\{T^n x_t : n \geq 1\}} = X$), there exists a positive l such that $\varphi^{(n_t)}(T^l x_t) - \varphi^{(n_t)}(z) \in V_1$. Then $\varphi^{(n_t)}(T^l x_t) \notin g + V$. Let k be the smallest positive integer such that $\varphi^{(n_t)}(T^k x_t) \notin g + V$. Then $\varphi^{(n_t)}(T^{k-1}x_t) \in g + V$. For each i, by (5) we have

$$\varphi^{(n_t)}(T^{k-i+1}x_t) - \varphi^{(n_t)}(T^{k-i}x_t) = \varphi(T^{n_t+k-i}x_t) - \varphi(T^{k-i}) \in W. \tag{8}$$

Now observe that $k > M$. Indeed, if this is not the case then

$$\begin{aligned}\varphi^{(n_t)}(T^k x_t) - \varphi^{(n_t)}(x_t) &= \sum_{j=0}^{k-1} [\varphi^{(n_t)}(T^{j+1}x_t) - \varphi^{(n_t)}(T^j x_t)] \\ &= \sum_{j=0}^{k-1} [\varphi(T^j x_t) - \varphi(T^{n_t+j}x_t)] \in k \cdot W \subset M \cdot W \subset V_1\end{aligned}$$

by (8). Then, by (7),

$$\varphi^{(n_t)}(T^k x_t) \in \varphi^{(n_t)}(x_t) + V_1 \subset g + W + V_1 \subset g + V_1 + V_1 \subset g + V,$$

which is impossible because of the choice of k. Therefore $k > M$.

Consider now the points $T^{k-i}x_t$, $i = 1, \ldots, M$. Since $k > M$, all differences $k - i$, $i = 1, \ldots, M$, are positive and $\varphi^{(n_t)}(T^{k-i}x_t) \in g + V$, $i = 1, \ldots, M$. By (4), at least one of the $T^{k-i}x_t$ is in B, say $T^{k-j}x_t \in B$. Put $y = T^{k-j}x_t$. We will show that

$$y \in U \cap T^{-n_t}U \cap \{x \in X : \varphi^{(n_t)} \in K\},$$

which contradicts (3). By our choice, $y \in B \subset U$. By (6), $d(T^{n_t}y, y) < \delta$ so $T^{n_t}y$ belongs to the δ-neighbourhood of B. By definition of δ, $T^{n_t}y \in U$, i.e. $y \in T^{-n_t}y$. To finish the proof observe that $\varphi^{(n_t)}(y) \notin g + V_1$. Indeed, $y = T^{k-j}x_t$, where $j \le M$. By (8),

$$\begin{aligned}\varphi^{(n_t)}(y) - \varphi^{(n_t)}(T^k x_t) &= \varphi^{(n_t)}(T^{k-j}x_t) - \varphi^{(n_t)}(T^k x_t) \\ &\in j \cdot W \subset M \cdot W \subset V_1,\end{aligned}$$

so $\varphi^{(n_t)}(T^k x_t) \in \varphi^{(n_t)}(y) + V_1$.

If $\varphi^{(n_t)}(y) \in g + V_1$ then $\varphi^{(n_t)}(T^k x_t) \in g + V_1 + V_1 \subset g + V$, which is not true. Thus $\varphi^{(n_t)}(y) \notin g + V_1$, $\varphi^{(n_t)}(y) \in g + V$, so

$$\varphi^{(n_t)}(y) \in g + (V \setminus V_1) \subset g + (\overline{V} \setminus V_1) = K,$$

which finishes the proof. ∎

Now we are in a position to describe all possible groups of essential values for cocycles $\varphi : X \to G$ over minimal rotations in the case when G has no compact subgroups. By Remark 1, such a group is a direct sum of $\mathbb{R}^m$ and a discrete group.

REMARK 3. If G is a closed subgroup of $\mathbb{R}^m$, then, by [6, Theorem 6], G is of the form

$$G = \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_l \oplus \mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_k,$$

where $w_1, \ldots, w_l, v_1, \ldots, v_k \in \mathbb{R}^m$ are linearly independent vectors.

THEOREM 3.5. *Assume that T is a minimal rotation on a compact metric monothetic group X, and G is a locally compact Abelian group without* *compact subgroups. If $\varphi : X \to G$ is a continuous map then $E(\varphi)$ is a linear subspace of $\mathbb{R}^m \subset G$.*

Proof. First we will show that $E(\varphi) \subset \mathbb{R}^m$, where $\mathbb{R}^m \subset G$ is an open subgroup of G. To do this we will use Proposition 2.6 for $H = E(\varphi) \cap \mathbb{R}^m$. Since $G/\mathbb{R}^m$ is discrete, so is $E(\varphi)/H$. By Lemma 3.4, $E(\varphi) = H \subset \mathbb{R}^m$ and, by Remark 3,

$$E(\varphi) = \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_l \oplus \mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_k$$

for some linearly independent vectors $w_1, \ldots, w_l, v_1, \ldots, v_k$, $l + k \leq m$. Now apply Proposition 2.6 for $H = \mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_k$ to deduce that $E(\widetilde{\varphi}) = \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_l$ is a discrete group. By Lemma 3.4, $l = 0$ and $E(\varphi) = \mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_k$ is a linear subspace of $\mathbb{R}^m$. ■

4. Atkinson's theorem and regularity of cylinder flows. In the main theorem of [1] a condition for a conservative cylinder flow over a minimal rotation on a torus to be topologically transitive is given ([1, Theorem 1]). We will generalize this theorem to cylinder flows which are extensions of minimal rotations on any compact monothetic metric group. In Atkinson's proof the connectedness of the torus was used. We do not need it, using a method of "short steps", introduced in [4] to prove Proposition 4.1 there (the second relation in (10) of [4]).

In [4] the following description of the group of essential values for T being a minimal rotation and φ a real cocycle is given.

FACT 4.1. *Let T be a minimal rotation on a compact metric monothetic group X equipped with a probability Haar measure μ, and $\varphi : X \to \mathbb{R}$ a continuous map.*

If $\int_X \varphi \, d\mu \neq 0$, then $E_\infty(\varphi) = \{0, \infty\}$ and each point in $X \times \mathbb{R}$ is wandering.

If $\int_X \varphi \, d\mu = 0$, then φ is either topologically ergodic (with $E(\varphi) = \mathbb{R}$) or a coboundary (with $E_\infty(\varphi) = \{0\}$).

In this paper we generalize this theorem to any locally compact Abelian metric group without compact subgroups.

We start with a version of [1, Lemma 4]. The differences are that the torus is replaced by a minimal rotation, and the sphere $\{v \in \mathbb{R}^m : \|v\| = r\}$ by a "ring" $K(a, b) = \{v \in \mathbb{R}^m : a \leq \|v\| \leq b\}$, where $\|\cdot\|$ denotes a norm in $\mathbb{R}^m$. The fact that we are assuming that $E(\varphi) = \{0\}$ is not essential in view of Theorem 3.5.

LEMMA 4.2. *Let T be a minimal rotation on a compact metric monothetic group X, and $\varphi : X \to \mathbb{R}^m$ a continuous map such that $E(\varphi) = \{0\}$. Then for any positive numbers $a < b$ there exists a positive $\delta = \delta(a, b)$ such* *that if* $d(T^n, \mathrm{Id}) < \delta$ *then either* $\varphi^{(n)}(X) \subset B(0,a)$ *or* $\varphi^{(n)}(X) \subset \overline{B(0,b)}^{\mathrm{c}} = X \times \mathbb{R}^m \setminus \overline{B(0,b)}$.

Proof. For any $x_0 \in X$ and $i = 1, 2, \ldots$ let

$$A_i(x_0) = \overline{\{\varphi^{(n)}(x) : x \in B(x_0, i^{-1}),\, d(T^n, \mathrm{Id}) < i^{-1}\}},$$
$$A(x_0) = \bigcap_{i=1}^{\infty} A_i(x_0).$$

By [4, Proposition 4.1], $A(x_0) \subset E(\varphi)$, so either $A(x_0) = \emptyset$ or $A(x_0) = \{0\}$. In particular, if we put

$$K(a,b) = \{v \in \mathbb{R}^m : a \le \|v\| \le b\} = \overline{B(0,b)} \setminus B(0,a),$$

then $A(x_0) \cap K(a,b) = \emptyset$, i.e.

$$\bigcap_{i=1}^{\infty} [A_i(x_0) \cap K(a,b)] = \emptyset.$$

Since $A_1(x_0) \supset A_2(x_0) \supset \ldots$ and $K(a,b)$ is a compact set, some of the sets $A_i(x_0) \cap K(a,b)$ must be empty. Let $A_j \cap K(a,b) = \emptyset$. In particular

$$\{\varphi^{(n)}(x) : x \in B(x_0, j^{-1}),\, d(T^n, \mathrm{Id}) < j^{-1}\} \cap K(a,b) = \emptyset.$$

This is true for each $x_0 \in X$, because

$$X = \bigcup_{x \in X} B(x, i_x^{-1}),$$

where for each $x \in X$,

$$\{\varphi^{(n)}(y) : y \in B(x, i_x^{-1}),\, d(T^n, \mathrm{Id}) < i_x^{-1}\} \cap K(a,b) = \emptyset.$$

Since X is compact, there are points $x_1, \ldots, x_k$ and integers $i_1, \ldots, i_k$ such that

$$X = \bigcup_{j=1}^{k} B(x_j, i_j^{-1})$$

and for each $j = 1, \ldots, k$,

$$\{\varphi^{(n)}(y) : y \in B(x_j, i_j^{-1}),\, d(T^n, \mathrm{Id}) < i_j^{-1}\} \cap K(a,b) = \emptyset.$$

The map φ is uniformly continuous, thus there exists a $\delta_1 > 0$ such that for any $x', x'' \in X$, if $d(x', x'') < \delta_1$ then $\|\varphi(x') - \varphi(x'')\| < b - a$. Let

$$\delta = \delta(a,b) = \min\{\delta_1, i_1^{-1}, \ldots, i_k^{-1}\}.$$

Then

$$\{\varphi^{(n)}(y) : y \in B(x_j, i_j^{-1}),\, d(T^n, \mathrm{Id}) < \delta\} \cap K(a,b) = \emptyset, \qquad j = 1, \ldots, k.$$

Since the balls $B(x_j, i_j^{-1})$, $j = 1, \ldots, k$, cover the whole X, for each $x \in X$ and for each n satisfying $d(T^n, \mathrm{Id}) < \delta$ we have $\varphi^{(n)}(x) \notin K(a,b)$.

Now fix n satisfying $d(T^n, \mathrm{Id}) < \delta$ and assume that $\|\varphi^{(n)}(x)\| < a$ for some $x \in X$. Then for any integer j,

$$\varphi^{(n)}(T^{j+1}x) - \varphi^{(n)}(T^j x) = \varphi(T^{n+j}x) - \varphi(T^j x) = \varphi(T^n(T^{j+1}x)) - \varphi(T^j x).$$

Since $d(T^n, \mathrm{Id}) < \delta$, we have $d(T^n(T^j x), T^j x) < \delta$, thus

$$\|\varphi(T^n(T^{j+1}x)) - \varphi(T^j x)\| < b - a,$$

and therefore $\|\varphi^{(n)}(T^{j+1}x) - \varphi^{(n)}(T^j x)\| < b - a$. This means that the distances between consecutive points of the sequence $(\varphi^{(n)}(T^j x))_{j\in\mathbb{Z}}$ are less than $b-a$. Since none of them is in $K(a,b)$ and $\|\varphi^{(n)}(x)\| < a$, all $\varphi^{(n)}(T^j x)$, $j \in \mathbb{Z}$, are in $B(0,a)$. As $\varphi^{(n)}$ is a continuous map, $\varphi^{(n)}(X) \subset B(0,a)$. ∎

The next lemma is a version of [1, Theorem 2] for $m = 1$ only, but for an arbitrary minimal rotation on a compact monothetic group. Moreover, we formulate it as a necessary and sufficient condition and do not assume that the map φ has bounded variation. In Theorem 4.9 we will prove such a result for any m.

LEMMA 4.3. *Let T be a minimal rotation on a compact metric monothetic group X, and $\varphi : X \to \mathbb{R}$ a continuous map. Then T_φ is conservative iff $\int_X \varphi\, d\mu = 0$, where μ is the normalized Haar measure on X.*

The next three lemmas are generalizations of [1, Lemmas 5–7] to the case of any minimal rotation on a compact metric monothetic group. By Lemma 4.2, the original proofs work also in this case.

LEMMA 4.4. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi : X \to \mathbb{R}^m$ a continuous map such that $E(\varphi) = \{0\}$, and $(n_t)_{t\ge1}$ a sequence of integers. The following conditions are equivalent*:

(i) *For all $(x, v) \in X \times \mathbb{R}^m$ the sequence $(T_\varphi^{n_t}(x, v))_{t\ge1}$ converges.*

(ii) *For some $(x_0, v_0) \in X \times \mathbb{R}^m$ the sequence $(T_\varphi^{n_t}(x_0, v_0))_{t\ge1}$ converges.*

(iii) *The sequence $(\varphi^{(n_t)})_{t\ge1}$ converges uniformly and $(T^{n_t})_{t\ge1}$ converges.*

LEMMA 4.5. *Let T be a minimal rotation on a compact metric monothetic group X, and $\varphi : X \to \mathbb{R}^m$ a continuous map with $E(\varphi) = \{0\}$. Then every orbit closure under T_φ is minimal.*

LEMMA 4.6. *Let T be a minimal rotation on a compact metric monothetic group X, and $\varphi : X \to \mathbb{R}^m$ a continuous map such that $E(\varphi) = \{0\}$ and T_φ is conservative. Then φ is a coboundary.*

Now we are able to exhibit a key property of conservative cocycles.

PROPOSITION 4.7. *Let T be a minimal rotation on a compact metric monothetic group X, and $\varphi : X \to \mathbb{R}^m$ a continuous map such that T_φ is conservative. Then there exits a basis of $\mathbb{R}^m$ such that $\varphi = (\varphi_1, \ldots, \varphi_m)$, $E(\varphi) = E(\varphi_1, \ldots, \varphi_k) = \mathbb{R}^k$, and $(\varphi_{k+1}, \ldots, \varphi_m)$ is a coboundary.*

Proof. If T_φ is topologically transitive then $E(\varphi) = \mathbb{R}^m$ and the assertion follows. Assume that T_φ is not topologically transitive. Then, by Theorem 3.5, $E(\varphi)$ is a k-dimensional linear subspace of $\mathbb{R}^m$, and by Fact 2.2, $k < m$. Changing a basis of $\mathbb{R}^m$ if necessary we may assume that $\varphi = (\varphi_1, \ldots, \varphi_m)$ and $E(\varphi) = E(\varphi_1, \ldots, \varphi_k) = \mathbb{R}^k$. By Proposition 2.6, $E(\varphi_{k+1}, \ldots, \varphi_m) = \{0\}$. By Lemma 4.6, $(\varphi_{k+1}, \ldots, \varphi_m)$ is a coboundary. ∎

Now we are in a position to generalize [1, Theorem 1] to any minimal rotation on a compact metric monothetic group.

PROPOSITION 4.8. *Let T be a minimal rotation on a compact metric monothetic group X, and $\varphi : X \to \mathbb{R}^m$ a continuous map such that T_φ is conservative. Then T_φ is not topologically transitive if and only if there exists a non-zero linear functional $L : \mathbb{R}^m \to \mathbb{R}$ and a continuous function $f : X \to \mathbb{R}$ satisfying the functional equation*

$$L \circ \varphi + f - f \circ T = 0. \tag{9}$$

In [1] a sufficient condition is given for T_φ to be conservative, where φ is a cocycle defined on the one-dimensional torus with values in $\mathbb{R}^m$ ([1, Theorem 2]). The next theorem contains a generalization of that result. Note that it is a sufficient and necessary condition.

THEOREM 4.9. *Let T be a minimal rotation on a compact metric monothetic group X, and $\varphi : X \to \mathbb{R}^m$ a continuous map. Then the following conditions are equivalent*:

(i) *T_φ is conservative.*

(ii) *φ is regular.*

(iii) *If $\widetilde{\varphi} : X \to \mathbb{R}^m/E(\varphi)$ is given by $\widetilde{\varphi}(x) = \varphi(x) + E(\varphi)$, then $E_\infty(\widetilde{\varphi}) = \{0\}$.*

(iv) *$\int_X \varphi \, d\mu = 0$, where μ is the normalized Haar measure on X.*

Proof. Assume that (i) is true, i.e. T_φ is conservative. Clearly, if T_φ is topologically transitive then φ is regular. Suppose T_φ is not topologically transitive. By Fact 2.2 and Proposition 4.7 we may assume that $\varphi = (\varphi_1, \ldots, \varphi_m)$, $E(\varphi) = E(\varphi_1, \ldots, \varphi_k) = \mathbb{R}^k$, $k < m$, and $(\varphi_{k+1}, \ldots, \varphi_m)$ is a coboundary, i.e. $\varphi_j = f_j \circ T - f_j$ for some continuous functions $f_j : X \to \mathbb{R}$, $j = k+1, \ldots, m$. Define $f : X \to \mathbb{R}^m$ by

$$f(x) = (0, \ldots, 0, f_{k+1}(x), \ldots, f_m(x)).$$

Then $\psi = \varphi + f - f \circ T : X \to E(\varphi)$ and φ is regular.

The implication (ii)⇒(iii) is a part of Corollary 2.8.

Assume now that condition (iii) is true. Suppose to the contrary that $\int_X \varphi \, d\mu \neq 0$. Then setting $\varphi = (\varphi_1, \ldots, \varphi_m)$ we may assume that for instance $\int_X \varphi_1 \, d\mu > 0$. Since $(1/n)\varphi_1^{(n)} \to \int_X \varphi_1 \, d\mu$ uniformly, $\varphi_1^{(n_t)} \to \infty$ uniformly for each rigidity time $(n_t)_{t\geq 1}$, hence $E_\infty(\varphi) = \{0,\infty\}$ and $E(\varphi) = \{0\}$. In particular $E_\infty(\widetilde{\varphi}) = \{0,\infty\}$, which is a contradiction.

Assume now that (iv) is true. If $\int_X \varphi\, d\mu = 0$, then $\int_X \varphi_i\, d\mu = 0$, $i = 1,\dots,m$. Changing a basis of $\mathbb{R}^m$ if necessary we may assume that $\varphi_{k+1},\dots,\varphi_m$ are coboundaries and $T_{(\varphi_1,\dots,\varphi_k)}$ is topologically transitive. Then $T_{(\varphi_1,\dots,\varphi_k,\varphi_{k+1},\dots,\varphi_m)}$ is isomorphic to $T_{(\varphi_1,\dots,\varphi_k,0,\dots,0)}$ (as homeomorphisms of $X\times\mathbb{R}^m$). By Theorem 2.9, the flow $(X\times\mathbb{R}^m, T_{(\varphi_1,\dots,\varphi_k,0,\dots,0)})$ is a disjoint union of topologically transitive subflows, each isomorphic to $(X\times\mathbb{R}^k, T_{(\varphi_1,\dots,\varphi_k)})$. The space $X\times\mathbb{R}^k$ is perfect and the flow $T_{(\varphi_1,\dots,\varphi_k)}$ is topologically transitive, hence conservative, and so is T_φ. ■

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