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ON THE PRODUCT FORMULA ON NONCOMPACT GRASSMANNIANS

BY

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Abstract. We study the absolute continuity of the convolution $\delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ of two orbital measures on the symmetric space $\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$, $q>p$. We prove sharp conditions on $X,Y\in\mathfrak{a}$ for the existence of the density of the convolution measure. This measure intervenes in the product formula for the spherical functions. We show that the sharp criterion developed for $\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$ also serves for the spaces $\mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$ and $\mathbf{Sp}(p,q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q)$, $q>p$. We moreover apply our results to the study of absolute continuity of convolution powers of an orbital measure $\delta^\natural_{e^X}$.

1. Introduction. The spaces $\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$, where $q>p$ (which we will assume throughout the paper), are the noncompact duals of real Grassmannians. They are Riemannian symmetric spaces of noncompact type corresponding to root systems of type B_p. The harmonic analysis on these spaces has been intensively developed in recent years ([1, 10, 12, 13, 14]).

We use throughout the paper the usual notations of the harmonic analysis on Riemannian symmetric spaces. The books [7, 8, 9] constitute a standard reference on these topics.

Let $X,Y\in\mathfrak{a}$ and let m_K denote the Haar measure of the group K. We define

$$\delta^\natural_{e^X} = m_K \star \delta_{e^X} \star m_K.$$

The question of the absolute continuity of the convolution $\delta^\natural_{e^X}\star\delta^\natural_{e^Y}$ of two K-invariant orbital measures that we address in our paper has important applications in harmonic analysis itself (the product formula for the spherical functions) and in probability theory (random walks, I_0 characterization of Gaussian measures).

The spherical Fourier transform of the measure $\delta^\natural_{e^X}$ is equal to the spherical function $\phi_\lambda(e^X)$, where λ is a complex-valued linear form on $\mathfrak{a}$. Thus the product $\phi_\lambda(e^X)\phi_\lambda(e^Y)$ is the spherical Fourier transform of the convolution

$$m_{X,Y} = \delta^\natural_{e^X}\star\delta^\natural_{e^Y}.$$

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If we denote by $\mu_{X,Y}$ the projection of the measure $m_{X,Y}$ on $\mathfrak{a}$ via the Cartan decomposition $G = KAK$, then

$$\phi_\lambda(e^X)\phi_\lambda(e^Y) = \int_{\mathfrak{a}} \phi_\lambda(e^H)\, d\mu_{X,Y}(H).$$

Let δ be the density of the invariant measure on $\mathfrak{a}$ in polar coordinates. The existence of a kernel in the last product formula

$$\phi_\lambda(e^X)\phi_\lambda(e^Y) = \int_{\mathfrak{a}^+} \phi_\lambda(e^H) k(H,X,Y)\delta(H)\, dH \tag{1}$$

is equivalent to the absolute continuity of the measure $\mu_{X,Y}$ with respect to the Lebesgue measure on $\mathfrak{a}$, and to the existence of the density of $m_{X,Y}$ on G with respect to the invariant measure dg. When the formula (1) holds, we say that we have a *product formula* for $X, Y \in \mathfrak{a}$. Provided that $X, Y \in \mathfrak{a}^+$, the product formula (1) has been shown previously (see [2] in the rank one case, [3] in the complex case and [4] in the general case). In [4] we were able to relax these conditions and show that $\mu_{X,Y}$ is absolutely continuous provided one of X or Y is in $\mathfrak{a}^+$ as long as the other is nonzero. The density can however exist in some cases when both X and Y are singular. It is a challenging problem to characterize all such pairs X and Y.

This problem was solved in [5] for symmetric spaces with root system of type A_n. We solve it in this paper for the space $\mathbf{SO}_0(p,q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$: we give a definition of an eligible pair (X,Y) (Definition 2.3) and next we prove the necessity (Proposition 3.2) and the sufficiency (Proposition 4.4 and Theorem 4.8) of this property for the absolute continuity of $m_{X,Y}$.

By [3, 4], the density $k(H,X,Y)$ exists if and only if the support $\mathcal{S}_{X,Y} = a(e^X K e^Y)$ of the measure $\mu_{X,Y}|_{\overline{\mathfrak{a}^+}}$ has nonempty interior. Similarly, the density of the measure $m_{X,Y}$ exists if and only if its support $Ke^X Ke^Y K$ has nonempty interior as seen in [5]. These facts are crucial in the proofs of the results of this paper.

We show in Corollary 5.1 that the result for $\mathbf{SO}_0(p,q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ also implies the result for the symmetric spaces $\mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ and $\mathbf{Sp}(p,q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ of type BC_p. We conclude the paper with two further applications of our main result. One of them is a characterization of an optimal convolution power l of the measure $\delta^\natural_{e^X}$, which is absolutely continuous for any $X \neq 0$, $X \in \mathfrak{a}$. Theorem 5.3 solves, on noncompact Grassmannians, a problem raised by Ragozin in [11].

2. Basic properties. We start by reviewing some useful information on the Lie group $\mathbf{SO}_0(p,q)$, its Lie algebra $\mathfrak{so}(p,q)$ and the corresponding root system. Most of this material was given in [14]. For the convenience of the reader, we gather below the properties we will need.

In this paper, E_{ij} is a rectangular matrix with 0's everywhere except at position (i,j) where it is 1.

Recall that $\mathbf{SO}(p,q)$ is the group of matrices $g \in \mathbf{SL}(p+q,\mathbb{R})$ such that $g^T\, I_{p,q}\, g = I_{p,q}$ where

$$I_{p,q} = \begin{bmatrix} -I_p & 0_{p\times q} \\ 0_{q\times p} & I_q \end{bmatrix}.$$

Unless otherwise specified, all 2×2 block decompositions in this paper follow the same pattern.

The group $\mathbf{SO}_0(p,q)$ is the connected component of $\mathbf{SO}(p,q)$ containing the identity. The Lie algebra $\mathfrak{so}(p,q)$ of $\mathbf{SO}_0(p,q)$ consists of the matrices

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

where A and D are skew-symmetric.

A very important element in our investigations is the Cartan decomposition of $\mathfrak{so}(p,q)$ and $\mathbf{SO}(p,q)$. The maximal compact subgroup K is the subgroup of $\mathbf{SO}(p,q)$ consisting of the matrices

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

of size $(p+q)\times(p+q)$ such that $A \in \mathbf{SO}(p)$ and $D \in \mathbf{SO}(q)$ (hence $K \simeq \mathbf{SO}(p)\times\mathbf{SO}(q)$). If $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the set of matrices

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

then the Cartan decomposition is given by $\mathfrak{so}(p,q) = \mathfrak{k}\oplus\mathfrak{p}$ with corresponding Cartan involution $\theta(X) = -X^T$.

The Cartan space $\mathfrak{a} \subset \mathfrak{p}$ is the set of matrices

$$H = \begin{bmatrix} 0_{p\times p} & \mathcal{D}_H & 0_{p\times(q-p)} \\ \mathcal{D}_H & 0_{p\times p} & 0_{p\times(q-p)} \\ 0_{(q-p)\times p} & 0_{(q-p)\times p} & 0_{(q-p)\times(q-p)} \end{bmatrix}$$

where $\mathcal{D}_H = \mathrm{diag}[H_1,\ldots,H_p]$. Its canonical basis is given by the matrices

$$A_i := E_{i,p+i} + E_{p+i,i}, \qquad 1 \le i \le p.$$

The restricted roots and associated root vectors for the Lie algebra $\mathfrak{so}(p,q)$ with respect to $\mathfrak{a}$ are given in Table 1.

The positive roots can be chosen as $\alpha(H) = H_i \pm H_j$, $1 \le i < j \le p$, and $\alpha(H) = H_i$, $i = 1, \ldots, p$. We therefore have the positive Weyl chamber

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : H_1 > \cdots > H_p > 0\}.$$

Table 1. Restricted roots and associated root vectors

Root α	Multiplicity	Root vectors X_α
$\alpha(H)=\pm H_i$ $1\le i\le p$	$q-p$	$X_{ir}^{\pm}=E_{i,2p+r}+E_{2,p+ri}\pm(E_{p+i,2p+r}-E_{2p+r,p+i})$ $r=1,\dots,q-p$
$\alpha(H)=\pm(H_i-H_j)$ $1\le i,j\le p,\ i<j$	1	$Y_{ij}^{\pm}=\pm(E_{ij}-E_{ji}+E_{p+i,p+j}-E_{p+j,p+i})$ $+E_{i,p+j}+E_{p+j,i}+E_{j,p+i}+E_{p+i,j}$
$\alpha(H)=\pm(H_i+H_j)$ $1\le i,j\le p,\ i<j$	1	$Z_{ij}^{\pm}=\pm(E_{ij}-E_{ji}-E_{p+i,p+j}+E_{p+j,p+i})$ $-(E_{i,p+j}+E_{p+j,i})+E_{j,p+i}+E_{p+i,j}$

The simple roots are given by $\alpha_i(H)=H_i-H_{i+1}$, $i=1,\dots,p-1$, and $\alpha_p(H)=H_p$.

The action of the Weyl group. The elements of the Weyl group W act as permutations of the diagonal entries of $\mathcal{D}_X$ with possible sign changes of any number of these entries.

The Lie algebra $\mathfrak{k}$ is generated by the vectors $X_\alpha+\theta X_\alpha$. We will use the notation

$$k_{X_\alpha}^t=e^{t(X_\alpha+\theta X_\alpha)}.$$

The linear space $\mathfrak{p}$ has a basis formed by $A_i\in\mathfrak{a}$, $1\le i\le p$, and by the symmetric matrices $X_\alpha^s:=\frac{1}{2}(X_\alpha-\theta X_\alpha)$ which have the following form:

$$\begin{aligned}
X_{ir}&:=E_{i,2p+r}+E_{2p+r,i}, && 1\le i\le p,\ 1\le r\le q-p,\\
Y_{ij}&:=E_{i,p+j}+E_{j,p+i}+E_{p+j,i}+E_{p+i,j}, && 1\le i<j\le p,\\
Z_{ij}&:=E_{i,p+j}-E_{j,p+i}+E_{p+j,i}-E_{p+i,j}, && 1\le i<j\le p.
\end{aligned}$$

If we were to follow the notation of [5], we should write $(X_{ir}^{+})^s$ etc., but we simplify the symbols to X_{ir}, Y_{ij} and Z_{ij}. If we write a matrix from the space $\mathfrak{p}$ in the form

$$\begin{bmatrix}0&B\\B^T&0\end{bmatrix}=\begin{bmatrix}0&B_1&B_2\\B_1^T&0&0\\B_2^T&0&0\end{bmatrix}$$

where B_1 is a square $p\times p$ matrix and B_2 is a $p\times(q-p)$ matrix, then the matrices

$$\begin{bmatrix}0&B_1&0\\B_1^T&0&0\\0&0&0\end{bmatrix}$$

are generated by the vectors A_i (for the diagonal entries of B_1 and B_1^T),

Y_{ij} and Z_{ij} (for the nondiagonal entries), whereas the matrices

$$\begin{bmatrix} 0 & 0 & B_2 \\ 0 & 0 & 0 \\ B_2^T & 0 & 0 \end{bmatrix}$$

are spanned by the vectors X_{ir}.

We now recall the useful matrix $S \in \mathbf{SO}(p+q)$ which allows us to diagonalize simultaneously all the elements of $\mathfrak{a}$. Let

$$S = \begin{bmatrix} \frac{\sqrt{2}}{2} I_p & 0_{p\times(q-p)} & \frac{\sqrt{2}}{2} J_p \\ \frac{\sqrt{2}}{2} I_p & 0_{p\times(q-p)} & -\frac{\sqrt{2}}{2} J_p \\ 0_{(q-p)\times p} & I_{q-p} & 0_{(q-p)\times p} \end{bmatrix}$$

where $J_p = (\delta_{i,p+1-i})$ is a matrix of size $p \times p$. If

$$H = \begin{bmatrix} 0 & \mathcal{D}_H & 0 \\ \mathcal{D}_H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $\mathcal{D}_H = \operatorname{diag}[H_1, \ldots, H_p]$ then

$$S^T H S = \operatorname{diag}[H_1, \ldots, H_p, \overbrace{0, \ldots, 0}^{q-p}, -H_p, \ldots, -H_1].$$

The "group" version of this result is as follows:

$$S^T e^H S = \operatorname{diag}[e^{H_1}, \ldots, e^{H_p}, \overbrace{1, \ldots, 1}^{q-p}, e^{-H_p}, \ldots, e^{-H_1}].$$

REMARK 2.1. The Cartan projection $a(g)$ on the group $\mathbf{SO}_0(p,q)$, defined as usual by

$$g = k_1 e^{a(g)} k_2, \qquad a(g) \in \overline{\mathfrak{a}^+},\ k_1, k_2 \in K,$$

is related to the singular values of $g \in \mathbf{SO}(p,q)$ in the following way. Recall that the singular values of g are defined as the nonnegative square roots of the eigenvalues of $g^T g$. Write $H = a(g)$. We have

$$g^T g = k_2^T e^{2H} k_2 = (k_2^T S)(S^T e^{2H} S)(S^T k_2)$$

where $S^T e^{2H} S$ is a diagonal matrix with nonzero entries

$$e^{2H_1}, \ldots, e^{2H_p}, \overbrace{1, \ldots, 1}^{q-p}, e^{-2H_p}, \ldots, e^{-2H_1}.$$

Write $a_j = e^{H_j}$. Thus the set of $p+q$ singular values of g contains the value 1 repeated $q-p$ times and the $2p$ values $a_1, \ldots, a_p$, $a_1^{-1}, \ldots, a_p^{-1}$.

Hence, in order to determine $a(g)$, we can compute the $p+q$ singular values of $g^T g$ and omit $q-p$ values 1 always appearing among them. The $2p$

remaining singular values may be ordered $a_1 \geq \cdots \geq a_p \geq a_p^{-1} \geq \cdots \geq a_1^{-1}$ with $a_1 \geq \cdots \geq a_p \geq 1$. Then

$$a(g) = \begin{bmatrix} 0 & \mathcal{D}_{a(g)} & 0 \\ \mathcal{D}_{a(g)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad \mathcal{D}_{a(g)} = \operatorname{diag}[\log a_1, \ldots, \log a_p].$$

Summarizing, if for $g \in \mathbf{SO}(p,q) \subset \mathbf{SL}(p+q)$ the $\mathbf{SL}(p+q)$-Cartan decomposition reads $g = k_1 e^{\tilde{a}(g)} k_2$, $k_1, k_2 \in \mathbf{SO}(p+q)$, then $\mathcal{D}_{a(g)} = \pi_p(\tilde{a}(g))$, where π_p denotes the projection $\pi_p(\operatorname{diag}[h_1, \ldots, h_{p+q}]) = \operatorname{diag}[h_1, \ldots, h_p]$.

Singular elements of $\mathfrak{a}$. In what follows, we will consider singular elements $X, Y \in \partial\mathfrak{a}^+$. As in [5], we need to control the irregularity of X and Y, i.e. consider the simple positive roots annihilating X and Y. Special attention must be paid to the last simple root α_p, different from the roots α_i, $i = 1, \ldots, p-1$, that generate a root subsystem of type A_{p-1}. We introduce the following definition of the configuration of $X \in \overline{\mathfrak{a}^+}$.

Definition 2.2. Let $X \in \overline{\mathfrak{a}^+}$. There exist nonnegative integers $s_1 \geq 1$, $\ldots, s_r \geq 1$, $u \geq 0$ such that

$$\mathcal{D}_X = \operatorname{diag}[\overbrace{x_1, \ldots, x_1}^{s_1}, \overbrace{x_2, \ldots, x_2}^{s_2}, \ldots, \overbrace{x_r, \ldots, x_r}^{s_r}, \overbrace{0, \ldots, 0}^{u}]$$

with $x_1 > \cdots > x_r > 0$ and $\sum s_i + u = p$. We say that $[s_1, \ldots, s_r; u]$ is the *configuration* of X. Writing $\mathbf{s} = (s_1, \ldots, s_r)$, we will shorten the notation of the configuration of X to $[\mathbf{s}; u]$. We will also write $X = X[\mathbf{s}; u]$.

Note that $X = 0$ is equivalent to $u = p$ and has configuration $[0; p]$. A regular $X \in \mathfrak{a}^+$ has the configuration $[\mathbf{1}; 0] = [1^p; 0]$. We extend naturally the definition of configuration to any $X \in \mathfrak{a}$, whose configuration is defined as that of the projection $\pi(X)$ of X on $\overline{\mathfrak{a}^+}$.

We will write $\max \mathbf{s} = \max_i s_i$ and $\max(\mathbf{s}, u) = \max(\max \mathbf{s}, u)$. We will show that in the case of the symmetric spaces $\mathbf{SO}_0(p,q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$, $q > p$, the criterion for the existence of the density of the convolution $\delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ is given by the following definition of an eligible pair X, Y:

Definition 2.3. Let $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v]$ be two elements of $\mathfrak{a}$. We say that X and Y are *eligible* if

$$\max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) \leq 2p.$$

Observe that if X and Y are eligible, then $X \neq 0$ and $Y \neq 0$.

3. Necessity of the eligibility condition. In the proof of the necessity of the eligibility condition we will use the result stated in [6, Step 1, p. 1767]:

LEMMA 3.1. *Consider two diagonal matrices U and V such that*

$$U = \mathrm{diag}[\overbrace{u_0,\dots,u_0}^{r}, u_1,\dots,u_{N-r}], \qquad V = \mathrm{diag}[\overbrace{v_0,\dots,v_0}^{N-s}, v_1,\dots,v_s]$$

where $s+1 \le r < N$, $s \ge 1$, and the u_i's and v_j's are arbitrary. Then each element of $\tilde a(e^U \mathbf{SU}(N,\mathbf{F})e^V)$ has at least $r-s$ entries equal to u_0+v_0.

We will use Lemma 3.1 with $N = p+q$ in the proofs of Proposition 3.2 and Theorem 5.3.

PROPOSITION 3.2. *If $X[\mathbf{s};u]$ and $Y[\mathbf{t},v]$ are not eligible then the measure $\mu_{X,Y}$ is not absolutely continuous with respect to the Lebesgue measure on $\mathfrak{a}$.*

Proof. Suppose $\max(\mathbf{s},2u)+\max(\mathbf{t},2v) > 2p$ and consider the matrices $a(e^X k e^Y)$, $k \in \mathbf{SO}(p)\times\mathbf{SO}(q)$. Applying Remark 2.1, the diagonal $p\times p$ matrix $\mathcal{D}_{a(e^X k e^Y)}$ contains the p largest diagonal terms of the matrix

$$\tilde a(e^X k e^Y) = \tilde a(\overbrace{(S^T e^X S)}^{e^{S^T X S}}\ \overbrace{(S^T k S)}^{\in \mathbf{SO}(p+q)}\ \overbrace{(S^T e^Y S)}^{e^{S^T Y S}}).$$

If $u+v>p$ then there are $r-s = r+(N-s)-N = (2u+q-p)+(2v+q-p)-(p+q) = 2(u+v-p)+(q-p)$ repetitions of $0+0=0$ in the coefficients of $\tilde a(e^X k e^Y)$. Therefore 0 occurs at least $u+v-p>0$ times as a diagonal entry of $\mathcal{D}_H$ for every $H \in a(e^X K e^Y)$, which implies that $a(e^X K e^Y)$ has empty interior.

If $2u+\max(\mathbf{t})>2p$ denote $t=\max(\mathbf{t})$. Let $Y_i \ne 0$ be repeated t times in $\mathcal{D}_Y$. Then there are $r-s = r+(N-s)-N = (2u+q-p)+t-(p+q) = 2u+t-2p$ repetitions of Y_i+0 in the coefficients of $\tilde a(e^X k e^Y)$. Therefore Y_i occurs at least $2u+t-2p>0$ times as a diagonal entry of $\mathcal{D}_H$ for every $H \in a(e^X K e^Y)$, which implies that $a(e^X K e^Y)$ has empty interior. ∎

4. Sufficiency of the eligibility condition. We use basic ideas and some results and notations of [5, Section 3].

PROPOSITION 4.1.

(i) *The density of the measure $m_{X,Y}$ exists if and only if its support $Ke^X K e^Y K$ has nonempty interior.*
(ii) *Consider the analytic map $T\colon K\times K\times K \to \mathbf{SO}_0(p,q)$ defined by*

$$T(k_1,k_2,k_3) = k_1 e^X k_2 e^Y k_3.$$

If the derivative of T is surjective for some choice of $\mathbf{k}=(k_1,k_2,k_3)$, then the set $T(K\times K\times K) = Ke^X K e^Y K$ contains an open set.

Proof. Part (i) follows for reasons explained in [4] in the case of the support of the measure $\mu_{X,Y}$, equal to $a(e^X K e^Y)$. Part (ii) is justified for example in [8, p. 479]. ∎

PROPOSITION 4.2. *Let $U_Z = \mathfrak{k} + \mathrm{Ad}(e^Z)\mathfrak{k}$. If there exists $k \in K$ such that*

$$U_{-X} + \mathrm{Ad}(k)U_Y = \mathfrak{g} \tag{2}$$

then the measure $m_{X,Y}$ is absolutely continuous.

Proof. We want to show that the derivative of T is surjective for some choice of $\mathbf{k} = (k_1, k_2, k_3)$.

Let $A, B, C \in \mathfrak{k}$. The derivative of T at $\mathbf{k}$ in the direction of (A, B, C) equals

$$\begin{aligned} dT_{\mathbf{k}}(A,B,C) &= \frac{d}{dt}\bigg|_{t=0} e^{tA}k_1e^Xe^{tB}k_2e^Ye^{tC}k_3 \\ &= Ak_1e^Xk_2e^Yk_3 + k_1e^XBk_2e^Yk_3 + k_1e^Xk_2e^YCk_3. \end{aligned} \tag{3}$$

We now transform the space of all matrices of the form (3) without modifying its dimension:

$$\begin{aligned} &\dim\{Ak_1e^Xk_2e^Yk_3 + k_1e^XBk_2e^Yk_3 + k_1e^Xk_2e^YCk_3 \colon A,B,C \in \mathfrak{k}\} \\ &\quad = \dim\{k_1^{-1}Ak_1e^Xk_2e^Y + e^XBk_2e^Y + e^Xk_2e^YC \colon A,B,C \in \mathfrak{k}\} \\ &\quad = \dim\{Ae^Xk_2e^Y + e^XBk_2e^Y + e^Xk_2e^YC \colon A,B,C \in \mathfrak{k}\} \\ &\quad = \dim\{e^{-X}Ae^X + B + k_2e^YCe^{-Y}k_2^{-1} \colon A,B,C \in \mathfrak{k}\}. \end{aligned}$$

The space in the last line equals $\mathfrak{k} + \mathrm{Ad}(e^{-X})(\mathfrak{k}) + \mathrm{Ad}(k_2)(\mathrm{Ad}(e^Y)(\mathfrak{k})) = U_{-X} + \mathrm{Ad}(k_2)U_Y$. ∎

In order to apply the condition (2), we will consider convenient root vectors and their symmetrizations. For $Z \in \mathfrak{a}$, we define the space

$$V_Z = \mathrm{span}\{X_\alpha^s \colon \alpha(Z) \neq 0\},$$

where $X_\alpha^s = X_\alpha - \theta X_\alpha$. Note that this space would be called V_Z^S in the notation of [5].

LEMMA 4.3. *Let $Z \in \mathfrak{a}$. The vector space $U_Z = \mathfrak{k} + \mathrm{Ad}(e^Z)(\mathfrak{k})$ contains the root vectors X_α for which $\alpha(Z) \neq 0$. Consequently, $V_Z = V_{-Z} \subset U_{\pm Z}$.*

Proof. Suppose α is a root such that $\alpha(Z) \neq 0$. Note that $[Z, X_\alpha] = \alpha(Z)X_\alpha$ and $[Z, \theta(X_\alpha)] = -\alpha(Z)\theta(X_\alpha)$. Let $U = X_\alpha + \theta(X_\alpha) \in \mathfrak{k}$. Now,

$$\begin{aligned} \mathrm{Ad}(e^Z)U &= e^{\mathrm{ad}\,Z}(X_\alpha + \theta(X_\alpha)) \\ &= \sum_{k=0}^{\infty} \frac{(\mathrm{ad}\,Z)^k}{k!}(X_\alpha + \theta(X_\alpha)) \\ &= \sum_{k=0}^{\infty} \frac{(\mathrm{ad}\,Z)^k}{k!}X_\alpha + \sum_{k=0}^{\infty} \frac{(\mathrm{ad}\,Z)^k}{k!}\theta(X_\alpha) \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha(Z))^k}{k!} X_\alpha + \sum_{k=0}^{\infty} \frac{(-1)^k(\alpha(Z))^k}{k!} \theta(X_\alpha)$$
$$= e^{\alpha(Z)} X_\alpha + e^{-\alpha(Z)} \theta(X_\alpha).$$

Therefore $X_\alpha = (e^{\alpha(Z)} - e^{-\alpha(Z)})^{-1}(-e^{-\alpha(Z)}U + \mathrm{Ad}(e^Z)U) \in \mathfrak{k} + \mathrm{Ad}(e^Z)(\mathfrak{k}) = U_Z$. The vector θX_α is a root vector for the root $-\alpha$, so we also have $\theta X_\alpha \in U_Z$. ∎

Proposition 4.4. *If there exists $k \in K$ such that*

$$V_X + \mathrm{Ad}(k) V_Y = \mathfrak{p} \tag{4}$$

then the measure $m_{X,Y}$ is absolutely continuous.

Proof. We want to prove formula (2). By Lemma 4.3, we know that $V_X = V_{-X} \subset U_{-X}$ and $V_Y \subset U_Y$. As $\mathfrak{k} \subset U_X$, we see that (4) implies (2). ∎

Later in this section, in Theorem 4.8, we will show that the hypotheses of Proposition 4.4 are always satisfied for X and Y eligible. For technical reasons, in order to make an induction proof work, we will show more, i.e. that a "better" matrix $k \in K$ exists such that the formula (4) holds. The meaning of a "better" k will be similar to the notion of a total matrix given in Definition 4.5. Here is a definition and a lemma about total matrices in K. The reasonning of the proof of this lemma will be used in a more general setting in Steps 2 and 3 of the proof of Theorem 4.8.

Definition 4.5. We say that a square $n \times n$ matrix k is *total* if by removing any $r < n$ rows and r columns of k we always obtain a nonsingular matrix.

Note that this definition of totality is more restrictive than in [5, Definition 3.7].

Lemma 4.6. *The set of matrices in $\mathbf{SO}(n)$ which are total is dense and open in $\mathbf{SO}(n)$.*

Proof. Consider first the set $M_{I,J} = M_{\{i_1,\ldots,i_r\},\{j_1,\ldots,j_r\}} \subset \mathbf{SO}(n)$ of orthogonal matrices which remain nonsingular once the rows of indices $i_1, \ldots, i_r$ and the columns of indices $j_1, \ldots, j_r$ are removed. To see that such matrices exist, take the identity matrix (whose determinant is 1 if we remove, say, the first r rows and columns). By taking convenient permutations of the rows and columns of the identity matrix, we obtain an element of $M_{I,J}$. Given that $\mathbf{SO}(n) \setminus M_{I,J}$ corresponds to the set of zeros of a certain determinant function, it must be closed and nowhere dense in $\mathbf{SO}(n)$.

To conclude, it suffices to notice that the set of total matrices in $\mathbf{SO}(n)$ is the finite intersection of all the sets $M_{I,J}$. ∎

In the proof of the main Theorem 4.8 we will need the following technical lemma.

LEMMA 4.7.

(i) *For the root vectors* X_{ir}^+, Z_{ij}^+, Y_{ij}^+, *we have*

$$\begin{aligned} \mathrm{Ad}(e^{t(X_{ir}^+ + \theta X_{ir}^+)})(X_{ir}) &= \cos(2t)X_{ir} + 2\sin(2t)A_i, \\ \mathrm{Ad}(e^{t(Y_{ij}^+ + \theta Y_{ij}^+)})(Y_{ij}) &= \cos(4t)Y_{ij} + 2\sin(4t)(A_i - A_j), \\ \mathrm{Ad}(e^{t(Z_{ij}^+ + \theta Z_{ij}^+)})(Z_{ij}) &= \cos(4t)Z_{ij} + 2\sin(4t)(A_i + A_j). \end{aligned}$$

(ii) *The functions* $\mathrm{Ad}(e^{t(X_{ir}^+ + \theta X_{ir}^+)})$, $\mathrm{Ad}(e^{t(Y_{ij}^+ + \theta Y_{ij}^+)})$ *and* $\mathrm{Ad}(e^{t(Z_{ij}^+ + \theta Z_{ij}^+)})$ *applied to the other symmetrized root vectors do not produce any components in* $\mathfrak{a}$.

Proof. This is just a matter of carefully evaluating

$$\mathrm{Ad}(e^{t(Z+\theta Z)})(W) = e^{t\,\mathrm{ad}(Z+\theta Z)}(W) = \sum_{k=0}^{\infty} (\mathrm{ad}(Z+\theta Z))^k(W)\frac{t^k}{k!}.$$

For (ii), use the well known properties of the root system: $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$ and $[X_\alpha, \theta X_\alpha] \in \mathfrak{a}$. ■

By Proposition 4.4, in order to justify the sufficiency of the eligibility condition, it is enough to prove the following theorem. This is the main result of this section.

THEOREM 4.8. *Let* $G = \mathbf{SO}_0(p,q)$ *and let* $X, Y \in \mathfrak{a}$. *If* X *and* Y *are eligible then there exists a matrix* $k \in K$ *such that*

$$V_X + \mathrm{Ad}(k)V_Y = \mathfrak{p}. \tag{5}$$

Proof. We will assume that $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v]$. Observe that the spaces V_X and V_Y depend on the Weyl chambers where X and Y belong. However (see [5, Lemma 3.3 and Reduction 1, p. 759]), the property (5) is equivalent to $V_{w_1X} + \mathrm{Ad}(k')V_{w_2Y} = \mathfrak{p}$ for any $w_1, w_2 \in W$ and a convenient $k' \in K$. Throughout the proof we will assume that the diagonal entries of $\mathcal{D}_X$ and $\mathcal{D}_Y$ are nonnegative and we will arrange (permute) them suitably.

To lighten the notation, for a matrix c of size $p \times q$, we will consider the $(p+q) \times (p+q)$ symmetric matrix

$$c^s = \begin{bmatrix} 0 & c \\ c^T & 0 \end{bmatrix} \in \mathfrak{p}.$$

The proof will be organized in the following way:

1. Proof for $q = p + 1$ using induction on p:

   (a) Proof for $p = 2$ and $q = 3$.

(b) Proof of the induction step:

(i) Proof in the case $u > 0$ or $v > 0$.

(ii) Proof in the case $X[p;0], Y[p;0]$.

2. Proof that the case (p,q) implies the case $(p,q+1)$.

1. Proof for $q = p+1$ using induction on p

(a) *Proof for $p = 2$ and $q = 3$.* This corresponds to the space $\mathbf{SO}_0(2,3)$. Only two configurations $[2;0]$ and $[1;1]$ may be realized by singular nonzero X and Y. When $Z \in \overline{\mathfrak{a}}^+$, we have $\mathcal{D}_{Z[1;1]} = \operatorname{diag}[z,0]$, and $\mathcal{D}_{Z[2;0]} = \operatorname{diag}[z,z]$, $z \neq 0$. It is easy to check that in all three possible cases:

(i) $X[2;0]$, $Y[2;0]$,

(ii) $X[2;0]$, $Y[1;1]$ or $X[1;1]$, $Y[2;0]$,

(iii) $X[1;1]$, $Y[1;1]$,

both X and Y are eligible. Note that

$$\mathfrak{p} = \left\{ \begin{bmatrix} h_1 & a & b \\ c & h_2 & d \end{bmatrix}^s : h_1, h_2, a, b, c, d \in \mathbb{R} \right\},$$

$$V_{Z[2;0]} = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \end{bmatrix}^s : a, b, c \in \mathbb{R} \right\}, \quad V_{Z[1;1]} = \left\{ \begin{bmatrix} 0 & a & c \\ b & 0 & 0 \end{bmatrix}^s : a, b, c \in \mathbb{R} \right\}.$$

If

$$k_1 = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

then

$$\operatorname{Ad}(k_1) V_{Z[2;0]} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} a & \frac{1}{2}(a-b+c) & \frac{1}{2}(a+b-c) \\ -\frac{\sqrt{2}}{2} a & \frac{1}{2}(a-b-c) & \frac{1}{2}(a+b+c) \end{bmatrix}^s : a, b, c \in \mathbb{R} \right\},$$

$$\operatorname{Ad}(k_1) V_{Z[1;1]} = \left\{ \begin{bmatrix} -\frac{\sqrt{2}}{2} b & \frac{1}{2}(a-c) & \frac{1}{2}(a+c) \\ \frac{\sqrt{2}}{2} b & \frac{1}{2}(a-c) & \frac{1}{2}(a+c) \end{bmatrix}^s : a, b, c \in \mathbb{R} \right\}.$$

If

$$k_2 = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$$

then

$$\mathrm{Ad}(k_2)V_{Z[1;1]} = \left\{ \begin{bmatrix} -\frac{1}{2}(b+c) & \frac{\sqrt{2}}{2}a & \frac{1}{2}(-b+c) \\ \frac{1}{2}(b-c) & \frac{\sqrt{2}}{2}a & \frac{1}{2}(b+c) \end{bmatrix}^s : a,b,c \in \mathbb{R} \right\}.$$

We easily verify that in the cases (i) and (iii) we have $V_X + \mathrm{Ad}(k_1)V_Y = \mathfrak{p}$. For $X[2;0]$ and $Y[1;1]$, we can see that $V_X + \mathrm{Ad}(k_2)V_Y = \mathfrak{p}$.

(b) *Proof of the induction step*

(i) *Proof in the case $u > 0$ or $v > 0$.* We consider the space

$$\mathbf{SO}_0(p,p+1)/\mathbf{SO}(p) \times \mathbf{SO}(p+1)$$

with $p > 2$ and the case when $X[\mathbf{s};u]$ and $Y[\mathbf{t};v]$ with $u > 0$ or $v > 0$. We assume $u \geq v$. We choose the predecessors in $\mathbf{SO}_0(p-1,p)/\mathbf{SO}(p-1)\times\mathbf{SO}(p)$ in the following way:

$$X' = X'[\mathbf{s};u-1], \qquad Y' = Y'[\mathbf{t}';v]$$

where $\mathbf{t}'$ means that we suppress one term from the longest block of size $\max\mathbf{t}$. Note that if $p > 2$ then $\mathbf{t}'$ is not the zero partition (otherwise, $\mathbf{t}$ would have been the partition $[1]$ meaning that $u \geq v = p-1$, which would make X and Y ineligible).

We arrange X, X', Y, Y' in the following way.

1. The first diagonal entry of $\mathcal{D}_X$ is zero and all the other zeros are at the end. The diagonal entries of $\mathcal{D}_{X'}$ are those of $\mathcal{D}_X$ without the first zero:

$$\mathcal{D}_X = \mathrm{diag}[0, \overbrace{x_1,\dots,x_{p-u}}^{\neq 0}, \overbrace{0,\dots,0}^{u-1}], \qquad \mathcal{D}_{X'} = \mathrm{diag}[\overbrace{x_1,\dots,x_{p-u}}^{\neq 0}, \overbrace{0,\dots,0}^{u-1}].$$

2. We put a longest block of size t of equal diagonal entries y_1 of $\mathcal{D}_Y$ in the beginning of Y. The diagonal entries of $\mathcal{D}_{Y'}$ are those of $\mathcal{D}_Y$ with the first entry omitted:

$$\mathcal{D}_Y = \mathrm{diag}[\overbrace{y_1,\dots,y_1}^{t}, y_2,\dots,y_s], \qquad \mathcal{D}_{Y'} = \mathrm{diag}[\overbrace{y_1,\dots,y_1}^{t-1}, y_2,\dots,y_s].$$

It is easy to check that if X, Y are eligible in $\mathbf{SO}_0(p,p+1)$ then X', Y' are eligible in $\mathbf{SO}_0(p-1,p)$.

STEP 1. By the induction hypothesis, there is $k_0 \in \mathbf{SO}(p-1)\times\mathbf{SO}(p)$ such that

$$V_{X'} + \mathrm{Ad}(k_0)V_{Y'} = \mathfrak{p}'. \tag{6}$$

We embed $K' = \mathbf{SO}(p-1)\times\mathbf{SO}(p)$ in $\mathbf{SO}(p)\times\mathbf{SO}(p+1)$:

$$K' = \begin{bmatrix} 1 & & & \\ & \mathbf{SO}(p-1) & & \\ & & 1 & \\ & & & \mathbf{SO}(p) \end{bmatrix} \subset \begin{bmatrix} \mathbf{SO}(p) & \\ & \mathbf{SO}(p+1) \end{bmatrix}.$$

Hence, we have (taking the natural embedding of $\mathfrak{p}'$ into $\mathfrak{p}$)

$$V_1 := V_{X'} + \mathrm{Ad}(k_0)V_{Y'} = \mathfrak{p}' = \begin{bmatrix} 0 & B' \\ B'^T & 0 \end{bmatrix} \tag{7}$$

where

$$B' = \left[\begin{array}{c|c} 0_{1\times 1} & 0_{1\times p} \\ \hline 0_{p\times 1} & B''_{(p-1)\times p} \end{array}\right]$$

and B'' is arbitrary (note that $\mathfrak{p}'$ is of dimension $(p-1)p$). We must show that for some $k \in K$, the space $V_X + \mathrm{Ad}(k)V_Y$ equals $\mathfrak{p}$, i.e.

(i) $V_X + \mathrm{Ad}(k)V_Y$ contains $\mathfrak{p}'$ embedded into $\mathfrak{p}$ as in (7);
(ii) $V_X + \mathrm{Ad}(k)V_Y$ contains all the matrices of the form

$$C = \left[\begin{array}{c|c} * & * \cdots * \\ \hline * & \\ \vdots & 0_{(p-1)\times p} \\ * & \end{array}\right]^s .$$

New vectors in V_X and V_Y. In order to prove the induction conclusion, we must now use the elements of V_X and V_Y which do not come from $V_{X'}$ or $V_{Y'}$. They appear by the interaction of, respectively, the first diagonal entry of $\mathcal{D}_X$ with the others of $\mathcal{D}_X$ and the interaction of the first diagonal entry of $\mathcal{D}_Y$ with the others of $\mathcal{D}_Y$. We see that the new independent root vectors in V_X and V_Y are respectively

$$N_X = \{Y_{1j}, Z_{1j} \colon , j = 2, \ldots, p+1-u\},$$
$$N_Y = \{X_1, Y_{1i}, Z_{1j} \colon i = t+1, \ldots, p,\ j = 2 \ldots, p\}$$

where $t = \max \mathbf{t} \geq 1$ and we wrote X_1 for X_{11}. Note that N_X has $2p-2u$ elements while N_Y has $2p-t$.

Step 2. *We show that there exists $k_0' \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ for which (6) holds, and with the following property:*

The space $V_2 := \mathrm{Ad}(k_0')\mathrm{span}(N_Y)$ is of dimension $2p-t$ and its elements can be written in the form

$$\left[\begin{array}{c|cccccc} 0 & \sigma_1 & \ldots & \sigma_r & a_1 & \ldots & a_{p-r} \\ \hline \tau_1 & & & & & & \\ \vdots & & & & & & \\ \tau_s & & & 0 & & & \\ a_{p-r+1} & & & & & & \\ \vdots & & & & & & \\ a_{2p-t} & & & & & & \end{array}\right]^s \tag{8}$$

with $r = [(t-1)/2]$, $s = t - 1 - r$, $a_i \in \mathbb{R}$ *arbitrary*, $\sigma_i = \sigma_i(a_1, \ldots, a_{2p-t})$ *and* $\tau_j = \tau_j(a_1, \ldots, a_{2p-t})$, $i \le r, j \le s$.

We will not need to write explicitly the functions σ_i and τ_j. Note that $s = r$ if t is odd and $s = r + 1$ if t is even.

To justify Step 2, we write

$$k_0 = \begin{bmatrix} 1 & & & \\ & k_{01} & & \\ & & 1 & \\ & & & k_{02} \end{bmatrix}$$

where $k_{01} \in \mathbf{SO}(p-1)$ and $k_{02} \in \mathbf{SO}(p)$. Let $\alpha_1, \ldots, \alpha_{p-1}$ be the columns of the matrix k_{01} and $\beta_1, \ldots, \beta_p$ the columns of the matrix k_{02}. A simple block multiplication to compute the action of $\mathrm{Ad}(k_0)$ on the elements of N_Y gives the linearly independent matrices

$$\begin{gathered} \mathrm{Ad}(k_0)X_1 = \left[\begin{array}{c|c} 0 & \beta_p^T \\ \hline 0 & 0 \end{array}\right]^s, \quad \mathrm{Ad}(k_0)Y_{1i} = \left[\begin{array}{c|c} 0 & \beta_{i-1}^T \\ \hline \alpha_{i-1} & 0 \end{array}\right]^s, \quad i = t+1, \ldots, p, \\ \mathrm{Ad}(k_0)Z_{1i} = \left[\begin{array}{c|c} 0 & \beta_{i-1}^T \\ \hline -\alpha_{i-1} & 0 \end{array}\right]^s, \quad i = 2, \ldots, p. \end{gathered} \tag{9}$$

Let us write β_i' for a column β_i from which we have removed the first r entries, and α_i' for a column α_i with the first s entries omitted. In order to prove the statement of Step 2, we must show that the matrices obtained by replacing β_i by β_i' and α_i by α_i' in (9) are still linearly independent. This is equivalent to the linear independence of the matrices

$$\begin{gathered} \left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline -\alpha_i' & 0 \end{array}\right]^s, \quad i = 1, \ldots, t-1, \\ \left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array}\right]^s, \quad i = t, \ldots, p, \qquad \left[\begin{array}{c|c} 0 & 0 \\ \hline \alpha_i' & 0 \end{array}\right]^s, \quad i = t, \ldots, p-1. \end{gathered} \tag{10}$$

We will reason in the same way as in Lemma 4.6.

It is enough to show that there exists at least one choice of matrices k_{01} and k_{02} such that the matrices in (10) are linearly independent. Then, as in Lemma 4.6, it will follow that such matrices form a dense open subset in $\mathbf{SO}(p-1) \times \mathbf{SO}(p)$. By choosing k_0' with the matrices in (10) linearly independent and close enough to k_0, property (6) will be preserved for k_0'.

Pick $k_{01} = I_{p-1}$, which implies that $\alpha_i' = \mathbf{0}$ for $i = 1, \ldots, s$ and $\alpha_i' = \mathbf{e}_{i-s}$ for $i > s$. With this choice, (10) becomes

$$\left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array}\right]^s, \quad i = 1, \ldots, s, \qquad \left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline -\mathbf{e}_{i-s} & 0 \end{array}\right]^s, \quad i = s+1, \ldots, t-1,$$
$$\left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array}\right]^s, \quad i = t, \ldots, p, \qquad \left[\begin{array}{c|c} 0 & 0 \\ \hline \mathbf{e}_{i-s} & 0 \end{array}\right]^s, \qquad i = t, \ldots, p-1,$$

which are linearly independent provided that

$$\left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array}\right]^s, \quad i = 1, \ldots, s, \qquad \left[\begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array}\right]^s, \quad i = t, \ldots, p,$$

are linearly independent. This is the case for a total matrix $k_{02} \in \mathbf{SO}(p)$ or by taking convenient permutations of the rows and columns of the identity matrix I_p.

Step 3. We show that there is a proper subset N_X' of N_X such that if

$$V_3 := \operatorname{span}(N_X') + V_{X'} + \operatorname{Ad}(k_0')V_Y = \operatorname{span}(N_X') + V_1 + V_2,$$

then $\dim V_3 = pq - 1 = \dim \mathfrak{p} - 1$ and V_3 is given by

$$V_3 = \left\{ \left[\begin{array}{c|cccc} 0 & a_1 & a_2 & \ldots & a_p \\ \hline a_{p+1} & & & & \\ \vdots & & & \mathfrak{p}' & \\ a_{2p-1} & & & & \end{array}\right]^s : a_1, \ldots a_{2p-1} \in \mathbb{R} \right\}. \tag{11}$$

Note that in matrices from the space V_2, there are $r = [(t-1)/2]$ pairs (σ_i, τ_i) plus possibly an extra τ_s if t is even and therefore $s = r+1$. Note also that $t + 2u \leq 2p$ implies that $p - u \geq s \geq r$. For $j \leq r \leq p-u$, each pair

$$Y_{1j} = \left[\begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline \mathbf{e}_{j-1} & 0 \end{array}\right]^s, \qquad Z_{1j} = \left[\begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline -\mathbf{e}_{j-1} & 0 \end{array}\right]^s$$

of elements of N_X allows us to replace σ_j and τ_j by independent variables. If t is odd, all the σ_j's and τ_j's will be taken care of and at least two elements of N_X will remain off N_X'. If t is even, all the σ_j's and τ_j's, $1 \leq j \leq r$, will be replaced by independent variables and only τ_s will remain. Now, letting the coefficient a_1 "vis-à-vis" the remaining τ_s be equal to 1 and all the other variables a_i equal to 0, we get either $\tau_s = 1$ or -1 or $\tau_s \neq \pm 1$. If $\tau_s = 1$ then Z_{1s} allows us to introduce the missing independent variable, if $\tau_s = -1$ then adding Y_{1s} to N_X' will do the trick. In the case $\tau_s \neq \pm 1$ we choose indifferently between Y_{1s} and Z_{1s}. In all cases the set $N_X \setminus N_X'$ has at least one element.

Step 4. *Let v_1 be the positive root vector corresponding to an element of $N_X \setminus N_X'$. We denote $k_1^t = k_{v_1}^t$. There exists $\epsilon > 0$ such that for $t \in (0, \epsilon)$,*

$$V_4^t := \operatorname{Ad}(k_1^t)(\operatorname{span}(N_X') + V_{X'}) + \operatorname{Ad}(k_0')V_Y = V_3.$$

Observe that v_1 is equal to Z_{1j}^+ or Y_{1j}^+ for one of the remaining Z_{1j} or Y_{1j} that was not used in the preceding step. We have $V_4^t \subset V_4^0$ for all t according to Lemma 4.7(ii).

Recall the definition of $k_{X_\alpha}^t = e^{t(X_\alpha+\theta X_\alpha)}$, $t > 0$. Let $d(t) = \dim V_4^t$; for $t = 0$ we have $k_1^0 = \mathrm{Id}$, and $\mathrm{Ad}(k_1^0)(\mathrm{span}(N'_X) + V_{X'}) + \mathrm{Ad}(k'_0)V_Y = V_3$ is of dimension $pq-1$, so $d(0) = pq-1$. The equality $d(t) = d(0)$ is equivalent to nonnullity of an appropriate determinant continuous in t. Thus $d(t) = pq-1$ holds for $t \in (0,\epsilon)$ for some $\epsilon > 0$. As $V_4^t \subset V_4^0$, the statement of Step 4 follows.

STEP 5. *Generation of A_1.* By Lemma 4.7, we have $\mathrm{Ad}(k_1^t)v_1^s = a_t v_1^s + b_t A_1 + c_t A_j$ with $j \neq 1$ and $b_t \neq 0$ for $t \in (0,\epsilon)$ with ϵ small enough. Consequently,

$$\mathrm{Ad}(k_1^t)\mathrm{span}(v_1^s) + V_4^t = \mathfrak{p}.$$

CONCLUSION. We have $\mathfrak{p} = \mathrm{Ad}(k_1^t)(\mathbb{R}v_1^s + \mathrm{span}(N'_X) + V_{X'}) + \mathrm{Ad}(k'_0)V_Y \subset \mathrm{Ad}(k_1^t)V_X + \mathrm{Ad}(k'_0)V_Y$, so $\mathrm{Ad}(k_1^t)V_X + \mathrm{Ad}(k'_0)V_Y = \mathfrak{p}$. It follows that

$$V_X + \mathrm{Ad}((k_1^t)^{-1}k'_0)V_Y = \mathfrak{p}.$$

(ii) *Proof in the case $X[p;0], Y[p;0]$.* This case must be treated separately because the predecessors X', Y' and consequently the sets N_X and N_Y are different from those in case **(i)**. The structure of the induction proof is identical as in **(i)**, with Steps 2 and 3 executed together.

We choose both predecessors $X'[p-1;0], Y'[p-1;0]$ and arrange X, X', Y, Y' in the same way we did in the first part of the proof with $Y[\mathbf{t};v]$ and $Y'[\mathbf{t}';v]$. In that case,

$$N_X = \{X_1, Z_{12}, \ldots, Z_{1p}\} = N_Y$$

and the space $\mathrm{Ad}(k'_0)(N_Y)$ is generated by

$$\left[\begin{array}{c|c} 0 & \beta_i^T \\ \hline -\alpha_i & 0 \end{array}\right]^s, \quad i = 1, \ldots, p-1, \quad \text{and} \quad \left[\begin{array}{c|c} 0 & \beta_p^T \\ \hline 0 & 0 \end{array}\right]^s. \tag{12}$$

Recall that

$$Z_{1j} = \left[\begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline -\mathbf{e}_{j-1} & 0 \end{array}\right]^s, \qquad j = 2, \ldots, p. \tag{13}$$

We want to show that the matrices in (12) together with those of (13) are linearly independent for a $k'_0 \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$ for which the equality (6) holds. Note that if

$$k'_0 = \begin{bmatrix} -I_{p-1} & 0 \\ 0 & I_p \end{bmatrix} \quad (p \text{ odd}) \quad \text{or} \quad k'_0 = \begin{bmatrix} I_{p-1} & 0 \\ 0 & -I_p \end{bmatrix} \quad (p \text{ even})$$

then the matrices (12) and (13) are linearly independent. Using once more the reasoning in Lemma 4.6 we find that the set of matrices k_0' for which this is true is open and dense in $\mathbf{SO}(p-1)\times\mathbf{SO}(p)$.

We conclude that if $N_X' = N_X \setminus \{X_1\}$ then $\operatorname{span}(N_X' + V_{X'}) + \operatorname{Ad}(k_0')V_Y$ has the form given in (11).

We reproduce the previous Step 4 and Step 5 using $v_1 = X_1^+$. The rest follows.

2. Proof that the case (p,q) implies the case $(p,q+1)$. We will show by induction that for any $q>p$, there exists a matrix $k\in K$ such that (5) holds. We know by the first part of the proof that this is true for $\mathbf{SO}_0(p,p+1)$.

Assume that X and Y are eligible in $\mathbf{SO}_0(p,q+1)$. Their configurations are eligible in $\mathbf{SO}_0(p,q)$. We write X', Y' when we work in $\mathbf{SO}_0(p,q)$.

We embed $K' = \mathbf{SO}(p)\times\mathbf{SO}(q)$ in $K=\mathbf{SO}(p)\times\mathbf{SO}(q+1)$ in the following way:

$$K' = \begin{bmatrix} \mathbf{SO}(p) & & \\ & \mathbf{SO}(q) & \\ & & 1 \end{bmatrix} \subset \begin{bmatrix} \mathbf{SO}(p) & \\ & \mathbf{SO}(q+1) \end{bmatrix}.$$

The space $\mathfrak{p}'$ is formed by the matrices

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where B are $p\times q$ matrices. We embed $\mathfrak{p}'$ in $\mathfrak{p}$ by adding a last column of zeros to B.

Step 1. We suppose that there exists a matrix $k_0\in K'$ such that

$$V_{X'} + \operatorname{Ad}(k_0)V_{Y'} = \mathfrak{p}'. \tag{14}$$

Then, by [5, Lemma 3.3], for any permutations s_1 and s_2 of the diagonal entries of $\mathcal{D}_{X'} = \mathcal{D}_X$ and $\mathcal{D}_{Y'} = \mathcal{D}_Y$, there exists $k_0\in K'$ such that

$$V_{s_1X'} + \operatorname{Ad}(k_0)V_{s_2Y'} = \mathfrak{p}'$$

so we can permute the elements of X' and Y' in a convenient way and still have the equality (14). We will arrange them in the following way (where the stars denote nonzero entries):

$$\mathcal{D}_{X'} = \operatorname{diag}[\overbrace{0,\dots,0}^{u},\star,\dots,\star],\quad \mathcal{D}_{Y'} = \operatorname{diag}[\star,\dots,\star,\overbrace{0,\dots,0}^{v}].$$

Let us denote by $k_{01}\in\mathbf{SO}(p)$ and $k_{02}\in\mathbf{SO}(q)$ the matrices composing k_0 corresponding in (14) to such X' and Y'. We can suppose that the matrix k_{01} is total.

By the eligibility of X and Y, $u+v\le p$, so no two zeros in $\mathcal{D}_{X'}$ and $\mathcal{D}_{Y'}$ are at the same position.

Let $N=\{X_{i,q+1}\}_{i=1}^p$. We set

$$N_X := V_X\cap N=\{X_{u+1,q+1},\dots,X_{p,q+1}\},$$
$$N_Y := V_Y\cap N=\{X_{1,q+1},\dots,X_{p-v,q+1}\}.$$

We have $p-v\ge u$.

STEP 2. Let

$$k_1=\begin{bmatrix} k_{01} & & \\ & k_{02} & \\ & & 1\end{bmatrix}$$

where $k_{01}\in\mathbf{SO}(p)$ and $k_{02}\in\mathbf{SO}(q)$ are the blocks composing k_0. We then have

$$V_{X'}+\mathrm{Ad}(k_1)V_{Y'}=[\mathfrak{p}'\quad \mathbf{0}]^s. \tag{15}$$

The space $V_X+\mathrm{Ad}(k_1)V_Y$ contains, in addition to the matrices in (15), the linear span of $N_X+\mathrm{Ad}(k_1)N_Y$.

Denote the columns of the matrix k_{01} by $\mathbf{c}_1,\dots,\mathbf{c}_p$. By block multiplication in $\mathbf{SO}_0(p,q+1)$, we obtain

$$\mathrm{Ad}(k_1)X_{j,q+1}=[0_{p\times q}\quad \mathbf{c}_j]^s.$$

This implies that the linear span of $N_X+\mathrm{Ad}(k_1)N_Y$ contains the following symmetric matrices:

$$[0_{p\times q}\quad \mathbf{c}_1]^s,\dots,[0_{p\times q}\quad \mathbf{c}_u]^s,[0_{p\times q}\quad \mathbf{e}_{u+1}]^s,\dots,[0_{p\times q}\quad \mathbf{e}_p]^s,$$

which are linearly independent by the totality of k_{01}. So $V_X+\mathrm{Ad}(k_1)V_Y=\mathfrak{p}$. ∎

We conclude this section with an example to illustrate our proof.

EXAMPLE 4.9. Consider $X=X[2;1]$, $Y=Y[1,1;1]$ in $\mathfrak{so}(3,4)$. We write X and Y in such a way that $\mathcal{D}_X=\mathrm{diag}[0,a,a]$ and $\mathcal{D}_Y=\mathrm{diag}[b,c,0]$. Their predecessors in $\mathfrak{so}(2,3)$ are X' and Y' such that $\mathcal{D}_{X'}=\mathrm{diag}[a,a]$ and $\mathcal{D}_{Y'}=\mathrm{diag}[c,0]$.

Note that X and Y form an eligible pair and so are $X'=X[2;0]$ and $Y'=Y'[1;1]$. In Step 1, we show that there exists a matrix

$$k_0=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & k_{0,1} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & k_{02}\end{bmatrix}$$

with $k_{01} \in \mathbf{SO}(2)$ and $k_{0,2} \in \mathbf{SO}(3)$ such that

$$V_{X'} + \mathrm{Ad}(k_0)V_{Y'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}^s$$

where $*$ designates an arbitrary element. We have

$$N_X = \{Z_{12}, Y_{12}, Z_{13}, Y_{13}\}, \qquad N_Y = \{X_1, Z_{12}, Y_{12}, Z_{13}, Y_{13}\}.$$

In Step 2, we observe that

$$\mathrm{Ad}(k_0)\mathrm{span}(N_Y) = \left\{ \left[\begin{array}{c|ccc} 0 & a_1 & a_2 & a_3 \\ \hline a_4 & & & \\ a_5 & & 0 & \end{array}\right]^s : \quad a_1, \dots a_6 \in \mathbb{R} \right\}$$

since the matrices

$$\left[\begin{array}{c|c} 0 & \beta_1^T \\ \hline -\alpha_1 & 0 \end{array}\right]^s, \quad \left[\begin{array}{c|c} 0 & \beta_1^T \\ \hline \alpha_1 & 0 \end{array}\right]^s, \quad \left[\begin{array}{c|c} 0 & \beta_2^T \\ \hline -\alpha_2 & 0 \end{array}\right]^s, \quad \left[\begin{array}{c|c} 0 & \beta_2^T \\ \hline \alpha_2 & 0 \end{array}\right]^s, \quad \left[\begin{array}{c|c} 0 & \beta_3^T \\ \hline 0 & 0 \end{array}\right]^s$$

are linearly independent. Note that in this case, there are no σ_i and no τ_i.

Now,

$$V_X = \mathrm{span}\overbrace{\{Z_{12}, Y_{12}, Z_{13}, Y_{13}\}}^{N_X} \cup V_{X'}$$

while

$$V_Y = \mathrm{span}\overbrace{\{X_1, Z_{12}, Z_{13}, Z_{14}, Y_{13}\}}^{N_Y} \cup V_{Y'}.$$

We can show that

$$\mathrm{Ad}(e^{t(Z_{1,2}^+ + \theta Z_{1,2}^+)})(V_{X'}) + \mathrm{Ad}(k_0)(\mathrm{span} N_Y \cup V_{Y'}) = \begin{bmatrix} 0 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}^s$$

for t small enough (with t small enough, the dimension will not decrease). Now,

$$\mathrm{Ad}(e^{t(Z_{1,2}^+ + \theta Z_{1,2}^+)})\overbrace{(\mathrm{span}\{Z_{12}\} \cup V_{X'})}^{\subset V_X} + \mathrm{Ad}(k_0)\overbrace{(\mathrm{span} N_Y \cup V_{Y'})}^{V_Y}$$

$$= \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}^s = \mathfrak{p}$$

for t close to 0 since

$$\mathrm{Ad}(e^{t(Z_{12}^+ + \theta Z_{12}^+)})(Z_{12}) = \cos(4t)Z_{12} + 2\sin(4t)(A_1 + A_2).$$

Therefore,

$$V_X + \mathrm{Ad}(\overbrace{e^{-t(Z_{1,2}^+ + \theta Z_{1,2}^+)} k_0}^{k}) V_Y = \mathrm{Ad}(e^{-t(Z_{1,2}^+ + \theta Z_{1,2}^+)})\mathfrak{p} = \mathfrak{p},$$

which means that the density exists.

5. Applications. We now extend our results to the symmetric spaces of type BC_p, i.e. to the complex and quaternion cases.

Recall that $\mathbf{SU}(p,q)$ is the subgroup of elements $g \in \mathbf{SL}(p+q,\mathbf{C})$ such that $g^* I_{p,q} g = I_{p,q}$ while $\mathbf{Sp}(p,q)$ is the subgroup of elements $g \in \mathbf{SL}(p+q,\mathbf{H})$ such that $g^* I_{p,q} g = I_{p,q}$. Their respective maximal compact subgroups are $\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ and $\mathbf{Sp}(p) \times \mathbf{Sp}(q) \equiv \mathbf{SU}(p,\mathbf{H}) \times \mathbf{SU}(q,\mathbf{H})$.

Their subspaces $\mathfrak{p}$ can be described as

$$\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

where B is an arbitrary complex (respectively quaternionic) matrix of size $p \times q$. The Cartan subalgebra $\mathfrak{a}$ is chosen in the same way as for $\mathfrak{so}(p,q)$.

COROLLARY 5.1. *Consider the symmetric spaces* $\mathbf{SO}_0(p,q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$, $\mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ *and* $\mathbf{Sp}(p,q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$, $q > p$. *Let* $X, Y \in \mathfrak{a}$. *Then the measure* $\delta_{e^X}^{\natural} \star \delta_{e^Y}^{\natural}$ *is absolutely continuous if and only if* X *and* Y *are eligible, as defined in Definition* 2.3.

Proof. Let $X, Y \in \mathfrak{a}$. If they are eligible then since

$$\begin{aligned} a(e^X(\mathbf{SO}(p) \times \mathbf{SO}(q))e^Y) &\subset a(e^X \mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))e^Y) \\ &\subset a(e^X(\mathbf{Sp}(p) \times \mathbf{Sp}(q))e^Y), \end{aligned}$$

it follows from Theorem 4.8 that these sets have nonempty interior. Hence the density exists in all three cases.

On the other hand, given Lemma 3.1, one can reproduce Proposition 3.2 using $\mathbf{F} = \mathbf{C}$ and $\mathbf{F} = \mathbf{H}$ to show that the eligibility condition is necessary in the complex and quaternionic cases. ∎

We will conclude this paper with two further applications.

PROPOSITION 5.2. *Let* $X, Y \in \mathfrak{a}$ *be such that* $(\delta_{e^X}^{\natural})^{*2}$ *and* $(\delta_{e^Y}^{\natural})^{*2}$ *are absolutely continuous. Then* $\delta_{e^X}^{\natural} * \delta_{e^Y}^{\natural}$ *is absolutely continuous.*

Proof. Let $X = X[\mathbf{s}; u]$ and $Y = Y[\mathbf{t}; v]$. We know that the couple (X, X) is eligible; therefore

$$2 \max\{\mathbf{s}, 2u\} \le 2p.$$

In the same manner, $\max\{\mathbf{t}, 2v\} \le p$. Hence,

$$\max\{\mathbf{s}, 2u\} + \max\{\mathbf{t}, 2v\} \le p + p = 2p$$

which means that X and Y are eligible. Consequently, $\delta^\natural_{e^X} * \delta^\natural_{e^Y}$ is absolutely continuous. ∎

If $X \in \mathfrak{a}$ and $X \neq 0$, it is important to know for which convolution powers l the measure $(\delta^\natural_{e^X})^l$ is absolutely continuous. This problem is equivalent to the study of the absolute continuity of convolution powers of uniform orbital measures $\delta^\natural_g = m_K * \delta_g * m_K$ for $g \notin K$.

It was proved in [6, Corollary 7] that it is always the case for $l \geq r+1$, where r is the rank of the symmetric space G/K. It was also conjectured ([6, Conjecture 10]) that $r+1$ is optimal for this property, which was effectively proved for symmetric spaces of type A_n([6, Corollary 18]). In the following theorem, the conjecture is shown not to hold on symmetric spaces of type B_p, where $r = p$. Thanks to the rich structure of the root system B_p, already all pth powers of orbital measures are absolutely continuous and p is optimal for this property.

THEOREM 5.3. *Consider the symmetric spaces* $\mathbf{SO}_0(p,q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$, $\mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ *and* $\mathbf{Sp}(p,q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$, $q > p$.

Then, for every nonzero $X \in \mathfrak{a}$, *the measure* $(\delta^\natural_{e^X})^p$ *is absolutely continuous. Moreover,* p *is the smallest value for which this is true: if* X *has the configuration* $[1; p-1]$ *then the measure* $(\delta^\natural_{e^X})^{p-1}$ *is singular.*

Proof. We will write S^l_X for the set $a(e^X K e^X \ldots K e^X)$ where the factor e^X appears l times. Note that $(\delta^\natural_{e^X})^l$ is absolutely continuous if and only if S^l_X has nonempty interior.

We prove first that for $l < p$, the measure $(\delta^\natural_{e^X})^l$ may not be absolutely continuous. Let $X = X[1; p-1]$. Using Lemma 3.1 repeatedly, as in the proof of Proposition 3.2, we show that for $l < p$, there are at least $p-l$ diagonal entries of $\mathcal{D}_H$ which are equal to 0 for every $H \in S^l_X$. Consequently, S^l_X has empty interior and $(\delta^\natural_{e^X})^l$ is not absolutely continuous when $l \leq p-1$.

We will now show that $(\delta^\natural_{e^X})^p$ has a density for every $X \neq 0$.

If $X = X[\mathbf{s}; 0]$ then the measure $(\delta^\natural_{e^X})^2$ is already absolutely continuous (the couple (X, X) is eligible). Suppose then that $X = X[\mathbf{s}; u] \in \overline{\mathfrak{a}^+}$, $u > 0$.

We remark that if $H \in S^l_X$ then $a(e^X K e^H) \subset S^{l+1}_X$. Indeed, we have $e^X k_1 e^X \ldots k_{l-1} e^X = k_a e^H k_b$ and therefore $a(e^X K e^H) = a(e^X K k_a e^H k_b)$ $= a(e^X K e^X k_1 \ldots k_{l-1} e^X) \subset S^{l+1}_X$.

We claim that there exists $H \in S^{p-1}_X$ such that $H = H[1^{p-1}; 1]$ or $H \in \mathfrak{a}^+$.

We prove the claim using induction on p. If $p = 2$ then $S^{p-1}_X = \{X\}$ and the result follows (in that case, u cannot be higher than 1 for $X \neq 0$).

Suppose that the claim is true for $p-1 \geq 2$. Let

$$K_0 = \begin{bmatrix} \mathbf{SO}(p-1) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{SO}(q-1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the set $B = a(e^X K_0 e^X \dots e^X)$ with $p-1$ factors e^X. By the induction hypothesis, there exists $H_0 \in B$ with $H_0 = H_0[1^{p-2}; 2]$ or $H_0 = H_0[1^{p-1}; 1]$. In the second case, we are done.

If $H_0 = H_0[1^{p-2}; 2] \in B$, we can assume that the diagonal entries which are 0 in $\mathcal{D}_X$ and in H_0 are at the end. We note that X and H_0 considered without their last entries are eligible in $\mathbf{SO}_0(p-1, q-1)$, their configurations being $[\mathbf{s}; u-1]$ and $[1^{p-2}; \underline{1}]$ respectively. Hence $a(e^X K_0 e^{H_0})$ has nonempty interior in the subspace $\overline{\mathfrak{a}^+} \cap \{H_p = 0\}$. Therefore, there exists $H \in a(e^X K_0 e^{H_0}) \subset S_X^{p-1}$ with $H = H[1^{p-1}; 1]$, which proves the claim.

To conclude, we take $H \in S_X^{p-1}$ with $H \in \mathfrak{a}^+$ or $H = H[1^{p-1}; 1]$. In both cases, X and H are eligible, so by Corollary 5.1 the set $a(e^X K e^H)$ has nonempty interior. As $a(e^X K e^H) \subset S_X^p$, this ends the proof. ∎

6. Conclusion. With this paper and with [5], we have now obtained sharp criteria on singular X and Y for the existence of the density of $\delta^\natural_{e^X} \star \delta^\natural_{e^Y}$ for the root systems of types A_n, B_p and BC_p. Thanks to [6] and Theorem 5.3 of the present paper, sharp criteria are now given for the lth convolution powers $(\delta^\natural_{e^X})^l$ to be absolutely continuous for any $X \neq 0$, $X \in \mathfrak{a}$.

Although there is considerable similarity between the criteria for both types of spaces, a characterization of eligibility that would be applicable for all Riemannian symmetric spaces of noncompact type has yet to emerge. The solution of the second problem in Theorem 5.3 seems to indicate that the answer may depend on the type of the symmetric space.

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