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ON THE COLLATZ CONJECTURE

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Abstract. We propose two conjectures which imply the Collatz conjecture. We give a numerical evidence for the second conjecture.

1. Introduction

1. For an odd integer n let

$$F(n) := \frac{3n+1}{2^{v_2(3n+1)}}$$

be the speeded-up Collatz function. Obviously, $F(n)$ is also an odd integer, and $F(n)$ is positive iff n is positive.

For a positive odd integer n we define inductively the *Collatz sequence* $(a_0, a_1, \dots)$ as follows:

$$a_0 = n, \quad a_{k+1} = F(a_k) \quad \text{for } k \ge 0,$$

and the corresponding sequence of exponents $(r_0, r_1, \dots)$ by

$$r_k = v_2(3a_k + 1) \quad \text{for } k \ge 0.$$

Since a_k is odd, r_k is a positive integer.

2. We have $F(1) = 1$. The Collatz conjecture says (see e.g. Lagarias [1]) that in every Collatz sequence $(a_0, a_1, \dots)$ we have $a_k = 1$ for every sufficiently large k, i.e. $F^m(a_0) = 1$ for some m.

We say that a positive odd integer n (or the Collatz sequence $(a_0, a_1, \dots)$ with $a_0 = n$) *has a finite stopping time* if $a_k < a_0$ for some $k \ge 1$ (see [2]).

Let U be the set of all positive odd integers with finite stopping times. The Collatz conjecture is equivalent to the statement that every odd integer $n > 1$ belongs to U.

3. In the present paper we change the order. First we fix an arbitrary sequence $(r_0, r_1, \dots)$ of positive integers, and then ask whether it corresponds to some Collatz sequence $(a_0, a_1, \dots)$. Next we investigate under which conditions on $(r_0, r_1, \dots)$ the sequence $(a_0, a_1, \dots)$ satisfies the Collatz conjecture, or it has a finite stopping time.

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The set of all sequences of positive integers is not countable, and the set of all Collatz sequences is countable, since every such sequence is determined by its first element. Therefore there are uncountably many sequences of positive integers which do not correspond to any Collatz sequence.

To overcome this difficulty we consider real Collatz sequences.

4. Real Collatz sequences. For a sequence $(r_0, r_1, \dots)$ of positive integers and a positive real number x we define inductively the sequence $(a_0, a_1, \dots)$ of real numbers by

$$a_0 = x, \qquad a_{k+1} = \frac{3a_k + 1}{2^{r_k}} \quad \text{for } k \geq 0. \tag{1.1}$$

We ask under which conditions on $(r_0, r_1, \dots)$ and on x the sequence $(a_0, a_1, \dots)$ has a finite stopping time.

First we give explicit formulas for elements of the sequence $(a_0, a_1, \dots)$ defined by (1.1).

2. Main results

LEMMA 2.1. *From* (1.1) *it follows that*

$$a_k = \frac{3^k x + \sum_{i=1}^{k} 3^{k-i} 2^{\sum_{j=0}^{i-2} r_j}}{2^{\sum_{j=0}^{k-1} r_j}} \quad \textit{for } k \geq 0. \tag{2.1}$$

Conversely, the numbers defined by (2.1) *satisfy* (1.1).

Proof. The standard induction on k proves (2.1). To prove the second statement it is sufficient to substitute (2.1) into (1.1). ∎

The formula (2.1) can be rewritten in the form

$$a_k = A_k x + B_k, \tag{2.2}$$

where

$$A_k = \frac{3^k}{2^{\sum_{j=0}^{k-1} r_j}}, \qquad B_k = \frac{\sum_{i=1}^{k} 3^{k-i} 2^{\sum_{j=0}^{i-2} r_j}}{2^{\sum_{j=0}^{k-1} r_j}},$$

are positive rational numbers independent of x with a power of 2 in denominators.

LEMMA 2.2.

(i) *If $A_k < 1$ for some k, i.e. if*

$$k \log_2 3 < \sum_{j=0}^{k-1} r_j \quad \textit{for some } k, \tag{2.3}$$

then the sequence $(a_0, a_1, \dots)$ defined by (1.1) *has a finite stopping time for every sufficiently large x; namely, $a_k < a_0$ for $x > B_k/(1 - A_k)$.*

(ii) *If* $A_k > 1$ *for every* k, *then the sequence* $(a_0, a_1, \ldots)$ *defined by* (1.1) *does not have a finite stopping time.*

Proof. (i) If $A_k < 1$, then, by (2.2), we have

$$a_k = A_k x + B_k < x = a_0 \quad \text{for } x > \frac{B_k}{1 - A_k},$$

since $B_k > 0$ and $0 < A_k < 1$.

(ii) If $A_k > 1$ for every k, then

$$a_k = A_k x + B_k > x = a_0 \quad \text{for every } k,$$

so the sequence $(a_0, a_1, \ldots)$ does not have a finite stopping time. ∎

From Lemma 2.1 it follows that the exponents $(r_0, r_1, \ldots)$ of a Collatz sequence $(a_0, a_1, \ldots)$ cannot be very small. More precisely:

PROPOSITION 2.3. *Let* $(a_0, a_1, \ldots)$ *be the Collatz sequence with the sequence of exponents* $(r_0, r_1, \ldots)$. *Then* $r_j > 1$ *for infinitely many* $j \geq 0$.

Proof. Assume that $r_j = 1$ for every $j \geq j_0$ for some $j_0 \geq 0$. Omitting the first j_0 terms of the Collatz sequence we may assume that $r_j = 1$ for every $j \geq 0$. Then (2.1) takes the form

$$a_k = \frac{3^k x + (3^k - 2^k)}{2^k} = \frac{3^k(x+1)}{2^k} - 1,$$

where $x = a_0$ is a positive integer. It follows that a_k is not an integer for k satisfying $2^k > x + 1$. We get a contradiction, since all terms of the Collatz sequence are integers. ∎

In view of Lemma 2.2 it is natural to assume that (2.3) holds. This is stated in the following

CONJECTURE 2.4. *For every positive odd integer* n *there exists* $k \in \mathbb{N}$ *such that*

$$\lceil k \log_2 3 \rceil \leq \sum_{j=0}^{k-1} r_j, \tag{2.4}$$

where $(r_0, r_1, \ldots)$ *is the sequence of exponents of the Collatz sequence* $(a_0, a_1, \ldots)$, *with* $a_0 = n$.

REMARK. Conjecture 2.4 follows from the Collatz conjecture. Namely, by the Collatz conjecture we have $r_j = 2$ for every sufficiently large j, and $\log_2 3 < 2$. This implies (2.4) for large k.

Our second conjecture concerns solutions of linear Diophantine equations, but it gives some information on the stopping time of the Collatz sequence.

Conjecture 2.5. *Assume that a sequence* $(r'_0, r'_1, \dots, r'_{k-1})$ *of positive integers, where* $k \ge 1$, *satisfies*

$$\sum_{j=0}^{l-1} r'_j < \lceil l \log_2 3 \rceil \quad \text{for every } l,\ 1 \le l \le k-1, \tag{2.5}$$

$$\sum_{j=0}^{k-1} r'_j = \lceil k \log_2 3 \rceil. \tag{2.6}$$

Then the minimal solution (x'_0, y'_0) *of the equation*

$$2^{\sum_{j=0}^{k-1} r'_j} \cdot y' - 3^k x' = \sum_{i=1}^{k} 3^{k-i} 2^{\sum_{j=0}^{i-2} r'_j} \tag{2.7}$$

in positive integers satisfies $y'_0 < x'_0$ *or* $x'_0 = 1$.

Theorem 2.6. *Conjectures* 2.4 *and* 2.5 *imply the Collatz conjecture. More precisely, the stopping time of the Collatz sequence* $(a_0, a_1, \dots)$ *with* $a_0 > 1$ *is at the term* a_k *(i.e.* $a_k < a_0$*) where* k *is the minimal positive integer satisfying* (2.4).

Proof. Let n be a positive odd integer, and let $(a_0, a_1, \dots)$ be the Collatz sequence with $a_0 = n$ and the sequence of exponents $(r_0, r_1, \dots)$. Take the minimal positive integer k satisfying the condition (2.4) in Conjecture 2.4, and put

$$r'_j := \begin{cases} r_j & \text{for } j = 0, 1, \dots, k-2, \\ \lceil k \log_2 3 \rceil - \sum_{j=0}^{k-2} r_j & \text{for } j = k-1. \end{cases}$$

Then (2.5) holds, by the minimality of k, and (2.6) holds by the above definition of r'_{k-1}.

Let us observe that, by (2.5),

$$\lceil k \log_2 3 \rceil - \sum_{j=0}^{k-2} r'_j > \lceil k \log_2 3 \rceil - \lceil (k-1) \log_2 3 \rceil \ge 1,$$

since $\log_2 3 > 1$. Thus r'_{k-1} defined above is a positive integer, as it should be. So the assumptions of Conjecture 2.5 are satisfied.

Moreover, by (2.4),

$$r_{k-1} \ge \lceil k \log_2 3 \rceil - \sum_{j=0}^{k-2} r_j = r'_{k-1}.$$

From Lemma 2.1 it follows that $(x, y) = (a_0, a_k)$ satisfies the Diophantine equation

$$2^{\sum_{j=0}^{k-1} r_j} \cdot y - 3^k\, x = \sum_{i=1}^{k} 3^{k-i} 2^{\sum_{j=0}^{i-2} r_j},$$

which can be rewritten in the form

$$2^{\sum_{j=0}^{k-1} r'_j} \cdot 2^{r_{k-1}-r'_{k-1}} \cdot y - 3^k\, x = \sum_{i=1}^{k} 3^{k-i} 2^{\sum_{j=0}^{i-2} r_j}.$$

In other words,

$$(x', y') = (x, 2^{r_{k-1}-r'_{k-1}} y) = (a_0, 2^{r_{k-1}-r'_{k-1}} a_k) \tag{2.8}$$

satisfies (2.7).

All solutions of (2.7) in positive integers are given by

$$(x'_t, y'_t) = \big(x'_0 + t \cdot 2^{\sum_{j=0}^{k-1} r'_j}, y'_0 + 3^k t\big), \qquad \text{where } t \ge 0,\ t \in \mathbb{Z}.$$

From (2.6) it follows that $3^k < 2^{\sum_{j=0}^{k-1} r'_j}$. By Conjecture (2.5), we have $y'_0 < x'_0$ or $x'_0 = 1$. Therefore in the first case

$$y'_t = y'_0 + 3^k t < x'_0 + 2^{\sum_{j=0}^{k-1} r'_j} t = x'_t \tag{2.9}$$

for every $t \ge 0$. In particular, (x', y') defined in (2.8) satisfies (2.7), therefore $(x', y') = (x'_t, y'_t)$ for some $t \ge 0$. Hence

$$y'_t = y' = 2^{r_{k-1}-r'_{k-1}} y = 2^{r_{k-1}-r'_{k-1}} a_k,$$

by (2.9), is less than $x'_t = x' = x = a_0$. Then from $r_{k-1} \ge r'_{k-1}$ we conclude that $a_k < a_0$.

There remains the case $x'_0 = 1$ in Conjecture 2.5. Then, by (2.8), $a_0 = x' = x'_0 = 1$, and the Collatz sequence with $a_0 = 1$ has all its terms equal 1, so it does not have a finite stopping time! Fortunately, the case $a_0 = 1$ has been excluded in the statement of the theorem. ■

Remark. For a fixed positive integer k there are only a finite number of sequences $(r'_0, r'_1, \dots, r'_{k-1})$ satisfying the assumptions of Conjecture 2.5, i.e. (2.5) and (2.6). Thus to verify Conjecture 2.5 for a given k it is sufficient to determine minimal solutions of a finite number of linear Diophantine equations (2.7). We have done this for all $k \le 6$: see Table 1.

In the table, we give the value of k, the sequence of positive integers $(r'_0, r'_1, \dots, r'_{k-1})$ satisfying (2.5) and (2.6), the coefficients

$$A := 2^{\sum_{j=0}^{k-1} r'_j}, \quad B := 3^k, \quad C := \sum_{i=1}^{k} 3^{k-i} 2^{\sum_{j=0}^{i-2} r'_j}$$

of the linear equation (2.7), and the minimal solution (x'_0, y'_0) of (2.7).

Table 1

k	$(r'_0, r'_1, \dots, r'_{k-1})$	(A, B, C)	(x'_0, y'_0)
1	(2)	$(4, 3, 1)$	$(1, 1)$
2	$(1, 3)$	$(16, 9, 5)$	$(3, 2)$
3	$(1, 1, 3)$	$(32, 27, 19)$	$(23, 20)$
3	$(1, 2, 2)$	$(32, 27, 23)$	$(11, 10)$
4	$(1, 1, 1, 4)$	$(128, 81, 65)$	$(15, 10)$
4	$(1, 1, 2, 3)$	$(128, 81, 73)$	$(7, 5)$
4	$(1, 2, 1, 3)$	$(128, 81, 85)$	$(59, 38)$
5	$(1, 1, 1, 1, 4)$	$(256, 243, 211)$	$(95, 91)$
5	$(1, 1, 1, 2, 3)$	$(256, 243, 227)$	$(175, 167)$
5	$(1, 1, 1, 3, 2)$	$(256, 243, 259)$	$(79, 76)$
5	$(1, 1, 2, 1, 3)$	$(256, 243, 251)$	$(39, 38)$
5	$(1, 1, 2, 2, 2)$	$(256, 243, 283)$	$(199, 190)$
5	$(1, 2, 1, 1, 3)$	$(256, 243, 287)$	$(219, 209)$
5	$(1, 2, 1, 2, 2)$	$(256, 243, 319)$	$(123, 118)$
6	$(1, 1, 1, 1, 1, 5)$	$(1024, 729, 665)$	$(575, 410)$
6	$(1, 1, 1, 1, 2, 4)$	$(1024, 729, 697)$	$(287, 205)$
6	$(1, 1, 1, 1, 3, 3)$	$(1024, 729, 761)$	$(735, 524)$
6	$(1, 1, 1, 2, 1, 4)$	$(1024, 729, 745)$	$(367, 262)$
6	$(1, 1, 1, 2, 2, 3)$	$(1024, 729, 809)$	$(815, 581)$
6	$(1, 1, 1, 3, 1, 3)$	$(1024, 729, 905)$	$(975, 695)$
6	$(1, 1, 2, 1, 1, 4)$	$(1024, 729, 817)$	$(999, 712)$
6	$(1, 1, 2, 1, 2, 3)$	$(1024, 729, 881)$	$(423, 302)$
6	$(1, 1, 2, 2, 1, 3)$	$(1024, 729, 977)$	$(583, 416)$
6	$(1, 2, 1, 1, 1, 4)$	$(1024, 729, 925)$	$(923, 658)$
6	$(1, 2, 1, 1, 2, 3)$	$(1024, 729, 989)$	$(347, 248)$
6	$(1, 2, 1, 2, 1, 3)$	$(1024, 729, 1085)$	$(507, 362)$

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