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ON NEAR-PERFECT AND DEFICIENT-PERFECT NUMBERS

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Abstract. For a positive integer n, let $\sigma(n)$ denote the sum of the positive divisors of n. Let d be a proper divisor of n. We call n a near-perfect number if $\sigma(n) = 2n + d$, and a deficient-perfect number if $\sigma(n) = 2n - d$. We show that there is no odd near-perfect number with three distinct prime divisors and determine all deficient-perfect numbers with at most two distinct prime factors.

1. Introduction. For a positive integer n, let $\omega(n)$ and $\sigma(n)$ denote the number of distinct prime factors of n and the sum of the positive divisors of n, respectively. A positive integer n is called *abundant* if $\sigma(n) > 2n$ and *deficient* if $\sigma(n) < 2n$. In 2012, Pollack and Shevelev [Po] introduced the concept of a near-perfect number. A positive number n is called *near-perfect* if it is the sum of all of its proper divisors except one of them. Pollack and Shevelev presented an upper bound on the count of near-perfect numbers and constructed three types of near-perfect numbers. Recently, Ren and Chen [Re] determined all near-perfect numbers with two distinct prime factors, and one sees from this classification that all such numbers are even. On the other hand, D. Johnson found an explicit example of an odd near-perfect number with four distinct prime factors (see [Sl, A181595]). It is natural to consider whether or not there is an odd near-perfect number with three distinct prime divisors.

Motivated by the concept of a near-perfect number, we also study deficient-perfect numbers, a very special kind of deficient numbers. We call n a *deficient-perfect* number with *deficiency divisor* d if $\sigma(n) = 2n - d$, where d is a proper divisor of n. For related problems, see [Pom], [C80], [Co], [Ha], [T].

In this paper, we obtain the following results:

THEOREM 1.1. *There is no odd near-perfect number with three distinct prime divisors.*

THEOREM 1.2. *If n is deficient-perfect and $\omega(n) \le 2$, then*

(i) $n = 2^\alpha$ *with deficiency divisor* $d = 1$*;*

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(ii) $n = 2^\alpha(2^{\alpha+1}+2^s-1)$ *with deficiency divisor* $d = 2^s$, *where* $1 \le s \le \alpha$ *and* $2^{\alpha+1}+2^s-1$ *is an odd prime.*

2. Lemmas

LEMMA 2.1. *If* $n = p^\alpha$ *is a deficient-perfect number with deficiency divisor* $d = p^\beta$, *then* $n = 2^\alpha$ *and* $d = 1$.

Proof. Assume that $\sigma(p^\alpha) = 2p^\alpha - p^\beta$, where $0 \le \beta < \alpha$. Then $p^{\alpha+1} - 2p^\alpha - p^{\beta+1} + p^\beta = -1$, which implies that $\beta = 0$. Thus

$$p^{\alpha+1} - 2p^\alpha - p = p(p^\alpha - 2p^{\alpha-1} - 1) = -2,$$

and we have $p = 2$, $n = 2^\alpha$ and $d = 1$. ∎

LEMMA 2.2. *If* $n = 2^\alpha q^\beta$ *is a deficient number with deficiency divisor* d, *then* $n = 2^\alpha(2^{\alpha+1}+2^s-1)$ *and* $d = 2^s$, *where* $1 \le s \le \alpha$ *and* $2^{\alpha+1}+2^s-1$ *is an odd prime.*

Proof. Assume that

$$\sigma(2^\alpha q^\beta) = (2^{\alpha+1}-1)(1+q+\cdots+q^\beta) = 2^{\alpha+1}q^\beta - 2^s q^t, \tag{2.1}$$

where $s + t < \alpha + \beta$. Then

$$(2^{\alpha+1}-q)(1+q+\cdots+q^{\beta-1}) = 1 - 2^s q^t, \tag{2.2}$$

thus $t = 0$. In fact, if $t \ge 1$, then by (2.1) we have $2^{\alpha+1} - 1 \equiv 0 \pmod q$, $q < 2^{\alpha+1}$. Hence, the left side of (2.2) is positive and the right side of (2.2) is negative, a contradiction.

Now we consider the following two cases.

CASE 1: $\beta = 1$. Then $\sigma(n) = (2^{\alpha+1}-1)(1+q) = 2^{\alpha+1}q - 2^s$. Thus $s \ge 1$ and $q = 2^{\alpha+1} + 2^s - 1$.

CASE 2: $\beta \ge 2$. If $\beta \equiv 0 \pmod 2$, then $\sigma(n) \equiv 1 \pmod 2$, thus by (2.1) we have $s = 0$. By (2.2) we have $q = 2^{\alpha+1}$, which is impossible.

If $\beta \equiv 1 \pmod 4$, then

$$1+q+\cdots+q^\beta = (1+q)(1+q^2+q^4+\cdots+q^{\beta-1})$$

and

$$1+q^2+q^4+\cdots+q^{\beta-1} \equiv \frac{\beta+1}{2} \not\equiv 0 \pmod 2.$$

Noting that $t = 0$, by (2.1) we have

$$(2^{\alpha+1}-1)(1+q)(1+q^2+q^4+\cdots+q^{\beta-1}) = 2^s(2^{\alpha+1-s}q^\beta - 1).$$

Thus $2^s \mid 1+q$, $2^s - 1 \le q$. Since $\beta \ge 5$, we have

$$|(2^{\alpha+1}-q)(1+q+\cdots+q^{\beta-1})| > 1+q,$$

but $|1-2^s| \le q$, which contradicts (2.2).

If $\beta \equiv 3 \pmod 4$, then

$$1+q+\cdots+q^\beta = (1+q^2)(1+q+q^4+q^5+\cdots+q^{\beta-3}+q^{\beta-2})$$

and $4 \nmid 1+q^2$. It follows from (2.1) that $2^{s-1} \mid 1+q+q^4+q^5+\cdots+q^{\beta-3}+q^{\beta-2}$. Since $\beta \geq 3$, we have

$$|(2^{\alpha+1}-q)(1+q+\cdots+q^{\beta-1})| > 2(1+q+q^4+q^5+\cdots+q^{\beta-3}+q^{\beta-2}) \geq 2^s,$$

but $|1-2^s| < 2^s$, which contradicts (2.2). ∎

3. Proof of Theorem 1.1. Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ is an odd near-perfect number, then

$$\sigma(n) = 2n + d,$$

where $d \mid n$ and $d < n$. Since $\sigma(n) \equiv 1 \pmod 2$, we have $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2, 3$.

If $p_1 \geq 5$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} < 2,$$

which is impossible. Thus $p_1 = 3$. If $p_2 \geq 7$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10} < 2,$$

which is also impossible. Thus $p_2 = 5$. If $p_3 \geq 17$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{17}{16} < 2,$$

a contradiction. Thus $p_3 \leq 13$. Hence, if n is an odd near-perfect number with three distinct prime divisors, then

$$\sigma(n) = \frac{3^{\alpha_1+1}-1}{2} \cdot \frac{5^{\alpha_2+1}-1}{4} \cdot \frac{p_3^{\alpha_3+1}-1}{p_3-1} = 2 \cdot 3^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3} + 3^{\beta_1} 5^{\beta_2} p_3^{\beta_3}, \tag{3.1}$$

where $\beta_1 + \beta_2 + \beta_3 < \alpha_1 + \alpha_2 + \alpha_3$.

Since α_i's are even, we have

$$3^{\alpha_1+1}-1,\ 7^{\alpha_3+1}-1,\ 13^{\alpha_3+1}-1 \equiv 1, 2 \pmod 5, \tag{3.2}$$

$$3^{\alpha_1+1}-1,\ 5^{\alpha_2+1}-1 \equiv 2, 4, 5 \pmod 7, \tag{3.3}$$

$$5^{\alpha_2+1}-1,\ 11^{\alpha_3+1}-1 \equiv 1 \pmod 3. \tag{3.4}$$

Let

$$f(\alpha_1, \alpha_2, \alpha_3) = \left(1 - \frac{1}{3^{\alpha_1+1}}\right)\left(1 - \frac{1}{5^{\alpha_2+1}}\right)\left(1 - \frac{1}{p_3^{\alpha_3+1}}\right).$$

Now we consider the following three cases.

Case 1: $p_3 = 7$. Then by (3.1)–(3.3), we have $\beta_2 = \beta_3 = 0$ and

$$f(\alpha_1, \alpha_2, \alpha_3) = \frac{32}{35} + \frac{2^4 3^{\beta_1+1}}{3^{\alpha_1+1} 5^{\alpha_2+1} 7^{\alpha_3+1}}. \tag{3.5}$$

We have

$$f(\alpha_1,\alpha_2,\alpha_3) \ge \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{7^3}\right) = 0.952\ldots$$

and

$$\frac{32}{35}+\frac{2^4 3^{\beta_1+1}}{3^{\alpha_1+1}5^{\alpha_2+1}7^{\alpha_3+1}} \le \frac{32}{35}+\frac{2^4}{5^3 7^3} = 0.914\ldots,$$

a contradiction.

CASE 2: $p_3 = 11$. Then by (3.1) and (3.4), we have $\beta_1 = 0$ and

$$f(\alpha_1,\alpha_2,\alpha_3) = \frac{32}{33}+\frac{2^4 5^{\beta_2+1}11^{\beta_3}}{3^{\alpha_1+1}5^{\alpha_2+1}11^{\alpha_3+1}}. \tag{3.6}$$

If $\alpha_1 = 2$, then $f(\alpha_1,\alpha_2,\alpha_3) < 1-1/3^3 < 32/33$, thus (3.6) cannot hold. Hence $\alpha_1 \ge 4$ and

$$f(\alpha_1,\alpha_2,\alpha_3) \ge \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{11^3}\right) = 0.987\ldots$$

and

$$\frac{32}{33}+\frac{2^4 5^{\beta_2+1}11^{\beta_3}}{3^{\alpha_1+1}5^{\alpha_2+1}11^{\alpha_3+1}} \le \frac{32}{33}+\frac{2^4}{3^5 11} = 0.975\ldots,$$

a contradiction.

CASE 3: $p_3 = 13$. Then by (3.1) and (3.2), we have $\beta_2 = 0$ and

$$f(\alpha_1,\alpha_2,\alpha_3) = \frac{64}{65}+\frac{2^5 3^{\beta_1+1}13^{\beta_3}}{3^{\alpha_1+1}5^{\alpha_2+1}13^{\alpha_3+1}}. \tag{3.7}$$

If $\alpha_1 = 2$, then $f(\alpha_1,\alpha_2,\alpha_3) < 1-1/3^3 < 64/65$, thus (3.7) cannot hold. Hence $\alpha_1 \ge 4$.

If $\beta_3 = 0$, then

$$f(\alpha_1,\alpha_2,\alpha_3) \ge \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{13^3}\right) = 0.987\ldots$$

and

$$\frac{64}{65}+\frac{2^5 3^{\beta_1+1}13^{\beta_3}}{3^{\alpha_1+1}5^{\alpha_2+1}13^{\alpha_3+1}} \le \frac{64}{65}+\frac{2^5}{5^3 13^3} = 0.984\ldots,$$

a contradiction.

If $\beta_3 \ne 0$, then noting that $5^{\alpha_2+1}-1, 13^{\alpha_3+1}-1 \not\equiv 0 \pmod{13}$, by (3.1) we have $3^{\alpha_1+1}-1 \equiv 0 \pmod{13}$, thus $\alpha_1 \equiv 2 \pmod 6$. Since $\alpha_1 \ge 4$, we have $\alpha_1 \ge 8$. Hence

$$f(\alpha_1,\alpha_2,\alpha_3) \ge \left(1-\frac{1}{3^9}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{13^3}\right) = 0.9914\ldots.$$

Now we consider the following three subcases.

SUBCASE 3.1: $0 \le \beta_1 \le \alpha_1 - 1$. We have

$$\frac{64}{65} + \frac{2^5 3^{\beta_1+1} 13^{\beta_3}}{3^{\alpha_1+1} 5^{\alpha_2+1} 13^{\alpha_3+1}} \le \frac{64}{65} + \frac{2^5}{3 \cdot 5^3 13} = 0.9911\ldots < f(\alpha_1, \alpha_2, \alpha_3),$$

a contradiction.

SUBCASE 3.2: $\beta_1 = \alpha_1$, $1 \le \beta_3 \le \alpha_3 - 1$. We have

$$\frac{64}{65} + \frac{2^5 3^{\beta_1+1} 13^{\beta_3}}{3^{\alpha_1+1} 5^{\alpha_2+1} 13^{\alpha_3+1}} \le \frac{64}{65} + \frac{2^5}{5^3 13^2} = 0.986\ldots < f(\alpha_1, \alpha_2, \alpha_3),$$

a contradiction.

SUBCASE 3.3: $\beta_1 = \alpha_1$, $\beta_3 = \alpha_3$. If $\alpha_2 \ge 4$, then

$$\frac{64}{65} + \frac{2^5 3^{\beta_1+1} 13^{\beta_3}}{3^{\alpha_1+1} 5^{\alpha_2+1} 13^{\alpha_3+1}} \le \frac{64}{65} + \frac{2^5}{5^5 13} = 0.985\ldots < f(\alpha_1, \alpha_2, \alpha_3),$$

a contradiction. If $\alpha_2 = 2$, then

$$\frac{64}{65} + \frac{2^5 3^{\beta_1+1} 13^{\beta_3}}{3^{\alpha_1+1} 5^{\alpha_2+1} 13^{\alpha_3+1}} = \frac{64}{65} + \frac{2^5}{5^3 13} > 1 > f(\alpha_1, \alpha_2, \alpha_3),$$

a contradiction.

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2. By Lemmas 2.1 and 2.2, it is sufficient to show that there is no odd deficient-perfect number with two distinct prime divisors.

Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2}$ is an odd deficient-perfect number. Then

$$\sigma(n) = 2n - d,$$

where $d \,|\, n$ and $d < n$. Since $\sigma(n) \equiv 1 \pmod 2$, we have $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2$.

If $p_1 \ge 5$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{5}{4} \cdot \frac{7}{6} + \frac{1}{5} < 2,$$

which is impossible. Thus $p_1 = 3$. If $p_2 \ge 11$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{11}{10} + \frac{1}{3} < 2,$$

which is also impossible. Thus $p_2 = 5$ or 7. Hence, if n is an odd deficient-perfect number with two distinct prime divisors, then $\sigma(3^{\alpha_1} p_2^{\alpha_2}) = 2 \cdot 3^{\alpha_1} p_2^{\alpha_2} - 3^{\beta_1} p_2^{\beta_2}$, where $0 \le \beta_1 + \beta_2 < \alpha_1 + \alpha_2$ and $p_2 = 5$ or 7. We have $\beta_1 = \alpha_1 - 1$, $\beta_2 = \alpha_2$ or $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2 - 1$. Otherwise,

$$2 = \frac{\sigma(n)}{n} + \frac{1}{3^{\alpha_1 - \beta_1} p_2^{\alpha_2 - \beta_2}} < \frac{3}{2} \cdot \frac{5}{4} + \frac{1}{9} < 2.$$

Now we consider the following two cases.

Case 1: $\beta_1 = \alpha_1 - 1$, $\beta_2 = \alpha_2$. Then $\sigma(3^{\alpha_1} p_2^{\alpha_2}) = 5 \cdot 3^{\alpha_1 - 1} p_2^{\alpha_2}$. Noting that $3^{\alpha_1+1} - 1 \equiv 1, 2 \pmod 5$ and $3^{\alpha_1+1} - 1 \equiv 2, 4, 5 \pmod 7$, we have $p_2 \nmid \sigma(3^{\alpha_1})$. Moreover, $p_2 \nmid \sigma(p_2^{\alpha_2})$. Thus $p_2 \nmid \sigma(3^{\alpha_1} p_2^{\alpha_2})$, a contradiction.

Case 2: $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2 - 1$. Then $\sigma(3^{\alpha_1} p_2^{\alpha_2}) = 2 \cdot 3^{\alpha_1} p_2^{\alpha_2} - 3^{\alpha_1} p_2^{\alpha_2 - 1}$.

If $p_2 = 5$, then $\sigma(3^{\alpha_1} 5^{\alpha_2}) = 3^{\alpha_1+2} 5^{\alpha_2 - 1}$. Since $5^{\alpha_2+1} - 1 \equiv 1 \pmod 3$, we have $3 \nmid \sigma(5^{\alpha_2})$. Moreover, $3 \nmid \sigma(3^{\alpha_1})$. Thus $3 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2})$, a contradiction.

If $p_2 = 7$, then $\sigma(3^{\alpha_1} 7^{\alpha_2}) = 3^{\alpha_1} 7^{\alpha_2 - 1} 13$. Since $3^{\alpha_1+1} - 1 \equiv 2, 4, 5 \pmod 7$, we have $7 \nmid \sigma(3^{\alpha_1})$. Moreover, $7 \nmid \sigma(7^{\alpha_2})$. Thus $7 \nmid \sigma(3^{\alpha_1} 7^{\alpha_2})$, a contradiction.

This completes the proof of Theorem 1.2.

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