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# EXPONENTIALS OF BOUNDED NORMAL OPERATORS

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**Abstract.** The present paper is mainly concerned with equations involving exponentials of bounded normal operators. Conditions implying commutativity of normal operators are given, without using the known $2\pi i$-congruence-free hypothesis. This is a continuation of a recent work by the second author.

**1. Introduction.** First, we assume the reader is familiar with notions and results of bounded operator theory. Some important references are [3] and [15]. We suppose that all operators considered are linear and defined on a complex Hilbert space, designated by $H$. The set of all these operators is denoted by $B(H)$, which is a Banach algebra.

Let us just say a few words about notations. It is known that any bounded linear operator $T$ may be expressed as $A+iB$ where $A$ and $B$ are self-adjoint. In fact,

$$A = \frac{T+T^*}{2} \quad \text{and} \quad B = \frac{T-T^*}{2i}.$$

We call $A$ the real part of $T$ and denote it $\operatorname{Re} T$; and we call $B$ the imaginary part of $T$ and $\operatorname{Im} T$. It is also well-known that $T$ is normal if and only if $AB = BA$.

The following standard result will be useful.

Lemma 1.1. *Let $T$ be a self-adjoint operator such that $e^T = I$. Then $T = 0$.*

We include a proof for the reader's convenience.

*Proof.* Let $x \in \sigma(T)$. Then $e^x = 1$. Since $T$ is self-adjoint, $x$ is real and hence $e^x = 1$ implies $x = 0$. Since $\sigma(T)$ is never empty, $\sigma(T) = \{0\}$. Again, since $T$ is self-adjoint, by the spectral radius theorem we have $\|T\| = r(T) = 0$, so $T = 0$. ∎

We will also be using the celebrated Fuglede theorem, which we recall for the reader's convenience (for other versions, see [6, 8, 10]). For a proof, see e.g. [3] or [15].

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THEOREM 1.2 (Fuglede). *Let $A, N \in B(H)$. Assume that $N$ is normal. Then*

$$AN = NA \;\Rightarrow\; AN^* = N^*A.$$

The exponential of an operator appears in many areas of mathematics: when solving problems of the type $X' = AX$ where $A$ is an operator, when dealing with semigroups, the Stone theorem, the Lie product formula, the Trotter product formula, the Feynman–Kac formula, (bounded) wave operators, etc. See [5, 7, 12, 13, 14].

Commutativity of operators and its characterization is one of the most important topics in operator theory. Thus when the commutativity of exponentials implies that of operators becomes an interesting problem. In this paper we are mainly concerned with problems of this sort. Many authors have worked on similar questions (see [11, 16, 17, 18, 20]). However, they all used what is known as the $2\pi i$-congruence-free hypothesis (and similar hypotheses, see the above-mentioned references for definitions). G. Bourgeois [2] dropped that hypothesis but he only worked in low dimensions. Very recently, M. H. Mortad [9] gave a different approach to this problem for normal operators, bypassing the $2\pi i$-congruence-free hypothesis. He used the well-known cartesian decomposition of normal operators as $A+iB$ where $A$ and $B$ are commuting self-adjoint operators, so that the following result may be applied (whose proof may be found e.g. in [20]):

THEOREM 1.3. *Let $A$ and $B$ be two self-adjoint operators defined on a Hilbert space. Then*

$$e^Ae^B = e^Be^A \;\Leftrightarrow\; AB = BA.$$

Using a result on similarities (due to S. K. Berberian [1]) as well as the Riesz functional calculus, the following two results were obtained in [9]:

PROPOSITION 1.4. *Let $N$ be a normal operator with cartesian decomposition $A + iB$. Let $S$ be a self-adjoint operator. If $\sigma(B) \subset (0, \pi)$, then*

$$e^Se^N = e^Ne^S \;\Leftrightarrow\; SN = NS.$$

REMARK. Inspecting the proof of Proposition 1.4, we see that we may take $(-\pi/2, \pi/2)$ in lieu of $(0, \pi)$ without any problem. Hence the same results hold with this new interval. Thus any self-adjoint operator (remember that its imaginary part must then vanish) obeys the given condition on the spectrum.

THEOREM 1.5. *Let $N$ and $M$ be two normal operators with cartesian decompositions $A+iB$ and $C+iD$ respectively. If $\sigma(B), \sigma(D) \subset (0, \pi)$, then*

$$e^Me^N = e^Ne^M \;\Leftrightarrow\; MN = NM.$$

In this paper, we investigate this question further. Our proofs are very simple; moreover, in some cases, they may even be applied to prove known results which use the $2\pi i$-congruence-free hypothesis, for example.

Let us now give a *sample* of already known results on the topic of the present paper.

THEOREM 1.6 (Hille, [4]). *Let $A$ and $B$ be both in $B(H)$ such that $e^A = e^B$. If $\sigma(A)$ is incongruent modulo $2\pi i$, then $A$ and $B$ commute.*

THEOREM 1.7 (Schmoeger, [18]). *Let $A$ and $B$ be both in $B(H)$. Then:*

(1) *If $A+B$ is normal, $\sigma(A+B)$ is generalized $2\pi i$-congruence-free and*
$$e^A e^B = e^B e^A = e^{A+B},$$
*then $AB = BA$.*

(2) *If $A$ is normal, $\sigma(A)$ is generalized $2\pi i$-congruence-free and*
$$e^A = e^B,$$
*then $AB = BA$.*

THEOREM 1.8 (Schmoeger, [19]). *Let $A$ and $B$ be both in $B(H)$ such that $e^A = e^B$. Assume that $A$ is normal.*

(1) *If $r(A) < \pi$, then $AB = BA$ (where $r(A)$ is the spectral radius of $A$).*

(2) *If*
$$\sigma(A) \subseteq \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \pi\}$$
*and*
$$\sigma(A) \cap \sigma(A + 2\pi i) \subseteq \{i\pi\},$$
*then $A^2 B = BA^2$. If $i\pi \notin \sigma_p(A)$ or $-i\pi \notin \sigma_p(A)$, then $AB = BA$.*

Throughout this paper, the reader will see that with simpler hypotheses, we shall get the same conclusions as above.

**2. An example.** The following example, partly inspired by [20], will be needed in the next section, mainly as a counterexample.

EXAMPLE 2.1. Let
$$A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}.$$

Then $A$ is clearly normal. By computing integer powers of $A$ we may easily check that
$$e^A = \begin{pmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

Next, we have

$$\operatorname{Im} A = \frac{A - A^*}{2i} = \begin{pmatrix} 0 & -i\pi \\ i\pi & 0 \end{pmatrix}$$

and hence $\sigma(\operatorname{Im} A) = \{\pi, -\pi\}$. This signifies that $\sigma(\operatorname{Im} A)$ cannot be inside an open interval of length $\pi$ (a hypothesis which will play an important role in our proofs).

Now let

$$B = \begin{pmatrix} \pi & -2\pi \\ \pi & -\pi \end{pmatrix}.$$

We can also show that $e^B = -I$. Finally, it is easily verifiable that $A$ and $B$ do not commute.

**3. Main results.** We start with a result that will be of much use in the paper:

THEOREM 3.1. *Let $A$ be in $B(H)$. Let $N \in B(H)$ be normal and such that $\sigma(\operatorname{Im} N) \subset (0, \pi)$. Then*

$$Ae^N = e^N A \iff AN = NA.$$

*Proof.* Of course, we are only concerned with proving the implication "$\Rightarrow$". The normality of $N$ implies that of $e^N$, and so by the Fuglede theorem

$$Ae^N = e^N A \quad \text{yields} \quad Ae^{N^*} = e^{N^*} A,$$

so

$$A^* e^N = e^N A^*.$$

Hence

$$(A + A^*)e^N = e^N (A + A^*), \quad \text{i.e.} \quad (\operatorname{Re} A)e^N = e^N (\operatorname{Re} A),$$

so that

$$e^{\operatorname{Re} A} e^N = e^N e^{\operatorname{Re} A}.$$

But $\operatorname{Re} A$ is self-adjoint, so Proposition 1.4 applies and gives

$$(\operatorname{Re} A)N = N(\operatorname{Re} A).$$

Similarly, we find that

$$(\operatorname{Im} A)e^N = e^N (\operatorname{Im} A)$$

and as $\operatorname{Im} A$ is self-adjoint, similar arguments yield

$$(\operatorname{Im} A)N = N(\operatorname{Im} A).$$

Therefore, $AN = NA$, establishing the result. ∎

REMARK. The hypothesis $\sigma(\operatorname{Im} N) \subset (0,\pi)$ cannot merely be dropped. Take $N$ to be the operator $A$ in Example 2.1, and take $A$ to be any operator which does not commute with $N$. Then

$$Ae^N = e^N A = -A \quad \text{but} \quad AN \neq NA.$$

Next we give the first consequence of the previous result:

THEOREM 3.2. *Let $A$ and $B$ be both in $B(H)$. Assume that $A+B$ is normal such that $\sigma(\operatorname{Im}(A+B)) \subset (0,\pi)$. If*

$$e^A e^B = e^B e^A = e^{A+B},$$

*then $AB = BA$.*

*Proof.* We have

$$e^{A+B}e^A = e^B e^A e^A = e^A e^B e^A = e^A e^{A+B}.$$

Since $A+B$ is normal and $\sigma(\operatorname{Im}(A+B)) \subset (0,\pi)$, Theorem 3.1 gives

$$(A+B)e^A = e^A(A+B), \quad \text{so} \quad Be^A = e^A B,$$

for $A$ commutes with $e^A$. Now, right multiplying both sides of the previous equation by $e^B$ leads to

$$Be^A e^B = e^A B e^B = e^A e^B B,$$

or equivalently

$$Be^{A+B} = e^{A+B}B.$$

Applying again Theorem 3.1, we see that

$$B(A+B) = (A+B)B, \quad \text{so} \quad AB = BA. \ \blacksquare$$

COROLLARY 3.3. *Let $A \in B(H)$. Then*

$$e^A e^{A^*} = e^{A^*} e^A = e^{A+A^*} \ \Leftrightarrow \ A \textit{ is normal.}$$

*Proof.* We need only prove the implication "$\Rightarrow$". It is plain that $A+A^*$ is self-adjoint. Hence the remark following Proposition 1.4 combined with Theorem 3.2 gives us

$$AA^* = A^*A. \ \blacksquare$$

We have yet another consequence of Theorem 3.1 (cf. Theorem 1.8).

COROLLARY 3.4. *Let $A$ be normal such that $\sigma(\operatorname{Im} A) \subset (0,\pi)$. Let $B \in B(H)$. Then*

$$e^A = e^B \ \Rightarrow \ A^2 B = BA^2.$$

*Proof.* We obviously have

$$e^B(e^B B) = (Be^B)e^B.$$

So since $e^A = e^B$, we have

$$e^A(e^A B) = (Be^A)e^A.$$

By Theorem 3.1, we obtain

$$Ae^AB = Be^AA.$$

Hence

$$e^A(AB) = (BA)e^A.$$

Applying Theorem 3.1 once more yields

$$A(AB) = (BA)A, \quad \text{so} \quad A^2B = BA^2. \ ■$$

COROLLARY 3.5. *Let $A$ be normal such that $\sigma(\operatorname{Im} A) \subset (0,\pi)$. Let $B \in B(H)$. Then*

$$e^A = e^B \ \Rightarrow \ AB = BA.$$

REMARK. In Example 2.1, $e^A = e^B$ $(= -I)$, but $AB \neq BA$, showing again the importance of the assumption $\sigma(\operatorname{Im} A) \subset (0,\pi)$.

*Proof.* We obviously have

$$Be^B = e^BB,$$

so that

$$Be^A = e^AB.$$

Theorem 3.1 does the remaining job, i.e. it gives the commutativity of $A$ and $B$. ■

COROLLARY 3.6. *Let $A$ and $B$ be two self-adjoint operators. Then*

$$e^A = e^B \ \Leftrightarrow \ A = B.$$

*Proof.* By the remark after Proposition 1.4, we get

$$e^A = e^B \ \Rightarrow \ e^Ae^B = e^Be^A \ \Rightarrow \ AB = BA.$$

Hence

$$I = e^Ae^{-A} = e^Ae^{-B} = e^{A-B}$$

since $A$ and $B$ commute. But $A - B$ is obviously self-adjoint, so Lemma 1.1 gives $A = B$. ■

REMARK. The previous corollary is actually a consequence of Theorem 1.3. Other authors usually obtained it as a consequence of more complicated results. But, with the proof given here, we clearly see that we only need Theorem 1.3 and Lemma 1.1.

REMARK. Of course, the previous corollary also generalizes Lemma 1.1.

COROLLARY 3.7. *Let $A$ be normal. Then*

$$A \text{ is self-adjoint} \ \Leftrightarrow \ e^{iA} \text{ is unitary.}$$

*Proof.* The implication "$\Rightarrow$" is well-known. Let us prove the reverse implication. By the normality of $A$, we have

$$e^{iA-iA^*} = e^{iA}e^{-iA^*} = e^{iA}(e^{iA})^* = I.$$

Since $iA - iA^*$ is self-adjoint, Lemma 1.1 gives $A = A^*$, which completes the proof. ∎

We now come to a result that appeared in [11] and [18]: If $A$ is self-adjoint, $\sigma(A) \subseteq [-\pi, \pi]$, and if $e^{iA} = e^B$ and $B$ is normal, then $B^* = -B$. Here is an improvement of that result.

PROPOSITION 3.8. *If $A$ is self-adjoint, and if $e^{iA} = e^B$ and $B$ is normal, then $B^* = -B$.*

*Proof.* It is clear that $e^{iA}$ is unitary. We also have

$$e^{B^*} = e^{-iA} \quad \text{and} \quad e^{-B} = e^{-iA}.$$

Thus

$$e^{-B} = e^{B^*} \quad \text{so that} \quad e^{B+B^*} = I$$

because $B$ is normal. However, $B + B^*$ is always self-adjoint, whence $B^* = -B$ by Lemma 1.1. ∎

**4. Conclusion.** The results of this paper as well as those of [9] could be easily generalized to unital $C^*$-algebras.

Theorem 3.1 is important here. The simple and interesting proof of Corollary 3.4 or Corollary 3.5 could not have been achieved if we did not have Theorem 3.1 in hand. Also, as mentioned in the introduction, most of the proofs, for instance that of Corollary 3.6, may be adopted to prove the results that use the $2\pi i$-congruence-free hypothesis and similar hypotheses.

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