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WEIGHTED BOUNDEDNESS OF TOEPLITZ TYPE OPERATORS RELATED TO SINGULAR INTEGRAL OPERATORS WITH NON-SMOOTH KERNEL

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Abstract. Some weighted sharp maximal function inequalities for the Toeplitz type operator $T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2}$ are established, where $T^{k,1}$ are a fixed singular integral operator with non-smooth kernel or $\pm I$ (the identity operator), $T^{k,2}$ are linear operators defined on the space of locally integrable functions, $k = 1, \ldots, m$, and $M_b(f) = bf$. The weighted boundedness of T_b on Morrey spaces is obtained by using sharp maximal function inequalities.

1. Introduction. Developing the thory of singular integral operators (see [GR], [S]), their commutators have been extensively studied. In [CRW], [PER], [PT], the authors proved that commutators generated by singular integral operators and BMO functions were bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo [C] proved a similar result with singular integral operators replaced by fractional integral operators. In [J], [PA], the boundedness of commutators generated by singular integral operators and Lipschitz functions on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ and Triebel–Lizorkin spaces was obtained. In [B], [HG], the boundedness of commutators generated by singular integral operators and weighted BMO and Lipschitz functions on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ was obtained (see also [HEW]). In [KRL], [LIL], [LM], some Toeplitz type operators related to singular integral operators and strongly singular integral operators were introduced, and the boundedness of their commutators with BMO and Lipschitz functions was obtained. In [DUM], [MA], some singular integral operators with non-smooth kernel were introduced, and the boundedness of these operators and their commutators was obtained (see [DEY], [DUM], [DY1], [DY2], [LIU2], [ZL]).

On the other hand, the classical Morrey spaces were introduced by Morrey [MO] to investigate the local behavior of solutions to second order elliptic partial differential equations (see also [P]). As Morrey spaces may be considered as extensions of Lebesgue spaces, it is natural and important to

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study the boundedness of operators on Morrey spaces. The boundedness of maximal operators, singular integral operators, fractional integral operators and their commutators on Morrey spaces has been studied by many authors (see [DR1], [DR2], [KOS], [MI]). In [KOS], Komori and Shirai studied the boundedness of these operators on weighted Morrey spaces.

Motivated by these, in this paper, we will study Toeplitz type operators related to singular integral operators with non-smooth kernel and weighted Lipschitz and BMO functions, and prove their weighted boundedness on Morrey spaces.

2. Preliminaries. In this paper, we will study some singular integral operators (see [DUM], [MA]).

DEFINITION 2.1. A family of operators D_t, $t > 0$, is said to be an *approximation to the identity* if, for every $t > 0$, D_t can be represented by a kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y)\, dy$$

for every $f \in L^p(\mathbb{R}^n)$ with $p \geq 1$, and $a_t(x, y)$ satisfies

$$|a_t(x, y)| \leq h_t(x, y) = Ct^{-n/2} \rho(|x - y|^2/t),$$

where ρ is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon} \rho(r^2) = 0 \quad \text{for some } \epsilon > 0.$$

DEFINITION 2.2. A linear operator T is called a *singular integral operator with non-smooth kernel* if T is bounded on $L^2(\mathbb{R}^n)$ and associated with a kernel $K(x, y)$ such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy$$

for every continuous function f with compact support, and for almost all x not in the support of f. Moreover, we assume:

(1) There exists an approximation to the identity $\{B_t, t > 0\}$ such that TB_t has kernel $k_t(x, y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1 t^{1/2}} |K(x, y) - k_t(x, y)|\, dx \leq c_2 \quad \text{for all } y \in \mathbb{R}^n.$$

(2) There exists an approximation to the identity $\{A_t, t > 0\}$ such that $A_t T$ has the associated kernel $K_t(x, y)$ which satisfies

$$|K_t(x, y)| \leq c_4 t^{-n/2} \quad \text{if } |x - y| \leq c_3 t^{1/2},$$
$$|K(x, y) - K_t(x, y)| \leq c_4 t^{\delta/2} |x - y|^{-n-\delta} \quad \text{if } |x - y| \geq c_3 t^{1/2},$$

for some $\delta > 0$ and $c_3, c_4 > 0$.

Let b be a locally integrable function on $\mathbb{R}^n$ and T be a singular integral operator with non-smooth kernel. A *Toeplitz type operator associated to* T is defined by

$$T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2},$$

where each $T^{k,1}$ is either T or $\pm I$ (the identity operator), each $T^{k,2}$ is a linear operator defined on the space of locally integrable functions, $k = 1, \dots, m$ and $M_b(f) = bf$.

Note that if $m = 2$, then $T_b = T^{1,1} M_b T^{1,2} + T^{2,1} M_b T^{2,2}$; if one takes $T^{1,1} = I$, $T^{1,2} = T$, $T^{2,1} = T$ and $T^{2,2} = -I$, then $T_b(f) = M_b T(f) - T M_b(f) = bT(f) - T(bf)$, which is the commutator $[b,T](f)$ generated by T and b. Thus commutator can be considered a particular case of the Toeplitz type operator T_b, and the latter is a non-trivial generalization. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [PT]). In [DUM], [MA], the boundedness of singular integral operators with non-smooth kernel was obtained. In [DEY], [DUM], [DY1], [DY2], [ZL], the boundedness of the commutator associated to a singular integral operator with non-smooth kernel was obtained. The main purpose of this paper is to prove sharp maximal inequalities for T_b. As an application, we obtain the weighted boundedness of T_b on Morrey spaces.

Now, let us introduce some notation. Throughout this paper, Q will denote a cube in $\mathbb{R}^n$ with sides parallel to the axes. For any locally integrable function f, let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 \le \eta < n$, $1 \le p < \infty$ and a non-negative weight function w, set

$$M_{\eta,p,w}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{w(Q)^{1-p\eta/n}} \int_Q |f(y)|^p w(y)\, dy \right)^{1/p}.$$

We write $M_{\eta,p,w}(f) = M_{p,w}(f)$ if $\eta = 0$.

The sharp maximal function $M_A^{\#}(f)$ associated with the approximation to the identity $\{A_t, t > 0\}$ is defined by

$$M_A^{\#}(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)|\, dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. For $\eta > 0$, let $M_{A,\eta}^{\#}(f) = M_A^{\#}(|f|^\eta)^{1/\eta}$.

The set of A_p *weights* is defined, for $1 < p < \infty$, by (see [GR])

$$A_p = \left\{ w \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\, dx \right)^{p-1} < \infty \right\};$$

moreover,

$$A_1 = \{ w \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : M(w)(x) \le Cw(x), \text{ a.e.} \},$$
$$A_\infty = \bigcup_{p \ge 1} A_p.$$

For $0 < \beta < 1$ and a non-negative weight function w, the *weighted Lipschitz space* $\mathrm{Lip}_\beta(w)$ is the space of functions b such that

$$\|b\|_{\mathrm{Lip}_\beta(w)} = \sup_Q \frac{1}{w(Q)^{\beta/n}} \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p}\, dy \right)^{1/p} < \infty,$$

and the *weighted BMO space* $\mathrm{BMO}(w)$ is the space of functions b such that

$$\|b\|_{\mathrm{BMO}(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p}\, dy \right)^{1/p} < \infty.$$

REMARK. (1) It is known (see [G]) that for $b \in \mathrm{Lip}_\beta(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^jQ}| \le Cj \|b\|_{\mathrm{Lip}_\beta(w)} w(x) w(2^jQ)^{\beta/n}.$$

(2) It is known (see [G]) that for $b \in \mathrm{BMO}(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^jQ}| \le Cj \|b\|_{\mathrm{BMO}(w)} w(x).$$

(3) Let $b \in \mathrm{Lip}_\beta(w)$ or $b \in \mathrm{BMO}(w)$, and $w \in A_1$. By [G], [HG], we know that the spaces $\mathrm{Lip}_\beta(w)$ or $\mathrm{BMO}(w)$ coincide and the norms $\|b\|_{\mathrm{Lip}_\beta(w)}$ or $\|b\|_{\mathrm{BMO}(w)}$ for different values $1 \le p < \infty$ are all equivalent.

Throughout this paper, φ will denote a positive, increasing function on $\mathbb{R}^+$ for which there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \quad \text{for } t \ge 0.$$

Let w be a non-negative weight function on $\mathbb{R}^n$ and f be a locally integrable function on $\mathbb{R}^n$. Set, for $1 \le p < \infty$,

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n,\, d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p w(y)\, dy \right)^{1/p},$$

where $Q(x,d) = \{ y \in \mathbb{R}^n : |x - y| < d \}$. The *generalized weighted Morrey space* is defined by

$$L^{p,\varphi}(\mathbb{R}^n, w) = \{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\varphi}(w)} < \infty \}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(\mathbb{R}^n, w) = L^{p,\delta}(\mathbb{R}^n, w)$, which is the classical Morrey space (see [PE1], [PE2]). If $\varphi(d) = 1$, then $L^{p,\varphi}(\mathbb{R}^n, w) = L^p(\mathbb{R}^n, w)$, which is a weighted Lebesgue space (see [GR]).

3. Some lemmas. We begin with the following lemmas.

LEMMA 3.1 (see [GR, p. 485]). *Let $0 < p < q < \infty$ and $1/r = 1/p - 1/q$. For any function $f \geq 0$ define*

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : f(x) > \lambda\}|^{1/q},$$
$$N_{p,q}(f) = \sup_Q \|f\chi_Q\|_{L^p}/\|\chi_Q\|_{L^r},$$

where the sup is taken over all measurable sets Q with $0 < |Q| < \infty$. Then

$$\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} \|f\|_{WL^q}.$$

LEMMA 3.2 (see [DUM], [MA]). *Let T be a singular integral operator as in Definition 2.2. Then T is bounded on $L^p(\mathbb{R}^n, w)$ for $w \in A_p$ with $1 < p < \infty$, and weak (L^1, L^1) bounded.*

LEMMA 3.3 (see [G], [GR]). *Let $0 \leq \eta < n$, $1 \leq s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A_1$. Then*

$$\|M_{\eta,s,w}(f)\|_{L^q(w)} \leq C\|f\|_{L^p(w)}.$$

LEMMA 3.4 (see [DUM], [MA]). *Let $\{A_t, t > 0\}$ be an approximation to the identity. For any $\gamma > 0$, there exists a constant $C > 0$ independent of γ such that*

$$|\{x \in \mathbb{R}^n : M(f)(x) > D\lambda,\ M_A^{\#}(f)(x) \leq \gamma\lambda\}| \leq C\gamma |\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}|$$

for $\lambda > 0$, where D is a fixed constant which only depends on n. Thus, for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, $0 < \eta < \infty$ and $w \in A_1$,

$$\|M_\eta(f)\|_{L^p(w)} \leq C\|M_{A,\eta}^{\#}(f)\|_{L^p(w)}.$$

LEMMA 3.5. *Let $\{A_t, t > 0\}$ be an approximation to the identity, $0 < D < 2^n$ and $w \in A_1$. Then*

(a) $\|M_\eta(f)\|_{L^{p,\varphi}(w)} \leq C\|M_{A,\eta}^{\#}(f)\|_{L^{p,\varphi}(w)}$ *for $1 < p < \infty$, $0 < \eta < \infty$ and $w \in A_1$;*

(b) $\|M_{\eta,s,w}(f)\|_{L^{q,\varphi}(w)} \leq C\|f\|_{L^{p,\varphi}(w)}$ *for $0 \leq \eta < n$, $1 \leq s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A_1$.*

Proof. (a) For any cube $Q = Q(x_0, d)$ in $\mathbb{R}^n$, we know $M(w\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [CR]. Then, by Lemma 3.4, we have

$$\begin{aligned}
\int\limits_Q |M_\eta(f)(y)|^p w(y)\,dy &= \int\limits_{\mathbb{R}^n} |M_\eta(f)(y)|^p w(y)\chi_Q(y)\,dy \\
&\le \int\limits_{\mathbb{R}^n} |M_\eta(f)(y)|^p M(w\chi_Q)(y)\,dy \le C\int\limits_{\mathbb{R}^n} |M^\#_{A,\eta}(f)(y)|^p M(w\chi_Q)(y)\,dy \\
&= C\int\limits_Q |M^\#_{A,\eta}(f)(y)|^p M(w\chi_Q)(y)\,dy \\
&\quad + C\sum_{k=0}^\infty \int\limits_{2^{k+1}Q\setminus 2^kQ} |M^\#_{A,\eta}(f)(y)|^p M(w\chi_Q)(y)\,dy \\
&\le C\Big(\int\limits_Q |M^\#_{A,\eta}(f)(y)|^p w(y)\,dy + \sum_{k=0}^\infty \int\limits_{2^{k+1}Q\setminus 2^kQ} |M^\#_{A,\eta}(f)(y)|^p \frac{w(Q)}{|2^{k+1}Q|}\,dy\Big) \\
&\le C\Big(\int\limits_Q |M^\#_{A,\eta}(f)(y)|^p w(y)\,dy + \sum_{k=0}^\infty \int\limits_{2^{k+1}Q} |M^\#_{A,\eta}(f)(y)|^p \frac{M(w)(y)}{2^{n(k+1)}}\,dy\Big) \\
&\le C\Big(\int\limits_Q |M^\#_{A,\eta}(f)(y)|^p w(y)\,dy + \sum_{k=0}^\infty \int\limits_{2^{k+1}Q} |M^\#_{A,\eta}(f)(y)|^p \frac{w(y)}{2^{nk}}\,dy\Big) \\
&\le C\|M^\#_{A,\eta}(f)\|^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty 2^{-nk}\varphi(2^{k+1}d) \\
&\le C\|M^\#_{A,\eta}(f)\|^p_{L^{p,\varphi}(w)} \sum_{k=0}^\infty (2^{-n}D)^k\varphi(d) \le C\|M^\#_{A,\eta}(f)\|^p_{L^{p,\varphi}(w)}\varphi(d),
\end{aligned}$$

thus

$$\|M_\eta(f)\|_{L^{p,\varphi}(w)} \le C\|M^\#_{A,\eta}(f)\|_{L^{p,\varphi}(w)}.$$

This finishes the proof of (a).

The proof of (b) is similar to that of (a) by Lemma 3.3; we omit the details. ∎

4. Results and their proofs

THEOREM 4.1. *Let T be a singular integral operator with non-smooth kernel as in Definition 2.2, $w \in A_1$, $0 < \eta < 1$, $1 < s < \infty$, $0 < \beta < 1$ and $b \in \mathrm{Lip}_\beta(w)$. If $T1 = 0$ on $L^r(\mathbb{R}^n)$ $(1 < r < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(\mathbb{R}^n)$ and $\tilde{x} \in \mathbb{R}^n$,*

$$M^\#_{A,\eta}(T_b(f))(\tilde{x}) \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) \sum_{k=1}^m M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}).$$

Proof. It suffices to prove that for $f \in C_0^\infty(\mathbb{R}^n)$,

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - A_{t_Q}(T_b(f))(x)|^\eta\, dx\right)^{1/\eta} \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) \sum_{k=1}^m M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}),$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. Without loss of generality, we may assume $T^{k,1} = T$ $(k = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Since $T_1(g) = 0$, we have

$$\begin{aligned} T_b(f)(x) &= T_{b-b_{2Q}}(f)(x) = T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) \\ &= U_1(x) + U_2(x) \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - A_{t_Q}(T_b(f))(x)|^\eta\, dx\right)^{1/\eta} \\ &\quad \le \left(\frac{C}{|Q|}\int_Q |U_1(x)|^\eta\, dx\right)^{1/\eta} + \left(\frac{C}{|Q|}\int_Q |A_{t_Q}(U_1)(x)|^\eta\, dx\right)^{1/\eta} \\ &\qquad + \left(\frac{C}{|Q|}\int_Q |U_2(x) - A_{t_Q}(U_2)(x)|^\eta\, dx\right)^{1/\eta} \\ &\quad = I_1 + I_2 + I_3. \end{aligned}$$

For I_1, by the weak (L^1, L^1) boundedness of T (see Lemma 3.2) and Kolmogorov's inequality (see Lemma 3.1), we obtain

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|^\eta\, dx\right)^{1/\eta} \\ &\quad \le \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}} \frac{\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\chi_Q\|_{L^\eta}}{\|\chi_{2Q}\|_{L^{\eta/(1-\eta)}}} \\ &\quad \le \frac{C}{|Q|}\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{WL^1} \le \frac{C}{|Q|}\int_{\mathbb{R}^n} |M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|\, dx \\ &\quad \le \frac{C}{|Q|}\int_{2Q} |b(x) - b_{2Q}|w(x)^{-1/s}|T^{k,2}(f)(x)|w(x)^{1/s}\, dx \\ &\quad \le \frac{C}{|Q|}\left(\int_{2Q} |b(x) - b_{2Q}|^{s'} w(x)^{1-s'}\, dx\right)^{1/s'} \left(\int_{2Q} |T^{k,2}(f)(x)|^s w(x)\, dx\right)^{1/s} \\ &\quad \le \frac{C}{|Q|}\|b\|_{\mathrm{Lip}_\beta(w)} w(2Q)^{1/s'+\beta/n} w(2Q)^{1/s-\beta/n} \end{aligned}$$

$$\times \left(\frac{1}{w(Q)^{1-s\beta/n}} \int_{2Q} |T^{k,2}(f)(x)|^s w(y)\, dy \right)^{1/s}$$
$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} \frac{w(2Q)}{|2Q|} M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})$$
$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}),$$

thus

$$I_1 \le C \sum_{k=1}^m \left(\frac{1}{|Q|} \int_Q |T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(x)|^\eta \, dx \right)^{1/\eta}$$
$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) \sum_{k=1}^m M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}).$$

For I_2, by the condition on h_{t_Q} and noticing that if $x \in Q$ and $y \in 2^{j+1}Q \setminus 2^jQ$, then $h_{t_Q}(x,y) \le C t_Q^{-n/2} \rho(2^{2(j-1)})$, we have

$$\left[\frac{1}{|Q|} \int_Q |A_{t_Q}(T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f))(x)|^\eta \, dx \right]^{1/\eta}$$
$$\le C \left[\frac{1}{|Q|} \int_Q \left(\int_{\mathbb{R}^n} h_{t_Q}(x,y) |T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(y)|\, dy \right)^\eta dx \right]^{1/\eta}$$
$$\le C \left[\frac{1}{|Q|} \int_Q \left(\int_{2Q} h_{t_Q}(x,y) |T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(y)|\, dy \right)^\eta dx \right]^{1/\eta}$$
$$+ C \left[\frac{1}{|Q|} \int_Q \left(\int_{(2Q)^c} h_{t_Q}(x,y) |T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(y)|\, dy \right)^\eta dx \right]^{1/\eta}$$
$$\le C \int_{2Q} t_Q^{-n/2} |T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(y)|\, dy$$
$$+ C \sum_{j=1}^\infty t_Q^{-n/2} \rho(2^{2(j-1)}) \int_{2^{j+1}Q \setminus 2^jQ} |T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(y)|\, dy$$
$$= I_2^{(1)} + I_2^{(2)}.$$

Since $w \in A_1$, w satisfies the reverse Hölder inequality:

$$\left(\frac{1}{|Q|} \int_Q w(x)^{p_0}\, dx \right)^{1/p_0} \le \frac{C}{|Q|} \int_Q w(x)\, dx$$

for all cubes Q and some $1 < p_0 < \infty$ (see [GR]). Choose $q > 1$ such that $r = (p_0 - 1)/q + 1 < s$ and let $p > 1$ with $r/s + 1/p + 1/q = 1$; then $(r - r/s - 1/p)q = p_0$. We obtain, by Hölder's inequality and L^r-boundedness of T (see Lemma 3.2),

$$I_2^{(1)} \le C\bigg(\frac{1}{|Q|}\int_{\mathbb{R}^n}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^r\,dy\bigg)^{1/r}$$

$$\le C\bigg(\frac{1}{|Q|}\int_{\mathbb{R}^n}|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^r\,dy\bigg)^{1/r}$$

$$\le C|Q|^{-1/r}\Big(\int_{2Q}|b(y)-b_{2Q}|^r w(y)^{1/p-r}|T^{k,2}(f)(y)|^r w(y)^{r/s}w(y)^{r-r/s-1/p}\,dy\Big)^{1/r}$$

$$\le C|Q|^{-1/r}\Big(\int_{2Q}|b(y)-b_{2Q}|^{pr}w(y)^{1-pr}\,dy\Big)^{1/pr}$$

$$\times\Big(\int_{2Q}|T^{k,2}(f)(y)|^s w(y)\,dy\Big)^{1/s}\Big(\int_{2Q}w(y)^{(r-r/s-1/p)q}\,dy\Big)^{1/qr}$$

$$\le C|Q|^{-1/r}\|b\|_{\mathrm{Lip}_\beta(w)}w(2Q)^{\beta/n+1/pr}w(2Q)^{1/s-\beta/n}|2Q|^{1/qr}$$

$$\times\bigg(\frac{1}{w(2Q)^{1-s\beta/n}}\int_{2Q}|T^{k,2}(f)(y)|^s w(y)\,dy\bigg)^{1/s}\bigg(\frac{1}{|2Q|}\int_{2Q}w(y)^{p_0}\,dy\bigg)^{1/qr}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(w)}|Q|^{-1/r}w(2Q)^{1/pr}w(2Q)^{1/s}M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})$$

$$\times|2Q|^{1/qr}\bigg(\frac{1}{|2Q|}\int_{2Q}w(y)\,dy\bigg)^{p_0/qr}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(w)}\frac{w(2Q)}{|2Q|}M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(w)}w(\tilde{x})M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}),$$

and

$$I_2^{(2)} \le C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})\bigg(\frac{1}{|2^{j+1}Q|}\int_{\mathbb{R}^n}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^r\,dy\bigg)^{1/r}$$

$$\le C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})\bigg(\frac{1}{|2^{j+1}Q|}\int_{\mathbb{R}^n}|M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(y)|^r\,dy\bigg)^{1/r}$$

$$\le C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})(2^j l(Q))^{-n/r}$$

$$\times\Big(\int_{2Q}|b(y)-b_{2Q}|^r w(y)^{1/p-r}|T^{k,2}(f)(y)|^r w(y)^{r/s}w(y)^{r-r/s-1/p}\,dy\Big)^{1/r}$$

$$\le C\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})(2^j l(Q))^{-n/r}\Big(\int_{2Q}|b(y)-b_{2Q}|^{pr}w(y)^{1-pr}\,dy\Big)^{1/pr}$$

$$\times \Big(\int_{2Q} |T^{k,2}(f)(y)|^s w(y)\,dy\Big)^{1/s} \Big(\int_{2Q} w(y)^{(r-r/s-1/p)q}\,dy\Big)^{1/qr}$$

$$\le C\sum_{j=1}^{\infty} 2^{jn}\rho(2^{2(j-1)})(2^j l(Q))^{-n/r}\|b\|_{\mathrm{Lip}_\beta(w)} w(2Q)^{\beta/n+1/pr} w(2Q)^{1/s-\beta/n}$$

$$\times M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})|2Q|^{1/qr}\Big(\frac{1}{|2Q|}\int_{2Q} w(y)^{p_0}\,dy\Big)^{1/qr}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(w)}\sum_{j=1}^{\infty} 2^{jn}\rho(2^{2(j-1)})(2^j l(Q))^{-n/r} w(2Q)^{1/pr} w(2Q)^{1/s}$$

$$\times M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})|2Q|^{1/qr}\Big(\frac{1}{|2Q|}\int_{2Q} w(y)\,dy\Big)^{p_0/qr}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(w)}\frac{w(2Q)}{|2Q|} M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty} 2^{(j-1)(n+\epsilon)}\rho(2^{2(j-1)})2^{-j(\epsilon+n/r)}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}),$$

where $\sum_{j=1}^{\infty} 2^{(j-1)(n+\epsilon)}\rho(2^{2(j-1)})2^{-j(\epsilon+n/r)} < \infty$ since $\lim_{r\to\infty} r^{n+\epsilon}\rho(r^2) = 0$; thus

$$I_2 \le C\sum_{k=1}^{m}\Big(\frac{1}{|Q|}\int_{\mathbb{R}^n} |A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f))(x)|^{\eta}\,dx\Big)^{1/\eta}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})\sum_{k=1}^{m} M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}).$$

For I_3, notice $w \in A_1 \subset A_s$; by the definition of A_s, we get, for $x \in Q$,

$$|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x) - A_{t_Q}(T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,2}(f))(x)|$$

$$\le \int_{(2Q)^c} |b(y)-b_{2Q}|\,|K(x-y)-K_{t_Q}(x-y)|\,|T^{k,2}(f)(y)|\,dy$$

$$\le C\sum_{j=1}^{\infty}\int_{2^j d\le|y-x_0|<2^{j+1}d} \frac{l(Q)^{\delta}}{|y-x_0|^{n+\delta}}|b(y)-b_{2Q}|\,|T^{k,2}(f)(y)|\,dy$$

$$\le C\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}\int_{2^{j+1}Q} |b(y)-b_{2^{j+1}Q}+b_{2^{j+1}Q}-b_{2Q}|$$

$$\times w(y)^{-1/s}|T^{k,2}(f)(y)|w(y)^{1/s}\,dy$$

$$\le C\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}\Big(\int_{2^{j+1}Q} |b(y)-b_{2^{j+1}Q}|^{s'} w(y)^{1-s'}\,dy\Big)^{1/s'}$$

$$\times \Big(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s w(y)\,dy \Big)^{1/s}$$

$$+ C \sum_{j=1}^{\infty} \frac{d^\delta}{(2^{j+1}d)^{n+1}} |b_{2^{j+1}Q} - b_{2Q}| \Big(\int_{2^{j+1}Q} w(y)^{-1/(s-1)}\,dy \Big)^{1/s'}$$

$$\times \Big(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s w(y)\,dy \Big)^{1/s}$$

$$\le C \sum_{j=1}^{\infty} \frac{d^\delta}{(2^{j+1}d)^{n+\delta}} \|b\|_{\mathrm{Lip}_\beta(w)} w(2^{j+1}Q)^{1/s'+\beta/n} w(2^{j+1}Q)^{1/s-\beta/n} \times M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})$$

$$+ C \sum_{j=1}^{\infty} \frac{d^\delta}{(2^{j+1}d)^{n+\delta}} \|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) j w(2^{j+1}Q)^{\beta/n} w(2^{j+1}Q)^{1/s-\beta/n} \times M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})$$

$$\times \frac{|2^{j+1}Q|}{w(2^{j+1}Q)^{1/s}} \Big(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} w(y)\,dy \Big)^{1/s}$$

$$\times \Big(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} w(y)^{-1/(s-1)}\,dy \Big)^{(s-1)/s}$$

$$\le C \|b\|_{\mathrm{Lip}_\beta(w)} \sum_{j=1}^{\infty} \frac{w(2^{j+1}Q)}{|2^{j+1}Q|} 2^{-\delta j} M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})$$

$$+ C \|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) \sum_{j=1}^{\infty} j 2^{-\delta j} M_{\beta,s,w}(T^{k,2}(f))(\tilde{x})$$

$$\le C \|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}),$$

thus

$$I_3 \le \frac{C}{|Q|} \int_Q \sum_{k=1}^{m} |T^{k,1} M_{(b-b_{2Q})\chi_{(2Q)^c}} T^{k,2}(f)(x) - A_{t_Q}(T^{k,1} M_{(b-b_{2Q})\chi_{(2Q)^c}} T^{k,2}(f))(x)|\,dx$$

$$\le C \|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) \sum_{k=1}^{m} M_{\beta,s,w}(T^{k,2}(f))(\tilde{x}).$$

This completes the proof of Theorem 4.1. ∎

Theorem 4.2. *Let T be a singular integral operator with non-smooth kernel as in Definition 2.2, $w \in A_1$, $0 < \eta < 1$, $1 < s < \infty$ and $b \in \mathrm{BMO}(w)$. If $T_1 = 0$ on $L^r(\mathbb{R}^n)(1 < r < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(\mathbb{R}^n)$ and $\tilde{x} \in \mathbb{R}^n$,*

$$M^{\#}_{A,\eta}(T_b(f))(\tilde{x}) \le C \|b\|_{\mathrm{BMO}(w)} w(\tilde{x}) \sum_{k=1}^{m} M_{s,w}(T^{k,2}(f))(\tilde{x}).$$

Proof. It suffices to prove that for $f \in C_0^\infty(\mathbb{R}^n)$,

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - A_{t_Q}(T_b(f))(x)|^\eta\,dx\right)^{1/\eta} \le C\|b\|_{\mathrm{BMO}(w)} w(\tilde{x}) \sum_{k=1}^m M_{s,w}(T^{k,2}(f))(\tilde{x})),$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. Without loss of generality, we may assume $T^{k,1} = T$ $(k = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 4.1, we have

$$\begin{aligned} T_b(f)(x) &= T_{b-b_{2Q}}(f)(x) = T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) \\ &= V_1(x) + V_2(x) \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - A_{t_Q}(T_b(f))(x)|^\eta\,dx\right)^{1/\eta} \\ &\qquad \le \left(\frac{C}{|Q|}\int_Q |V_1(x)|^\eta\,dx\right)^{1/\eta} + \left(\frac{C}{|Q|}\int_Q |A_{t_Q}(V_1)(x)|^\eta\,dx\right)^{1/\eta} \\ &\qquad\quad + \left(\frac{C}{|Q|}\int_Q |V_2(x) - A_{t_Q}(V_2)(x)|^\eta\,dx\right)^{1/\eta} \\ &\qquad = I_4 + I_5 + I_6. \end{aligned}$$

By using the same argument as in the proof of Theorem 4.1, we get

$$\begin{aligned} I_4 &\le C\sum_{k=1}^m \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}} \frac{\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}} \\ &\le C\sum_{k=1}^m \frac{C}{|Q|}\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{WL^1} \\ &\le \sum_{k=1}^m \frac{C}{|Q|}\int_{\mathbb{R}^n} |M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)|\,dx \\ &\le \sum_{k=1}^m \frac{C}{|Q|}\int_{2Q} |b(x) - b_{2Q}|w(x)^{-1/s}|T^{k,2}(f)(x)|w(x)^{1/s}\,dx \\ &\le \sum_{k=1}^m \frac{C}{|Q|}\left(\int_{2Q} |b(x) - b_{2Q}|^{s'}w(x)^{1-s'}\,dx\right)^{1/s'}\left(\int_{2Q} |T^{k,2}(f)(x)|^s w(x)\,dx\right)^{1/s} \end{aligned}$$

$$\leq C \sum_{k=1}^{m} \frac{w(2Q)}{|2Q|} \left(\frac{1}{w(2Q)} \int_{2Q} |b(x) - b_{2Q}|^{s'} w(x)^{1-s'}\, dx \right)^{1/s'}$$
$$\times \left(\frac{1}{w(2Q)} \int_{2Q} |T^{k,2}(f)(x)|^{s} w(x)\, dx \right)^{1/s}$$
$$\leq C\|b\|_{\mathrm{BMO}(w)} w(\tilde{x}) \sum_{k=1}^{m} M_{s,w}(T^{k,2}(f))(\tilde{x}),$$

and

$$I_5 \leq C \sum_{k=1}^{m} \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(y)|^r\, dy \right)^{1/r}$$
$$+ C \sum_{k=1}^{m} \sum_{j=1}^{\infty} 2^{jn} \rho(2^{2(j-1)}) \left(\frac{1}{|2^{j+1}Q|} \int_{\mathbb{R}^n} |T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(y)|^r\, dy \right)^{1/r}$$
$$\leq C \sum_{k=1}^{m} |Q|^{-1/r}$$
$$\times \left(\int_{2Q} |b(y) - b_{2Q}|^r w(y)^{1/p-r} |T^{k,2}(f)(y)|^r w(y)^{r/s} w(y)^{r-r/s-1/p}\, dy \right)^{1/r}$$
$$+ C \sum_{k=1}^{m} \sum_{j=1}^{\infty} 2^{jn} \rho(2^{2(j-1)}) (2^j l(Q))^{-n/r}$$
$$\times \left(\int_{2Q} |b(y) - b_{2Q}|^r w(y)^{1/p-r} |T^{k,2}(f)(y)|^r w(y)^{r/s} w(y)^{r-r/s-1/p}\, dy \right)^{1/r}$$
$$\leq C \sum_{k=1}^{m} |Q|^{-1/r} \left(\int_{2Q} |b(y) - b_{2Q}|^{pr} w(y)^{1-pr}\, dy \right)^{1/pr}$$
$$\times \left(\int_{2Q} |T^{k,2}(f)(y)|^s w(y)\, dy \right)^{1/s} \left(\int_{2Q} w(y)^{(r-r/s-1/p)q}\, dy \right)^{1/qr}$$
$$+ C \sum_{k=1}^{m} \sum_{j=1}^{\infty} 2^{jn} \rho(2^{2(j-1)}) (2^j l(Q))^{-n/r} \left(\int_{2Q} |b(y) - b_{2Q}|^{pr} w(y)^{1-pr}\, dy \right)^{1/pr}$$
$$\times \left(\int_{2Q} |T^{k,2}(f)(y)|^s w(y)\, dy \right)^{1/s} \left(\int_{2Q} w(y)^{(r-r/s-1/p)q}\, dy \right)^{1/qr}$$
$$\leq C \sum_{k=1}^{m} |Q|^{-1/r} \|b\|_{\mathrm{BMO}(w)} w(2Q)^{1/pr} w(2Q)^{1/s} M_{s,w}(T^{k,2}(f))(\tilde{x})$$

$$
\begin{aligned}
&\times |2Q|^{1/qr}\left(\frac{1}{|2Q|}\int\limits_{2Q} w(y)^{p_0}\,dy\right)^{1/qr}\\
&+C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})(2^{j}l(Q))^{-n/r}\|b\|_{\mathrm{BMO}(w)}w(2Q)^{1/pr}w(2Q)^{1/s}\\
&\times M_{s,w}(T^{k,2}(f))(\tilde{x})|2Q|^{1/qr}\left(\frac{1}{|2Q|}\int\limits_{2Q} w(y)^{p_0}\,dy\right)^{1/qr}\\
\le{}& C\|b\|_{\mathrm{BMO}(w)}\sum_{k=1}^{m}|Q|^{-1/r}w(2Q)^{1/pr}w(2Q)^{1/s}M_{s,w}(T^{k,2}(f))(\tilde{x})\\
&\times |2Q|^{1/qr}\left(\frac{1}{|2Q|}\int\limits_{2Q} w(y)\,dy\right)^{p_0/qr}\\
&+C\|b\|_{\mathrm{BMO}(w)}\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{jn}\rho(2^{2(j-1)})(2^{j}l(Q))^{-n/r}w(2Q)^{1/pr}w(2Q)^{1/s}\\
&\times M_{s,w}(T^{k,2}(f))(\tilde{x})|2Q|^{1/qr}\left(\frac{1}{|2Q|}\int\limits_{2Q} w(y)\,dy\right)^{p_0/qr}\\
\le{}& C\sum_{k=1}^{m}\|b\|_{\mathrm{BMO}(w)}\frac{w(2Q)}{|2Q|}M_{s,w}(T^{k,2}(f))(\tilde{x})\\
&+C\sum_{k=1}^{m}\|b\|_{\mathrm{BMO}(w)}\frac{w(2Q)}{|2Q|}M_{s,w}(T^{k,2}(f))(\tilde{x})\\
&\times\sum_{j=1}^{\infty}2^{(j-1)(n+\epsilon)}\rho(2^{2(j-1)})2^{-j(\epsilon+n/r)}\\
\le{}& C\|b\|_{\mathrm{BMO}(w)}w(\tilde{x})\sum_{k=1}^{m}M_{s,w}(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

and

$$
\begin{aligned}
I_6 \le{}& C\sum_{k=1}^{m}\frac{1}{|Q|}\int\limits_{Q}\sum_{j=1}^{\infty}\int\limits_{2^{k}d\le|y-x_0|<2^{k+1}d}|b(y)-b_{2Q}|\frac{|x-x_0|^{\delta}}{|x_0-y|^{n+\delta}}|T^{k,2}(f)(y)|\,dy\,dx\\
\le{}& C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}\int\limits_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}+b_{2^{j+1}Q}-b_{2Q}|\\
&\times w(y)^{-1/s}|T^{k,2}(f)(y)|w(y)^{1/s}\,dy
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}\Big(\int\limits_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}|^{s'}w(y)^{1-s'}\,dy\Big)^{1/s'}\\
&\quad\times\Big(\int\limits_{2^{j+1}Q}|T^{j,2}(f)(y)|^{s}w(y)\,dy\Big)^{1/s}\\
&\quad+C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}|b_{2^{j+1}Q}-b_{2Q}|\Big(\int\limits_{2^{j+1}Q}w(y)^{-1/(s-1)}\,dy\Big)^{1/s'}\\
&\quad\times\Big(\int\limits_{2^{j+1}Q}|T^{k,2}(f)(y)|^{s}w(y)\,dy\Big)^{1/s}\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}\|b\|_{\mathrm{BMO}(w)}w(2^{j+1}Q)M_{s,w}(T^{k,2}(f))(\tilde{x})\\
&\quad+\sum_{k=1}^{m}\sum_{j=1}^{\infty}\frac{d^{\delta}}{(2^{j+1}d)^{n+\delta}}\|b\|_{\mathrm{BMO}(w)}w(\tilde{x})jw(2^{j+1}Q)^{1/s}M_{s,w}(T^{k,2}(f))(\tilde{x})\\
&\quad\times\frac{|2^{j+1}Q|}{w(2^{j+1}Q)^{1/s}}\bigg(\frac{1}{|2^{j+1}Q|}\int\limits_{2^{j+1}Q}w(y)\,dy\bigg)^{1/s}\\
&\quad\times\bigg(\frac{1}{|2^{j+1}Q|}\int\limits_{2^{j+1}Q}w(y)^{-1/(s-1)}\,dy\bigg)^{(s-1)/s}\\
&\le C\|b\|_{\mathrm{BMO}(w)}w(\tilde{x})\sum_{k=1}^{m}M_{s,w}(T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}j2^{-\delta j}\\
&\le C\|b\|_{\mathrm{BMO}(w)}w(\tilde{x})\sum_{k=1}^{m}M_{s,w}(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 4.2. ■

COROLLARY 4.3. *Let T be a singular integral operator with non-smooth kernel as in Definition 2.2, $w\in A_1$, $0<\beta<1$ and $b\in \mathrm{Lip}_\beta(w)$, $1<p<n/\beta$, $1/q=1/p-\beta/n$ and $0<D<2^n$. If $T_1=0$ on $L^r(\mathbb{R}^n)$ $(1<r<\infty)$ and $T^{k,2}$ are bounded linear operators on $L^{p,\varphi}(\mathbb{R}^n,w)$ for $1<p<\infty$ and $w\in A_1$ $(1\le k\le m)$, then T_b is bounded from $L^{p,\varphi}(\mathbb{R}^n,w)$ to $L^{q,\varphi}(\mathbb{R}^n,w^{1-q})$.*

Proof. Choose $1<s<p$ in Theorem 4.1 and notice $w^{1-q}\in A_1$. Then, by Lemma 3.5,

$$
\begin{aligned}
\|T_b(f)\|_{L^{q,\varphi}(w^{1-q})}&\le\|M_\eta(T_b(f))\|_{L^{q,\varphi}(w^{1-q})}\le C\|M^{\#}_{A,\eta}(T_b(f))\|_{L^{q,\varphi}(w^{1-q})}\\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)}\sum_{k=1}^{m}\|wM_{\beta,s,w}(T^{k,2}(f))\|_{L^{q,\varphi}(w^{1-q})}
\end{aligned}
$$

$$= C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{k=1}^{m} \|M_{\beta,s,w}(T^{k,2}(f))\|_{L^{q,\varphi}(w)}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^{p,\varphi}(w)} \le C\|b\|_{\mathrm{Lip}_\beta(w)} \|f\|_{L^{p,\varphi}(w)}. \ \blacksquare$$

COROLLARY 4.4. *Let T be a singular integral operator with non-smooth kernel as in Definition 2.2, $w \in A_1$, $0 < D < 2^n$, $1 < p < \infty$ and $b \in \mathrm{BMO}(w)$. If $T_1 = 0$ on $L^r(\mathbb{R}^n)$ $(1 < r < \infty)$ and $T^{k,2}$ are bounded linear operators on $L^{p,\varphi}(\mathbb{R}^n, w)$ for $1 < p < \infty$ and $w \in A_1$ $(1 \le k \le m)$, then T_b is bounded from $L^{p,\varphi}(\mathbb{R}^n, w)$ to $L^{p,\varphi}(\mathbb{R}^n, w^{1-p})$.*

Proof. Choose $1 < s < p$ in Theorem 4.2 and notice $w^{1-p} \in A_1$. Then, by Lemma 3.5,

$$\begin{aligned}\|T_b(f)\|_{L^{p,\varphi}(w^{1-p})} &\le \|M_\eta(T_b(f))\|_{L^{p,\varphi}(w^{1-p})} \le C\|M^{\#}_{A,\eta}(T_b(f))\|_{L^{p,\varphi}(w^{1-p})} \\ &\le C\|b\|_{\mathrm{BMO}(w)} \sum_{k=1}^{m} \|wM_{s,w}(T^{k,2}(f))\|_{L^{p,\varphi}(w^{1-p})} \\ &= C\|b\|_{\mathrm{BMO}(w)} \sum_{k=1}^{m} \|M_{s,w}(T^{k,2}(f))\|_{L^{p,\varphi}(w)} \\ &\le C\|b\|_{\mathrm{BMO}(w)} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^{p,\varphi}(w)} \\ &\le C\|b\|_{\mathrm{BMO}(w)} \|f\|_{L^{p,\varphi}(w)}. \ \blacksquare\end{aligned}$$

COROLLARY 4.5. *Let $[b,T](f) = bT(f) - T(bf)$ be the commutator generated by the singular integral operator T with non-smooth kernel as in Definition 2.2 and b. Then the conclusions of Theorems 4.1–4.2 and Corollaries 4.3–4.4 hold for $[b,T]$ in place of T_b.*

5. Applications. In this section we shall apply Theorems 4.1–4.2 and Corollaries 4.3–4.4 to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [DUM], [MA]). Given $0 \le \theta < \pi$, define

$$S_\theta = \{z \in \mathbb{C} : |\arg(z)| \le \theta\} \cup \{0\}$$

and denote by S_θ^0 its interior. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed operator L on some Banach space E is said to be *of type θ* if its spectrum $\sigma(L)$ is contained in S_θ and for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta| \, \|(\eta I - L)^{-1}\| \le C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, let

$$H_\infty(S_\mu^0) = \{f : S_\theta^0 \to \mathbb{C} : f \text{ is holomorphic and } \|f\|_{L^\infty} < \infty\},$$

where $\|f\|_{L^\infty} = \sup\{|f(z)| : z \in S_\mu^0\}$. Set

$$\Psi(S_\mu^0) = \left\{ g \in H_\infty(S_\mu^0) : \exists s, c > 0 \text{ such that } |g(z)| \leq c\frac{|z|^s}{1+|z|^{2s}} \right\}.$$

If L is of type θ and $g \in H_\infty(S_\mu^0)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1} \int_\Gamma (\eta I - L)^{-1} g(\eta)\, d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \geq 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-one and has dense range, then, for $f \in H_\infty(S_\mu^0)$,

$$f(L) = [h(L)]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. The operator L is said to have a *bounded holomorphic functional calculus* on the sector S_μ if

$$\|g(L)\| \leq N\|g\|_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S_\mu^0)$.

Now, let L be a linear operator on $L^2(\mathbb{R}^n)$ with $\theta < \pi/2$ so that $-L$ generates a holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [MA] and Theorems 4.1–4.2 and Corollaries 4.3–4.4, we get

Corollary 5.1. *Assume the following conditions are satisfied:*

(i) *The holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$, is represented by the kernels $a_z(x,y)$ which satisfy, for all $\nu > \theta$, an upper bound*

$$|a_z(x,y)| \leq c_\nu h_{|z|}(x,y)$$

for $x, y \in \mathbb{R}^n$ and $0 \leq |\arg(z)| < \pi/2 - \theta$, where

$$h_t(x,y) = Ct^{-n/2}s(|x-y|^2/t)$$

and s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0.$$

(ii) *The operator L has a bounded holomorphic functional calculus in $L^2(\mathbb{R}^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S_\mu^0)$, the operator $g(L)$ satisfies*

$$\|g(L)(f)\|_{L^2} \leq c_\nu \|g\|_{L^\infty}\|f\|_{L^2}.$$

Let $g(L)_b$ be the Toeplitz type operator associated to $g(L)$. Then the conclusion of Theorems 4.1–4.2 *and Corollaries* 4.3–4.4 *hold for $g(L)_b$ in place of T_b.*

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