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PROPERTIES OF EXTREMAL SEQUENCES FOR THE BELLMAN FUNCTION OF THE DYADIC MAXIMAL OPERATOR

BY

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Abstract. We prove a necessary condition that has every extremal sequence for the Bellman function of the dyadic maximal operator. This implies the weak-L^p uniqueness for such a sequence.

1. Introduction. The dyadic maximal operator on $\mathbb{R}^k$ is defined by

$$\mathcal{M}_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\,du : x \in Q,\ Q \subseteq \mathbb{R}^k \text{ is a dyadic cube}\right\} \tag{1.1}$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^k)$, where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^k$ and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^k$, $N = 0, 1, 2, \ldots$.

It is well known that $\mathcal{M}_d$ satisfies the following weak type $(1,1)$ inequality:

$$|\{x \in \mathbb{R}^k : \mathcal{M}_d\phi(x) \geq \lambda\}| \leq \frac{1}{\lambda}\int_{\{\mathcal{M}_d\phi \geq \lambda\}} |\phi(u)|\,du \tag{1.2}$$

for every $\phi \in L^1(\mathbb{R}^k)$ and $\lambda > 0$.

From (1.2) it is easy to deduce the L^p-inequality

$$\|\mathcal{M}_d\phi\|_p \leq \frac{p}{p-1}\|\phi\|_p. \tag{1.3}$$

It is easy to see that (1.2) is best possible, and (1.3) is also best possible as can be seen in [W]. (See also [B1] and [B2] for general martingales.)

A way of studying the dyadic maximal operator is to find certain refinements of the above inequalities. Concerning (1.2), refinements have been studied in [MN2], [N1] and [N2], while for (1.3) the Bellman function of two variables for $p > 1$ has been introduced in the following way:

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$$(1.4)\qquad T_p(f,F)=\sup\Big\{\frac{1}{|Q|}\int_Q(\mathcal{M}_d\phi)^p:\phi\geq 0,\ \frac{1}{|Q|}\int_Q\phi(u)\,du=f,\ \frac{1}{|Q|}\int_Q\phi^p(u)\,du=F\Big\},$$

where Q is a fixed dyadic cube on $\mathbb{R}^k$ and $0<f^p\leq F$.

The function given in (1.4) has been explicitly computed. Actually, this is done in a much more general setting of a non-atomic probability measure space (X,μ) where the dyadic sets are now given in a family $\mathcal{T}$ of sets, called a tree, which satisfies conditions similar to those that are satisfied by the dyadic cubes on $[0,1]^k$.

The associated dyadic maximal operator $\mathcal{M}_\mathcal{T}$ is defined by

$$(1.5)\qquad \mathcal{M}_\mathcal{T}\phi(x)=\sup\Big\{\frac{1}{\mu(I)}\int_I|\phi|\,d\mu:x\in I\in\mathcal{T}\Big\},$$

where $\phi\in L^1(X,\mu)$.

The *Bellman function* (for a given $p>1$) of two variables associated to $\mathcal{M}_\mathcal{T}$ is given by

$$(1.6)\qquad S_p(f,F)=\sup\Big\{\int_X(\mathcal{M}_\mathcal{T}\phi)^p\,d\mu:\phi\geq 0,\ \int_X\phi\,d\mu=f,\ \int_X\phi^p\,d\mu=F\Big\},$$

where $0<f^p\leq F$.

In [M], (1.6) has been found to be $S_p(f,F)=F\omega_p(f^p/F)^p$ where $\omega_p:[0,1]\to[1,p/(p-1)]$ is the inverse function H_p^{-1} of H_p defined on $[1,p/(p-1)]$ by $H_p(z)=-(p-1)z^p+pz^{p-1}$.

As a result the Bellman function is independent of the measure space (X,μ) and the underlying tree $\mathcal{T}$. Other approaches to the computation of (1.4) can be seen in [NM] and [SSV].

In this paper we study those sequences of functions, $(\phi_n)_n$, that are extremal for the Bellman function (1.6). That is, $\phi_n:(X,\mu)\to\mathbb{R}^+$, $n=1,2,\dots$, satisfy $\int_X\phi_n\,d\mu=f$, $\int_X\phi_n^p\,d\mu=F$ and

$$(1.7)\qquad \lim_n\int_X(\mathcal{M}_\mathcal{T}\phi_n)^p\,d\mu=F\omega_p(f^p/F)^p.$$

In Section 3 we prove the following

THEOREM 1.1. *Let $\phi_n:(X,\mu)\to\mathbb{R}^+$ be as above. Then for every $I\in\mathcal{T}$,*

$$(1.8)\qquad \lim_n\frac{1}{\mu(I)}\int_I\phi_n\,d\mu=f\quad\text{and}\quad\lim_n\frac{1}{\mu(I)}\int_I\phi_n^p\,d\mu=F.$$

Additionally,

$$\lim_n\frac{1}{\mu(I)}\int_I(\mathcal{M}_\mathcal{T}\phi_n)^p\,d\mu=F\omega_p(f^p/F)^p,$$

for every $I\in\mathcal{T}$. ■

An immediate consequence is that extremal functions do not exist for the Bellman function. Another corollary is the weak-L^p uniqueness of such a sequence in all interesting cases. In other words if $(\phi_n)_n$, $(\psi_n)_n$ are extremal sequences for (1.4), then $\lim_n \int_Q (\phi_n - \psi_n) h \, d\mu = 0$, for every $h \in L^p(Q)$, where $1/p + 1/q = 1$. We also need to mention that related results in connection with Kolmogorov's inequality have been treated in [MN1], while in [N3] a characterization of such extremal sequences is given. More precisely it is proved there that they actually behave approximately like eigenfunctions of the dyadic maximal operator for a specific eigenvalue.

2. Extremal sequences. Let (X, μ) be a non-atomic probability measure space.

DEFINITION 2.1. A set $\mathcal{T}$ of measurable subsets of X will be called a *tree* if the following are satisfied:

(i) $X \in \mathcal{T}$ and $\mu(I) > 0$ for every $I \in \mathcal{T}$.

(ii) To every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I)$ of $\mathcal{T}$ containing at least two elements such that

(a) the elements of $C(I)$ are disjoint subsets of I,

(b) $I = \bigcup C(I)$.

(iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$, where $\mathcal{T}_{(0)} = \{X\}$ and

$$\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} C(I).$$

(iv) $\lim_{m \to \infty} \sup_{I \in \mathcal{T}_{(m)}} \mu(I) = 0$. ∎

DEFINITION 2.2. Given a tree $\mathcal{T}$ we define the associated *maximal operator* by

$$\mathcal{M}_{\mathcal{T}} \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| \, d\mu : x \in I \in \mathcal{T} \right\}$$

for every $\phi \in L^1(X, \mu)$. ∎

From [M] we obtain

THEOREM 2.3.

$$\sup \left\{ \int_X (\mathcal{M}_{\mathcal{T}} \phi)^p \, d\mu : \phi \geq 0, \int \phi \, d\mu = f, \int_X \phi^p \, d\mu = F \right\} = F \omega_p (f^p/F)^p$$

for $0 < f^p \leq F$. ∎

Finally, we give

DEFINITION 2.4. Let $(\phi_n)_n$ be a sequence of non-negative measurable functions defined on X and $0 < f^p \leq F$, $p > 1$. Then $(\phi_n)_n$ is called

(p, f, F) *extremal*, or simply *extremal*, if

$$\int_X \phi_n \, d\mu = f, \qquad \int_X \phi_n^p \, d\mu = F \quad \text{for every } n = 1, 2, \ldots,$$
$$\lim_n \int_X (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu = F \omega_p (f^p/F)^p.$$

3. Main theorem

THEOREM 3.1. *Let $(\phi_n)_n$ be an extremal sequence. Then for every $I \in \mathcal{T}$:*

(i) $\displaystyle \lim_n \frac{1}{\mu(I)} \int_I \phi_n \, d\mu = f,$

(ii) $\displaystyle \lim_n \frac{1}{\mu(I)} \int_I \phi_n^p \, d\mu = F,$

(iii) $\displaystyle \lim \frac{1}{\mu(I)} \int_I (\mathcal{M}_{\mathcal{T}} \phi_n)^p \, d\mu = F \omega_p (f^p/F)^p.$

Proof. We recall that $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$. We prove the assertion for $I \in \mathcal{T}_{(1)}$. Then inductively it holds for every $I \in \mathcal{T}_{(m)}$, $m \geq 1$.

Suppose then that $\mathcal{T}_{(1)} = \{I_k : k = 1, 2, \ldots\}$ and $I = I_1$. We now set

$$f_{n,1} = \frac{1}{\mu(I_1)} \int_{I_1} \phi_n \, d\mu, \qquad f_{n,2} = \frac{1}{\mu(X \setminus I_1)} \int_{X \setminus I_1} \phi_n \, d\mu,$$

$$F_{n,1} = \frac{1}{\mu(I_1)} \int_{I_1} \phi_n^p \, d\mu, \quad F_{n,2} = \frac{1}{\mu(X \setminus I_1)} \int_{X \setminus I_1} \phi_n^p \, d\mu \quad \text{for } n = 1, 2, \ldots. \tag{3.1}$$

The above sequences are obviously bounded, so passing to a subsequence we may suppose that

$$\lim_n f_{n,i} = f_i \quad \text{and} \quad \lim_n F_{n,i} = F_i \quad \text{for } i = 1, 2.$$

For any $J \in \mathcal{T}$ define

$$\mathcal{M}_J \phi(t) = \sup \left\{ \frac{1}{\mu(K)} \int_K |\phi| \, d\mu : t \in K \in \mathcal{T}_J \right\} \quad \text{for } t \in J,$$

where $\mathcal{T}_J$ is defined by

$$\mathcal{T}_J = \{K \in \mathcal{T} : K \subseteq J\}.$$

Consider the measure space $(J, \mu(\cdot)/\mu(J))$, the tree $\mathcal{T}_J$ and the associated maximal operator $\mathcal{M}_J$. Then using Theorem 2.3, we have

$$\frac{1}{\mu(J)} \int_J (\mathcal{M}_J \phi)^p \, d\mu \leq \frac{1}{\mu(J)} \int_J \phi^p \, d\mu \cdot \omega_p \left(\frac{\left(\frac{1}{\mu(J)} \int_J \phi \, d\mu\right)^p}{\frac{1}{\mu(J)} \int_J \phi^p \, d\mu} \right)^p \tag{3.2}$$

for every $\phi \in L^p(J)$, where $\omega_p : [0,1] \to [1, p/(p-1)]$ is H_p^{-1}, with

$$H_p(z) = -(p-1)z^p + pz^{p-1}, \qquad z \in [1, p/(p-1)].$$

Since H_p is decreasing we conclude from (3.2) that

$$H_p\Big(\Big[\frac{\int_J (\mathcal{M}_J \phi)^p \, d\mu}{\int_J \phi^p \, d\mu}\Big]^{1/p}\Big) \geq \frac{1}{\mu(J)^{p-1}} \frac{(\int_J \phi \, d\mu)^p}{\int_J \phi^p \, d\mu},$$

which gives

$$(3.3) \qquad -(p-1)\int_J (\mathcal{M}_J \phi)^p \, d\mu + p\Big(\int_J \phi^p \, d\mu\Big)^{1/p} \cdot \Big(\int_J (\mathcal{M}_J \phi)^p \, d\mu\Big)^{1-1/p} = \frac{1}{\mu(J)^{p-1}}\Big(\int_J \phi \, d\mu\Big)^p + \delta_{\phi,J}$$

for some constant $\delta_{\phi,J} \geq 0$ depending on ϕ and J.

For $\phi = \phi_n$ and $J = I_i$, $i = 1, 2, \ldots$, from (3.3) we obtain

$$(3.4) \qquad -(p-1)\int_{I_i} (\mathcal{M}_{I_i} \phi_n)^p \, d\mu + p\Big(\int_{I_i} \phi_n^p \, d\mu\Big)^{1/p} \cdot \Big(\int_{I_i} (\mathcal{M}_{I_i} \phi_n)^p \, d\mu\Big)^{1-1/p} = \frac{1}{\mu(I_i)^{p-1}}\Big(\int_{I_i} \phi_n \, d\mu\Big)^p + \delta_{n,i} \quad \text{for } n, i = 1, 2, \ldots.$$

Summing (3.4) over $i \geq 2$ we obtain

$$(3.5) \qquad -(p-1)\sum_{i=2}^{\infty}\int_{I_i} (\mathcal{M}_{I_i} \phi_n)^p \, d\mu + p\sum_{i=2}^{\infty}\Big(\int_{I_i} \phi_n^p \, d\mu\Big)^{1/p}\Big(\int_{I_i} (\mathcal{M}_{I_i} \phi_n)^p \, d\mu\Big)^{1-1/p} = \sum_{i=2}^{\infty} \frac{1}{\mu(I_i)^{p-1}}\Big(\int_{I_i} \phi_n \, d\mu\Big)^p + \sum_{i=2}^{\infty} \delta_{n,i}.$$

In view now of Hölder's inequality

$$\sum_i a_i b_i \leq \Big(\sum_i a_i^p\Big)^{1/p}\Big(\sum_i b_i^q\Big)^{1/q},$$

for $a_i, b_i \geq 0$ and $q = p/(p-1)$, (3.5) gives

$$(3.6) \qquad -(p-1)A_2(n) + p\Big(\int_{X \setminus I_1} \phi_n^p \, d\mu\Big)^{1/p} \cdot [A_2(n)]^{1-1/p} \geq \sum_{i=2}^{\infty} \frac{1}{\mu(I_i)^{p-1}}\Big(\int_{I_i} \phi_n \, d\mu\Big)^p + \sum_{i=2}^{\infty} \delta_{n,i},$$

where

$$A_2(n) = \sum_{i=2}^{\infty} \int_{I_i} (\mathcal{M}_{I_i}\phi_n)^p \, d\mu. \tag{3.7}$$

(In the last inequality we used the fact that $X \setminus I_1 = \bigcup_{i=2}^{\infty} I_i$.)

We now use now Hölder's inequality in the following form:

$$\frac{(\lambda_1 + \cdots + \lambda_m)^p}{(\sigma_1 + \cdots + \sigma_m)^{p-1}} \le \frac{\lambda_1^p}{\sigma_1^{p-1}} + \cdots + \frac{\lambda_m^p}{\sigma_m^{p-1}},$$

where $\sigma_i, \lambda_i \ge 0$ for all $i = 1, 2, \ldots$, and obtain

$$\begin{aligned} \sum_{i=2}^{\infty} \frac{1}{\mu(I_i)^{p-1}} \Big(\int_{I_i} \phi_n \, d\mu \Big)^p &\ge \frac{1}{\mu(X \setminus I_1)^{p-1}} \Big(\int_{X \setminus I_1} \phi_n \, d\mu \Big)^p \\ &= \mu(X \setminus I_1) f_{n,2}. \end{aligned} \tag{3.8}$$

We also set

$$A_3(n) = \int_{X \setminus I_1} (\mathcal{M}_{\mathcal{T}}\phi_n)^p \, d\mu \quad \text{for } n = 1, 2, \ldots. \tag{3.9}$$

Then by definition of $\mathcal{M}_{I_i}$ we have

$$A_3(n) \ge A_2(n). \tag{3.10}$$

From the above we then deduce that

$$\begin{aligned} -(p-1)A_2(n) + p\mu(X \setminus I_1)^{1/p}(F_{n,2})^{1/p}[A_3(n)]^{1-1/p} \\ = \mu(X \setminus I_1)(f_{n,2})^p + \delta_n^{(1)}, \end{aligned} \tag{3.11}$$

where $\delta_n^{(1)} \ge \sum_{i=2}^{\infty} \delta_{n,i}$.

By passing to a subsequence we may suppose that $\lim_n A_3(n) = A_3$. ∎

We will now use the following lemma, the proof of which will be given at the end of this section.

Lemma 3.2. *If $(\phi_n)_n$ is extremal then*

$$\lim_n \mu(\{\mathcal{M}_{\mathcal{T}}\phi_n = f\}) = 0.$$

From this lemma and the definitions (3.7) and (3.9) we easily see that $\lim_n A_2(n) = \lim_n A_3(n) = A_3$, in view of the fact that $I_i \in \mathcal{T}_{(1)}$ for $i = 2, 3, \ldots$. Then from (3.11) we conclude that

(3.12)

$$\begin{aligned} -(p-1) \int_{X \setminus I_1} (\mathcal{M}_{\mathcal{T}}\phi_n)^p \, d\mu + p\mu(X \setminus I_1)^{1/p}(F_{n,2})^{1/p} \Big(\int_{X \setminus I_1} (\mathcal{M}_{\mathcal{T}}\phi_n)^p \, d\mu \Big)^{1-1/p} \\ = \mu(X \setminus I_i)(f_{n,2})^p + \delta_n'', \end{aligned}$$

where $\delta_n'' \ge \delta_n'$ for every $n \in \mathbb{N}$.

In the same way we obtain

$$-(p-1)\int_{I_1}(\mathcal{M}_{\mathcal{T}}\phi_n)^p\,d\mu + p\mu(I_1)^{1/p}(F_{n,1})^{1/p}\Big(\int_{I_1}(\mathcal{M}_{\mathcal{T}}\phi_n)^p\,d\mu\Big)^{1-1/p} = \mu(I_1)(f_{n,1})^p + \varepsilon''_n, \tag{3.13}$$

where ε''_n is such that $\varepsilon''_n \ge \delta_{n,1}$ for every $n \in \mathbb{N}$.

Summing now (3.12) and (3.13) and using Hölder's inequality in both previously mentioned forms we obtain

$$-(p-1)\int_X(\mathcal{M}_{\mathcal{T}}\phi_n)^p\,d\mu + pF^{1/p}\Big(\int_X(\mathcal{M}_{\mathcal{T}}\phi_n)^p\,d\mu\Big)^{1-1/p} \ge \mu(I_1)(f_{n,1})^p + \mu(X\setminus I_1)(f_{n,2})^p + \delta''_n + \varepsilon''_n \ge f^p + \delta''_n + \varepsilon''_n, \tag{3.14}$$

which gives

$$-(p-1)\int_X(\mathcal{M}_{\mathcal{T}}\phi_1)^p\,d\mu + pF^{1/p}\Big(\int_X(\mathcal{M}_{\mathcal{T}}\phi_n)^p\,d\mu\Big)^{1-1/p} = f^p + \vartheta_n, \tag{3.15}$$

where $\vartheta_n \ge \delta''_n + \varepsilon''_n$, $n = 1, 2, \ldots$.

The hypothesis on (ϕ_n) is now that

$$\lim_n \int_X(\mathcal{M}_{\mathcal{T}}\phi_n)^p\,d\mu = F\omega_p(f^p/F)^p.$$

This gives $\vartheta_n \to 0$ in (3.15) and so

$$\delta''_n \to 0, \qquad \varepsilon''_n \to 0.$$

As a consequence,

$$\mu(I_1)(f_1)^p + \mu(X\setminus I_1)(f_2)^p = f^p$$

because of equality in (3.14) as $n\to\infty$.

Since now $\mu(I_1)f_1 + \mu(X\setminus I_1)f_2 = f$ and $t \mapsto t^p$ is strictly convex on $(0,\infty)$ we have $f_1 = f_2 = f$.

Additionally $\delta''_n \to 0$, so because of (3.12) and the fact that $f_2 = f$ we immediately see that

$$\lim_n \frac{1}{\mu(X\setminus I_1)}\int_{X\setminus I_1}(\mathcal{M}_{\mathcal{T}}\phi_n)^p\,d\mu = F_2\omega_p(f^p/F_2)^p. \tag{3.16}$$

Similarly

$$\lim_n \frac{1}{\mu(I_1)}\int_{I_1}(\mathcal{M}_{\mathcal{T}}\phi_n)^p\,d\mu = F_1\omega_p(f^p/F_1)^p. \tag{3.17}$$

Since $(\phi_n)_n$ is extremal, the last two equations give

$$\mu(I_1)\cdot F_1\omega_p(f^p/F_1)^p + \mu(X\setminus I_1)\cdot F_2\omega_p(f^p/F_2)^p = F\omega(f^p/F). \tag{3.18}$$

But as we shall prove in Lemma 3.3 below, the function $t \mapsto t\omega_p(f^p/t)^p$, $t \in (f^p, \infty)$ is strictly concave. So since $\mu(I_1)F_1 + \mu(X \setminus I_1)F_2 = F$ we see from (3.18) that $F_1 = F_2 = F$. Then since (3.17) holds we conclude that

$$\lim_n \frac{1}{\mu(I)} \int_I (\mathcal{M}_{\mathcal{T}}\phi_n)^p \, d\mu = F\omega_p(f^p/F)^p,$$

and Theorem 3.1 is proved. ∎

We now prove the following

LEMMA 3.3. *Let $G : (1, \infty) \to \mathbb{R}^+$ be defined by $G(t) = t\omega_p(1/t)^p$. Then G is strictly concave.*

Proof. It is known from [M] that ω_p satisfies

$$\frac{d}{dx}[\omega_p(x)]^p = -\frac{1}{p-1} \frac{\omega_p(x)}{\omega_p(x) - 1}, \quad x \in [0, 1].$$

So we can easily see that

$$G'(t) = \omega_p(1/t)^p + \frac{1}{p-1} \frac{1}{t} \frac{\omega_p(1/t)}{\omega_p(1/t) - 1},$$

$$G''(t) = \frac{1}{p-1} \cdot \frac{1}{t} \left(\frac{g(t)}{g(t) - 1} \right)',$$

where g is defined on $(1, \infty)$ by $g(t) = \omega_p(1/t)$. Since $g'(t) > 0$ for all $t > 1$, we have $G''(t) < 0$ for all $t > 1$, and Lemma 3.3 is proved. ∎

We continue now with

Proof of Lemma 3.2. Suppose first that all ϕ_n are $\mathcal{T}$-simple functions, that is, for every n, there exists an m_n such that ϕ_n is constant on each $I \in \mathcal{T}_{(m_n)}$. As a consequence, ϕ_n is $\mathcal{T}$-good in the sense of [M], for every n. If we look at the proof of Lemma 9 in [M, pp. 324–326] we see that in inequalities (4.20), (4.22), (4.23), (4.24) there we should have equality in the limit. So as a result we must have

$$\frac{1}{(\beta + 1 - \beta\rho_X^n)^{p-1}} - \frac{(p-1)\beta\rho_X^n}{(\beta+1)^p} \to \frac{1}{(\beta+1)^{p-1}} \quad \text{for } \beta = \omega_p(f^p/F) - 1,$$

where $\rho_X^n = a_X^n/\mu(X) = a_X^n$ with $a_X^n = \mu(\{\mathcal{M}_{\mathcal{T}}\phi_n = f\})$. But this can happen only if $a_X^n \to 0$. So the proof is complete in the case of $\mathcal{T}$-simple functions.

As for the general case, it is not difficult to see that if $(\phi_n)_n$ is an extremal sequence of measurable functions, then we can construct a sequence $(\psi_n)_n$ of $\mathcal{T}$-simple functions such that $\int_X \psi_n \, d\mu = f$, $\int_X \psi_n^p \, d\mu \le F$ and

$$\lim_n \int_X \psi_n^p \, d\mu = F, \qquad \lim_n \int_X (\mathcal{M}_{\mathcal{T}}\psi_n)^p \, d\mu = F\omega_p(f^p/F)^p.$$

Additionally, we can arrange everything in such a way that $\{\mathcal{M}_{\mathcal{T}}\phi_n = f\} \subseteq \{\mathcal{M}_{\mathcal{T}}\psi_n = f\}$.

The same arguments used for $(\psi_n)_n$ give $\lim_n \mu(\{\mathcal{M}_{\mathcal{T}}\psi_n = f\}) = 0$. So $\lim_n \mu(\{\mathcal{M}_{\mathcal{T}}\phi_n = f\}) = 0$ and Lemma 3.2 is proved. ∎

We now give some applications of the above.

First we prove

Corollary 3.4. *If* $0 < f^p < F$ *then there do not exist extremal functions for the Bellman function* $T_p(f, F)$ *described in* (1.4).

Proof. Let ϕ be an extremal function for (1.4). Applying Theorem 3.1 we see that

$$\frac{1}{\mu(I)}\int_I \phi\, d\mu = f \quad \text{and} \quad \frac{1}{\mu(I)}\int_I \phi^p\, d\mu = F,$$

for every dyadic subcube I of Q.

As we can see in [G], inequality (1.2) implies that the base of dyadic sets of the tree $\mathcal{T}$ differentiates $L^1(Q)$. That is,

$$\phi(x) = f \quad \text{a.e.} \quad \text{and} \quad \phi^p(x) = F \quad \text{a.e.}$$

This gives $f^p = F$, which is a contradiction. ∎

Corollary 3.5. *Let* $T_p(f, F)$ *be described by* (1.4). *Then if* $(\phi_n)_n$, $(\psi_n)_n$ *are extremal sequences for this function, we have* $\phi_n - \psi_n \xrightarrow{w(L^p)} 0$ *as* $n \to \infty$.

Proof. Of course

$$\lim_n \frac{1}{|I|}\int_I \phi_n(u)\, du = \lim_n \frac{1}{|I|}\int_I \psi_n(u)\, du = f.$$

So $\lim_n \int_Q (\phi_n - \psi_n)\xi_I(u)\, du = 0$, for every dyadic subcube $I \subseteq Q$.

Since linear combinations of the characteristic functions of the dyadic subcubes of Q are dense in $L^q(Q)$ we should have $\lim_n \int_Q (\phi_n - \psi_n)h = 0$ for every $h \in L^q(Q)$, where $q = p/(p-1)$, that is, $\phi_n - \psi_n \xrightarrow{w(L^p)} 0$ as $n \to \infty$. ∎

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