C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 95 2003 NO. 1

WORD DISTANCE ON THE DISCRETE HEISENBERG GROUP

BY

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Abstract. We establish an exact formula for the word distance on the discrete Heisenberg group $\mathbb{H}_3$ with its standard set of generators. This formula is then used to prove the almost connectedness of the spheres for this distance.

1. Introduction. For any finitely generated group Γ with a finite symmetric generating set S, we define a *Cayley graph* as follows. The vertices are the elements of Γ and there is an edge between two vertices x and y if $x^{-1}y$ belongs to S. The *word distance* $d(x,y)$ between two elements x,y of Γ is the minimal number of generators we need to go from e (identity of Γ) to $x^{-1}y$. Thus, it is the natural distance on the Cayley graph.

If the volume of the balls $B(e,n)=\{x\in\Gamma : d(e,x)<n\}$ grows like a polynomial in n, then Γ is said to be of *polynomial growth.* In this class of groups, $\mathbb{Z}^d$ is the simplest example. Another typical example, outside the Abelian setting, is the Heisenberg group $\mathbb{H}_3$: the group of upper triangular integer-valued (3×3) matrices with 1's on the diagonal,

$$M=\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix},\qquad (x,y,z)\in\mathbb{Z}^3.$$

The group $\mathbb{H}_3$ is $\mathbb{Z}^3$ with the following product:

$$(x,y,z)(x',y',z')=(x+x',y+y',z+z'+xy').$$

On $\mathbb{Z}^d$, the word distance is easily computable for the standard generating set, but not in general. Here, our aim is to give an exact formula (Theorem 2.2) for the word distance on $\mathbb{H}_3$ with its standard generators $\{A^{\pm 1},B^{\pm 1}\}$, with

$$A=\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\qquad B=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}.$$

2000 *Mathematics Subject Classification*: 05C25, 20F65.

Key words and phrases: Heisenberg group, word distance.

The method uses the formalism of the theory of words adapted to the behavior of the coordinates along a path. Typically, the third coordinate grows quadratically when the first two grow linearly.

Then we use this formula to obtain the almost connectedness of the spheres $\mathcal{S}(e,n) = \{x \in \mathit{\Gamma} : d(e,x) = n\}$ of $\mathbb{H}_3$ (Proposition 3.1), always with the standard set of generators.

DEFINITION 1.1. We say that the spheres (for the distance d) are *almost connected* with constants (α, β) if there exists a constant N_0 such that for all $n > N_0$ and $x \in \mathit{\Gamma}$, any two points in the sphere $\mathcal{S}(e,n)$ can be joined by a path within the annulus $A(e, n-\alpha, n+\beta) = B(e, n+\beta) \setminus B(e, n-\alpha)$.

By translation invariance, this property does not depend on the choice of the center of the spheres.

On any discrete group with polynomial growth (for any generating set), the spheres are relatively connected, which is the same property with the constants multiplicative (see [2, 1]). Clearly, on $\mathbb{Z}^d$ with its standard generating set, the spheres are almost connected with constants $(0,1)$. However, we do not know whether or not the almost connectedness of the spheres is still true for any finite set of generators, as this notion is not, a priori, stable under rough isometry.

2. Word distance on $\mathbb{H}_3$. We write $d(x,y,z) = d(e,M)$ for the word distance associated to the set $\{A^{\pm 1}, B^{\pm 1}\}$ of generators. We give some classical definitions dealing with the theory of words.

DEFINITION 2.1. The set $\{A, A^{-1}, B, B^{-1}\}$ is our *alphabet*. Each of its elements is called a *letter* from which we compose *words*. A *subword* is a part of a word formed by successive letters. A *reduced word* is a word from which we have removed all the possible subwords of the form XX^{-1} where X is any letter. A *prefix* is a subword starting with the first letter and a *suffix* is a subword ending with the last letter.

We first reduce the domain of (x,y,z), by some symmetry properties of the distance function. By changing the A^{ε}'s into $A^{-\varepsilon}$'s ($\varepsilon = \pm 1$), and then the B^{ε}'s into $B^{-\varepsilon}$'s in a word of length $d(x,y,z)$ which represents (x,y,z), we see that

$$d(x,y,z) = d(-x,y,-z) = d(-x,-y,z). \tag{1}$$

Therefore, we can restrict ourselves to $z \geq 0$ and $x \geq 0$. Now, by changing the A^{ε}'s into B^{ε}'s, the B^{ε}'s into A^{ε}'s, and reversing the order of the letters, we see that $d(x,y,z) = d(y,x,z)$. Together with (1) we also get $d(x,y,z) = d(-y,-x,z)$. Therefore, we can also suppose $x \geq y \geq -x$. We denote by $\lceil k \rceil$ the least integer greater than k.

Theorem 2.2. *Let* $z \geq 0$, $x \geq 0$, $x \geq y \geq -x$ *and* $(x, y, z) \neq (0, 0, 0)$.

I. $y \geq 0$:

I.1. *If* $x \leq \sqrt{z}$, *then*

$$d(x, y, z) = 2\lceil 2\sqrt{z}\rceil - x - y.$$

I.2. *If* $x \geq \sqrt{z}$, *and*

I.2.1. $xy \geq z$, *then*

$$d(x, y, z) = x + y;$$

I.2.2. $xy \leq z$, *then*

$$d(x, y, z) = 2\lceil z/x\rceil + x - y$$

(*except when* $x = 0$, *but then* $z = 0 = y$).

II. $y < 0$:

II.1. *If* $x \leq \sqrt{z - xy}$, *then*

$$d(x, y, z) = 2\lceil 2\sqrt{z - xy}\rceil - x + y.$$

II.2. *If* $x \geq \sqrt{z - xy}$, *then*

$$d(x, y, z) = 2\lceil z/x\rceil + x - y.$$

The rest of this section is devoted to the proof of Theorem 2.2. We start with some remarks.

Remark 2.3. If an element (x, y, z) is represented by a word written with α A's and β B's (α and β positive), then the maximum value we can have for z is $\alpha\beta$, obtained by $A^\alpha B^\beta$.

Remark 2.4. The values of x and y do not depend on the order of the letters in a word. If we have α A's, α' A^{-1}'s, β B's and β' B^{-1}'s, we get $x = \alpha - \alpha'$ and $y = \beta - \beta'$.

Remark 2.5. For all $k \in \mathbb{Z}$, adding (or removing) a prefix that can be reduced to B^k, or a suffix that can be reduced to A^k, does not change the value of z.

In what follows, for any letter X, X^0 will be the empty word. We denote by $|W|$ the length of a word W. An element (x, y, z) of $\mathbb{H}_3$ can be represented by several reduced words, and we denote by $\mathcal{M} = \mathcal{M}(x, y, z)$ the set of reduced words with minimal length $d(x, y, z)$. We denote by $\mathcal{M}' = \mathcal{M}'(x, y, z)$ the set of subwords of all $M \in \mathcal{M}$ obtained by removing the longest prefix of the form B^k and the longest suffix of the form A^l. Note that if $z > 0$, then $\mathcal{M}'$ is not empty. Moreover, the element represented by a word $M' \in \mathcal{M}'$ is (x', y', z) for some integers x', y'. The value of z remains the same by Remark 2.5.

For each $z > 0$, we define a particular word $S(z)$, which represents (x, y, z), with minimal length disregarding the value of x and y. That is to say, $S(z)$ is one of the words with minimal length in $\bigcup_{x,y} \mathcal{M}(x, y, z)$. Let $n = \lceil \sqrt{z} \rceil - 1$. We have three cases.

- $z = (n+1)^2$. Then
$$S(z) = A^{n+1} B^{n+1}.$$
This comes from Remark 2.3 and $\min_{\alpha\beta=k^2}\{\alpha+\beta\} = 2k$.
- $\exists k$, $0 < k \leq n$, such that $z = n^2 + k$. Then
$$S(z) = A^k B A^{n-k} B^n.$$
This comes from $\min_{k^2<\alpha\beta\leq k(k+1)}\{\alpha+\beta\} = 2k+1$.
- $\exists k$, $0 < k \leq n$, such that $z = n^2 + n + k$. Then
$$S(z) = A^k B A^{n+1-k} B^n.$$
This comes from $\min_{k(k+1)<\alpha\beta<(k+1)^2}\{\alpha+\beta\} = 2k+2$.

We check that $|S(z)| = \lceil 2\sqrt{z} \rceil$.

We now exhibit a particular word $T = T(x, y, z)$ that represents (x, y, z) in case II.1. Set $n = \lceil \sqrt{z - xy} \rceil - 1$. Note that $-y \leq x \leq \sqrt{z - xy} \leq n + 1$. We have two cases.

- $\exists k$, $0 < k \leq n$, such that $z - xy = n^2 + k$. Then
$$T = A^{x-n} B^{-(n+1)} A^k B A^{n-k} B^{n+y}.$$
- $\exists k$, $0 \leq k \leq n$, such that $z = (n+1)^2 - k$. Then
$$T = A^{x-n} B^{-k} A^{-1} B^{k-(n+1)} A^{n+1} B^{n+1+y}.$$

We check that $|T| = 2\lceil 2\sqrt{z - xy} \rceil - x + y$.

We define a last word $U = U(x, y, z)$ when $z \geq xy$ and $x > 0$. The Euclidean division of z by x gives $z = ux + v$ with $v < x$. If $u \leq y - 1$, then $z \leq xy - x + v < xy$, which is impossible. So $u \geq y$, and
$$U = B^{y-u-1} A^v B A^{x-v} B^u.$$
We check that $|U| = 2\lceil z/x \rceil + x - y$.

Lemma 2.6. *If $z > 0$, $x \geq 0$ and $x \geq y \geq 0$, then the subset $\mathcal{M}'_+$ composed of those words in $\mathcal{M}'$ that use only A's and B's is not empty.*

Proof. Let $M \in \mathcal{M}$ and let M' be the corresponding word in $\mathcal{M}'$. Our aim is to write, from M', a word M'_+ which belongs to $\mathcal{M}'_+$. So, we first show that we can remove all the possible A^{-1}'s and B^{-1}'s from M' and then create a new word $\overline{M}'$ such that the corresponding $\overline{M}$ represents $(x, y, \bar{z})$ with $\bar{z} \geq z$ and $|\overline{M}| \leq |M|$. Then we will change $\overline{M}'$ into M'_+ by decreasing

$\bar{z}$ down to z with $|M_+| \leq |\overline{M}| \leq |M|$. And so, by the definition of $\mathcal{M}$, we will have $|M_+| = |M|$ and then $M'_+ \in \mathcal{M}'_+$.

At each step of the proof below, we can check that we always get words of length less than or equal to $|M|$. We remark that the following changes in the order of the letters increase the value of z by 1, without changing x and y:

$$BA \to AB;\ A^{-1}B \to BA^{-1};\ B^{-1}A^{-1} \to A^{-1}B^{-1};\ AB^{-1} \to B^{-1}A.$$

We apply these changes in M', from left to right, as long as any of them is possible. The result is a word M'_1 such that the corresponding M_1 represents (x_1, y_1, z_1) with $z_1 \geq z$. The M'_1 can be written as

$$A^{(-1)^{k+1}\alpha_k} B^{(-1)^{k+1}\beta_k} \dots A^{-\alpha_2} B^{-\beta_2} A^{\alpha_1} B^{\beta_1}, \tag{2}$$

with the α_i's and β_i's non-zero and of the same sign. The word M'_1 represents (x'_1, y'_1, z_1) where $x'_1 = \sum_{i=1}^{k}(-1)^{i+1}\alpha_i$ and $y'_1 = \sum_{i=1}^{k}(-1)^{i+1}\beta_i$.

We first suppose $x'_1 \geq 0$. If $\beta_1 < 0$, then removing B^{β_1}, and putting it on the left of M_1, will increase z_1 by $-\beta_1 x'_1$. Then, by definition of $\mathcal{M}'$ we get a word $\widetilde{M_1}$, where A^{α_1} does not belong to the corresponding $\widetilde{M'_1}$. Hence, $\widetilde{M'_1}$ can be written as in (2), ending with $B^{-\beta_2}$ $(-\beta_2 > 0)$. Since $\alpha_1 < 0$, we have $\widetilde{x}'_1 > x'_1 \geq 0$. Therefore, without loss of generality, we can take all the α_i's and β_i's positive in the representation of M'_1.

Our aim is to change M_1 into some word $\widetilde{M}$, which represents $(x, y, \widetilde{z})$ with $\widetilde{z} \geq z_1$. Moreover, we want the corresponding $\widetilde{M}'$ written as

$$A^{-\alpha_2} B^{-\beta_2} A^{\alpha_1} B^{\beta_1},$$

with $\alpha_1, \beta_1, \alpha_2$, and β_2 positive. When $k \geq 3$, we can either reduce k, or suppose $\alpha_2 > \alpha_3$, and $\beta_1 < \beta_2 < \beta_3$. To see this, suppose $\beta_2 \geq \beta_3$. Then the change

$$A^{\alpha_3} B^{\beta_3} A^{-\alpha_2} B^{-\beta_2} A^{\alpha_1} B^{\beta_1} \to B^{\beta_3} A^{-\alpha_2} B^{-\beta_2} A^{\alpha_3} A^{\alpha_1} B^{\beta_1}$$

increases z_1 by $(\beta_2 - \beta_3)\alpha_3 \geq 0$. Moreover, by the same construction used to get M_1, we can change the new word into some $\widetilde{M_1}$ such that the corresponding $\widetilde{M'_1}$ can be written as in (2) for some $\widetilde{k} \leq k - 1$. Therefore, we can take $\beta_2 < \beta_3$.

Now, suppose $\alpha_2 \leq \alpha_3$. Then the change

$$A^{\alpha_3} B^{\beta_3} A^{-\alpha_2} B^{-\beta_2} \to B^{-\beta_2} A^{\alpha_3} B^{\beta_3} A^{-\alpha_2}$$

increases z_1 by $(\alpha_3 - \alpha_2)\beta_2 \geq 0$. Likewise, that allows us to take $\alpha_2 > \alpha_3$. Moreover, by the change

$$A^{-\alpha_2} B^{-\beta_2} A^{\alpha_1} B^{\beta_1} \to B^{-\beta_2} A^{\alpha_1} B^{\beta_1} A^{-\alpha_2}$$

we can also take $\beta_1 < \beta_2$.

Now, we suppose $k \geq 4$. The change

$$A^{-\alpha_4}B^{-\beta_4}A^{\alpha_3}B^{\beta_3}A^{-\alpha_2}B^{-\beta_2} \to A^{\alpha_3}B^{\beta_3}A^{-\alpha_2}B^{-\beta_2}A^{-\alpha_4}B^{-\beta_4}$$

increases z_1 by $(\alpha_2 - \alpha_3)\beta_4 - (\beta_2 - \beta_3)\alpha_4 \geq 0$. By the same construction used to get M_1, we can change the new word into some $\widetilde{M_1}$ such that the corresponding $\widetilde{M_1'}$ can be written as in (2) for some $\widetilde{k} \leq k-1$, with the same properties as M_1'. Therefore, we can take $k \leq 3$ and write M_1' as

$$A^{\alpha_3}B^{\beta_3}A^{-\alpha_2}B^{-\beta_2}A^{\alpha_1}B^{\beta_1}.$$

Then we can change M_1' into

$$A^{\alpha_1}A^{\alpha_3}B^{\beta_3}A^{-\alpha_2}B^{-\beta_2}B^{\beta_1}.$$

We increase z_1 by $(\beta_3 - \beta_2)\alpha_1 \geq 0$. As before, by the same construction used to get M_1, we can change this new word into M_2, which represents (x, y, z_2), such that the corresponding M_2' can be written as in (2) with $k \leq 2$. If $k = 1$, we take $\overline{M} = M_2$. If $k = 2$, then

$$M_2 = B^{y-(\beta_1-\beta_2)}A^{-\alpha_2}B^{-\beta_2}A^{\alpha_1}B^{\beta_1}A^{x-(\alpha_1-\alpha_2)}$$

with $\alpha_1 > \alpha_2$ (since $x_1' \geq 0$ and the preceding changes always increase x') and $\beta_2 > \beta_1$. We define

$$\overline{M} = B^{y-\beta_2}A^{\alpha_1}B^{\beta_2}A^{x-\alpha_1}.$$

We only need to check that $\overline{M}$ represents $(x, y, \overline{z})$ with $\overline{z} \geq z_2$ and $|\overline{M}| \leq |M_2|$. First note that

$$\overline{z} = \alpha_1\beta_2 = (\alpha_1 - \alpha_2)\beta_2 + \alpha_2\beta_2 \geq (\alpha_1 - \alpha_2)\beta_1 + \alpha_2\beta_2 = z_2.$$

Now, as $y \geq 0 > \beta_1 - \beta_2$, we have

$$|M_2| = |x - (\alpha_1 - \alpha_2)| + y + \alpha_1 + \alpha_2 + 2\beta_2,$$
$$|\overline{M}| = |x - \alpha_1| + |y - \beta_2| + \alpha_1 + \beta_2.$$

If $x - (\alpha_1 - \alpha_2) \geq 0$, then we have

- $y \geq \beta_2$:
  - $x \geq \alpha_1$: $|\overline{M}| \leq |M_2| \Leftrightarrow \alpha_2 + \beta_2 \geq 0$.
  - $x \leq \alpha_1$: $|\overline{M}| \leq |M_2| \Leftrightarrow x + \alpha_2 + \beta_2 \geq \alpha_1 \Leftarrow x - (\alpha_1 - \alpha_2) \geq 0$.
- $y \leq \beta_2$:
  - $x \geq \alpha_1$: $|\overline{M}| \leq |M_2| \Leftrightarrow y + \alpha_2 \geq 0$.
  - $x \leq \alpha_1$: $|\overline{M}| \leq |M_2| \Leftrightarrow x + y + \alpha_2 \geq \alpha_1 \Leftarrow x - (\alpha_1 - \alpha_2) \geq 0$ and $y \geq 0$.

If $x - (\alpha_1 - \alpha_2) \leq 0$, then $x \leq \alpha_1$. So, we have

- $y \geq \beta_2$: $|\overline{M}| \leq |M_2| \Leftrightarrow \beta_2 \geq 0$.
- $y \leq \beta_2$: $|\overline{M}| \leq |M_2| \Leftrightarrow y \geq 0$.

Thus, $|\overline{M}| \leq |M_2|$ in every case. Finally, $|\overline{M}| \leq |M|$ and $\overline{z} \geq z$. As $\overline{M}' = A^{\alpha_1}B^{\beta_2}$, we can reduce $\overline{z}$ down to z, by the changes $AB \to BA$. Hence, we have the desired M'_+.

Now, we consider the case $x'_1 < 0$. We first suppose $y'_1 \leq y$. We change the A^ε's into $A^{-\varepsilon}$'s and the B^ε's into $B^{-\varepsilon}$'s ($\varepsilon = \pm 1$) in M'_1. Then we get a word $\widetilde{M}'_1$ which represents $(-x'_1, -y'_1, z_1)$ and

$$\widetilde{M}_1 = B^{y+y'_1}\widetilde{M}'_1 A^{x+x'_1}.$$

We compare $|M_1|$ and $|\widetilde{M}_1|$. Since

$$|M_1| = |M'_1| + |y - y'_1| + |x - x'_1| = |M'_1| + y - y'_1 + x - x'_1,$$
$$|\widetilde{M}_1| = |\widetilde{M}'_1| + |y + y'_1| + |x + x'_1| = |M'_1| + |y + y'_1| + |x + x'_1|,$$

we have

- $y + y'_1 \geq 0$:
  - $x + x'_1 \geq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow y'_1 + x'_1 \leq 0$.
  - $x + x'_1 \leq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow y'_1 \leq x \Leftarrow y'_1 \leq y \leq x$.
- $y + y'_1 \leq 0$:
  - $x + x'_1 \geq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow x'_1 \leq y \Leftarrow x'_1 \leq 0 \leq y$.
  - $x + x'_1 \leq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow x + y \geq 0$.

So, $|\widetilde{M}_1| \leq |M_1|$ in every case, and starting with $\widetilde{M}_1$, we can apply the above proof, since $-x'_1 \geq 0$.

Now, suppose $y'_1 \geq y$. We change the A^ε's into B^ε's and the B^ε's into A^ε's ($\varepsilon = \pm 1$) in M'_1 and we reverse the order of the letters. Then we get a word $\widetilde{M}'_1$ which represents (y'_1, x'_1, z_1) and

$$\widetilde{M}_1 = B^{y-x'_1}\widetilde{M}'_1 A^{x-y'_1}.$$

We compare $|M_1|$ and $|\widetilde{M}_1|$. Since

$$|M_1| = |M'_1| - y + y'_1 + x - x'_1, \qquad |\widetilde{M}_1| = |M'_1| + y - x'_1 + |x - y'_1|,$$

we have

- $x - y'_1 \geq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow y \leq y'_1$.
- $x - y'_1 \leq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow y \leq x$.

So, $|\widetilde{M}_1| \leq |M_1|$ in every case, and starting with $\widetilde{M}_1$, we can apply the above proof, since $y'_1 \geq y \geq 0$. ■

We denote by $\mathcal{M}'_\pm$ the subset of $\mathcal{M}'$ composed of the words $M'_\pm$ that can be written as $M'_- M'_+$ where M'_- is written only with A^{-1}'s and B^{-1}'s, and M'_+ only with A's and B's (both M'_- and M'_+ can be the empty word).

We also define $\mathcal{M}_\pm \subset \mathcal{M}$ as the set of words $M \in \mathcal{M}$ such that the corresponding M' belongs to $\mathcal{M}'_\pm$.

LEMMA 2.7. *If* $z \geq 0$, $\sqrt{z - xy} \geq x \geq 0$, *and* $0 \geq y \geq -x$, *then* $\mathcal{M}_\pm$ *is not empty.*

Proof. The proof of Lemma 2.6 does not use $y \geq 0$ to construct M_2 in the case $x'_1 \geq 0$. When $x'_1 \leq 0$, we first suppose $y'_1 \geq 0$. We change the A^ε's into $A^{-\varepsilon}$'s and the B^ε's into $B^{-\varepsilon}$'s ($\varepsilon = \pm 1$) in M'_1. Then we get a word $\widetilde{M}'_1$ which represents $(-x'_1, -y'_1, z_1)$ and

$$\widetilde{M}_1 = B^{y+y'_1} \widetilde{M}'_1 A^{x+x'_1}.$$

We compare $|M_1|$ and $|\widetilde{M}_1|$. Since

$$|M_1| = |M'_1| + |y - y'_1| + |x - x'_1| = |M'_1| - y + y'_1 + x - x'_1,$$
$$|\widetilde{M}_1| = |\widetilde{M}'_1| + |y + y'_1| + |x + x'_1| = |M'_1| + |y + y'_1| + |x + x'_1|,$$

we have

- $y + y'_1 \geq 0$:
  - $x + x'_1 \geq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow y \leq -x'_1 \Leftarrow y \leq 0 \leq -x'_1$.
  - $x + x'_1 \leq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow y \leq x$.
- $y + y'_1 \leq 0$:
  - $x + x'_1 \geq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow x'_1 \leq y'_1 \Leftarrow x'_1 \leq 0 \leq y'_1$.
  - $x + x'_1 \leq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow -x \leq y'_1 \Leftarrow -x \leq 0 \leq y'_1$.

So, $|\widetilde{M}_1| \leq |M_1|$ in every case, and starting with $\widetilde{M}_1$, we can construct M_2, since $-x'_1 \geq 0$.

Now, suppose $y'_1 \leq 0$. We change the A^ε's into $B^{-\varepsilon}$'s and the B^ε's into $A^{-\varepsilon}$'s ($\varepsilon = \pm 1$) in M'_1 and we reverse the order of the letters. Then we get a word $\widetilde{M}'_1$ which represents $(-y'_1, -x'_1, z_1)$ and

$$\widetilde{M}_1 = B^{y+x'_1} \widetilde{M}'_1 A^{x+y'_1}.$$

We compare $|M_1|$ and $|\widetilde{M}_1|$. Since

$$|M_1| = |M'_1| + |y - y'_1| + x - x'_1, \qquad |\widetilde{M}_1| = |M'_1| - y - x'_1 + |x + y'_1|,$$

we have

- $y - y'_1 \geq 0$:
  - $x + y'_1 \geq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow y - y'_1 \geq 0$.
  - $x + y'_1 \leq 0$: $|\widetilde{M}_1| \leq |M_1| \Leftrightarrow -x \leq y$.
- $y - y'_1 \leq 0$: then $x + y'_1 \geq x + y \geq 0$. So, $|\widetilde{M}_1| \leq |M_1|$.

So, $|\widetilde{M_1}| \leq |M_1|$ in every case, and starting with $\widetilde{M_1}$, we can construct M_2, since $-y_1' \geq 0$. Now,

$$M_2' = A^{-\alpha_2} B^{-\beta_2} A^{\alpha_1} B^{\beta_1},$$

so we can reduce z_2 down to z by the changes $AB \to BA$ and $A^{-1}B^{-1} \to B^{-1}A^{-1}$. These changes keep the word in $\mathcal{M}_\pm$, so we have the desired $M_\pm'$. ∎

LEMMA 2.8. *If* $z \geq 0$, $x \geq \sqrt{z - xy}$, *and* $0 \geq y \geq -x$, *then* $\mathcal{M}_+'$ *is not empty.*

Proof. First note that, except when $(x, y, z) = (0, 0, 0)$, the conditions of the lemma imply $x > 0$. Lemma 2.7 works also in this case. So, we start with the preceding M_2', which represents (x_2', y_2', z_2) with $z_2 \geq z$. First, we suppose $x \geq \sqrt{z_2 - xy}$. We prove that $|M_2| \geq |U|$. We have

$$\begin{aligned} |M_2| &= |x - (\alpha_1 - \alpha_2)| + |y - (\beta_1 - \beta_2)| + \alpha_1 + \alpha_2 + \beta_1 + \beta_2, \\ |U| &= x - y + 2\lceil z/x \rceil. \end{aligned}$$

We easily see that

$$|M_2| \geq |x - (\alpha_1 - \alpha_2)| + \alpha_1 + \alpha_2 + 2\beta_1 - y. \tag{3}$$

Recall that none of the α_i's or β_i's are null, that $z = (\alpha_1 - \alpha_2)\beta_1 + \alpha_2\beta_2$, and that $x \geq \sqrt{z - xy}$, so $z/x \leq x + y$.

- $x - (\alpha_1 - \alpha_2) \geq 0$: we first suppose $x \leq \beta_2 + \alpha_2$.
  - $y - (\beta_1 - \beta_2) \geq 0$:

    $$\begin{aligned} |U| \leq |M_2| &\Leftrightarrow y + \beta_2 + \alpha_2 \geq \lceil z/x \rceil \\ &\Leftrightarrow y + \beta_2 + \alpha_2 \geq z/x \\ &\Leftarrow \beta_2 + \alpha_2 \geq x. \end{aligned}$$

    The second equivalence comes from the fact that $y + \beta_2 + \alpha_2$ is an integer.
  - $y - (\beta_1 - \beta_2) \leq 0$:

    $$\begin{aligned} |U| \leq |M_2| &\Leftrightarrow \beta_1 + \alpha_2 \geq z/x \\ &\Leftarrow \beta_1 + \alpha_2 \geq x + y \\ &\Leftarrow \beta_1 + \alpha_2 \geq x. \end{aligned}$$

  Now suppose $x > \beta_2 + \alpha_2$. Then by (3),

  $$\begin{aligned} |U| \leq |M_2| &\Leftarrow \beta_1 + \alpha_2 \geq z/x \\ &\Leftarrow (\beta_2 + \alpha_2)(\beta_1 + \alpha_2) \geq \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_2\beta_1 \\ &\Leftrightarrow \alpha_2^2 + \beta_1\beta_2 + 2\alpha_2\beta_1 \geq \alpha_1\beta_1 \\ &\Leftarrow \beta_2 + 2\alpha_2 \geq \alpha_1. \end{aligned}$$

And, when $\alpha_1 - \alpha_2 > \beta_2 + \alpha_2$,

$$|U| \leq |M_2| \Leftarrow (\alpha_1 - \alpha_2)(\beta_1 + \alpha_2) \geq \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_2\beta_1 \Leftrightarrow \alpha_1 - \alpha_2 \geq \beta_2.$$

- $x - (\alpha_1 - \alpha_2) \leq 0$: we first suppose $2x \leq \beta_2 + \alpha_1$.
  - $y - (\beta_1 - \beta_2) \geq 0$:
  $$|U| \leq |M_2| \;\Leftrightarrow\; y + \beta_2 + \alpha_1 \geq z/x + x \;\Leftarrow\; \beta_2 + \alpha_1 \geq 2x.$$
  - $y - (\beta_1 - \beta_2) \leq 0$:
  $$\begin{aligned} |U| \leq |M_2| &\Leftrightarrow \beta_1 + \alpha_1 \geq z/x + x \\ &\Leftarrow \beta_1 + \alpha_1 \geq 2x + y \\ &\Leftarrow \beta_2 + \alpha_1 \geq 2x. \end{aligned}$$

Now, suppose $2x > \beta_2 + \alpha_1$. Then

$$\alpha_1 - 2\alpha_2 \geq \beta_2. \tag{4}$$

And so, by (3),

$$\begin{aligned} |U| \leq |M_2| &\Leftarrow \beta_1 + \alpha_1 \geq z/x + x \\ &\Leftarrow x(\beta_1 + \alpha_1) \geq z + x^2 \\ &\Leftrightarrow x^2 - x(\beta_1 + \alpha_1) + \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_2\beta_1 \leq 0. \end{aligned}$$

The discriminant of this polynomial in x is

$$\Delta = (\beta_1 + \alpha_1)^2 - 4(\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_2\beta_1).$$

Using (4) yields $\Delta \geq (\alpha_1 - 2\alpha_2 - \beta_1)^2$, where $\alpha_1 - 2\alpha_2 - \beta_1 \geq 0$. So the polynomial is negative if $\beta_1 + \alpha_2 \leq x \leq \alpha_1 - \alpha_2$, which is true, since by (4),

$$2x > \beta_2 + \alpha_1 \geq 2\beta_2 + 2\alpha_2 \geq 2(\beta_1 + \alpha_2).$$

In U', by the changes $AB \to BA$, we can reduce z_2 down to z. Then we get the desired $M'_+ \in \mathcal{M}'_+$.

Now, suppose $x \leq \sqrt{z_2 - xy}$. We prove that $|M_2| \geq |T|$. We have

$$\begin{aligned} |M_2| &= |x - (\alpha_1 - \alpha_2)| + |y - (\beta_1 - \beta_2)| + \alpha_1 + \alpha_2 + \beta_1 + \beta_2, \\ |T| &= 2\lceil 2\sqrt{z - xy}\,\rceil - x + y. \end{aligned}$$

- $y - (\beta_1 - \beta_2) \geq 0$:
  - $x - (\alpha_1 - \alpha_2) \geq 0$:
  $$\begin{aligned} |T| \leq |M_2| &\Leftrightarrow \lceil 2\sqrt{z - xy}\,\rceil \leq x + \alpha_2 + \beta_2 \\ &\Leftrightarrow 2\sqrt{z - xy} \leq x + \alpha_2 + \beta_2 \\ &\Leftrightarrow 4[\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_2\beta_1 - xy] \leq (x + \alpha_2 + \beta_2)^2 \\ &\Leftarrow 0 \leq x^2 + 2x(\alpha_2 - \beta_2) + 4\beta_1[x - (\alpha_1 - \alpha_2)] \\ &\qquad + (\alpha_2 - \beta_2)^2 \\ &\Leftarrow 0 \leq (x + \alpha_2 - \beta_2)^2. \end{aligned}$$

The second equivalence comes from the fact that $x + \alpha_2 + \beta_2$ is an integer.

- $x - (\alpha_1 - \alpha_2) \leq 0$:

$$\begin{aligned} |T| \leq |M_2| &\Leftrightarrow 2\sqrt{z - xy} \leq \alpha_1 + \beta_2 \\ &\Leftarrow 4[\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_2\beta_1 - (\alpha_1 - \alpha_2)(\beta_1 - \beta_2)] \\ &\qquad \leq (\alpha_1 + \beta_2)^2 \\ &\Leftrightarrow 0 \leq (\beta_2 - \alpha_1)^2. \end{aligned}$$

- $y - (\beta_1 - \beta_2) \leq 0$:
  - $x - (\alpha_1 - \alpha_2) \geq 0$:

$$\begin{aligned} |T| \leq |M_2| &\Leftrightarrow 2\sqrt{z - xy} \leq x - y + \alpha_2 + \beta_1 \\ &\Leftarrow 4[(x + \alpha_2)\beta_1 + \alpha_2(\beta_1 - y) - \alpha_2\beta_1 - xy] \\ &\qquad \leq (x - y + \alpha_2 + \beta_1)^2 \\ &\Leftrightarrow 0 \leq (x + y + \alpha_2 - \beta_1)^2. \end{aligned}$$

  - $x - (\alpha_1 - \alpha_2) \leq 0$:

$$\begin{aligned} |T| \leq |M_2| &\Leftrightarrow 2\sqrt{z - xy} \leq -y + \alpha_1 + \beta_1 \\ &\Leftarrow 4[\alpha_1\beta_1 + \alpha_2(\beta_1 - y) - \alpha_2\beta_1 - (\alpha_1 - \alpha_2)y] \\ &\qquad \leq (\alpha_1 + \beta_1 - y)^2 \\ &\Leftrightarrow 0 \leq (y + \alpha_1 - \beta_1)^2. \end{aligned}$$

Note that $T' = T \in \mathcal{M}'_{\pm}$. In the prefix of T composed only of A^{-1}'s and B^{-1}'s, we change $A^{-1}B^{-1} \to B^{-1}A^{-1}$ until we get a word W with $W' \in \mathcal{M}'_{+}$. Then W represents (x, y, z_w).

If there exists k, $0 < k \leq n$, such that $z_2 - xy = n^2 + k$, then

$$z_w = z_2 - n(n+1) + x(n+1) = xy + k - n + x(n+1).$$

Hence, as $1 \leq x \leq n$,

$$z_w - xy \geq -n + x(n+1) \geq x^2 + (x-1)(n-x) \geq x^2.$$

So, we still have $\sqrt{z_w - xy} \geq x$.

If there exists k, $0 \leq k \leq n$, such that $z_2 - xy = (n+1)^2 - k$, then

$$z_w = z_2 + k - n(n+1) + x(n+1) = xy + (x+1)(n+1).$$

Hence, as $x \leq n+1$, we still have $\sqrt{z_w - xy} \geq x$.

As $\sqrt{z_w - xy} \geq x$, we need to reduce z_w down to z by the changes $AB \to BA$. So, we get the desired $M'_+ \in \mathcal{M}'_+$. ∎

Case I.1. As we exclude $(x, y, z) = (0, 0, 0)$, we have $z > 0$. We denote by (a, b, z) the element represented by $S(z)$ and we remark that $|S(z)| = a + b$.

Our aim is to prove that $S(z)$ belongs to $\mathcal{M}'_+$ and then to say that

$$W = B^{-(b-y)}S(z)A^{-(a-x)} \in \mathcal{M}.$$

As $x \leq \sqrt{z}$ we see that $y \leq x \leq \min\{a, b\}$. Therefore we will obtain $d(x, y, z) = 2(a + b) - x - y = 2\lceil 2\sqrt{z}\,\rceil - x - y$.

Let $M' \in \mathcal{M}'_+$ (non-empty by Lemma 2.6) and (x', y', z) be the element represented by M'. We will prove that there is an $M' \in \mathcal{M}'_+$ such that $x' = a$ and $y' = b$. We define two integers α, β by $x' = a + \alpha$ and $y' = b + \beta$. We will prove that the cases $(\alpha, \beta) \neq (0, 0)$ are either impossible or the word

$$B^{y-y'}M'A^{x-x'}$$

has the same length as W.

By Lemma 2.6, we have $|M'| = a + \alpha + b + \beta$. By Remark 2.3,

$$(a + \alpha)(b + \beta) \geq z. \tag{5}$$

We first suppose $\alpha \geq 1$. If $\beta \geq 0$, then $|M'| > |S(z)|$, $|y' - y| \geq b - y$ and $|x' - x| \geq a - x$. So,

$$|M| > |B^{y-b}S(z)A^{x-a}|,$$

which is impossible. Now suppose $\beta \leq -1$. By the construction of $S(z)$, we get $z > a(b - 1)$. So, (5) implies

$$a(\alpha + \beta) > -a + \alpha(-\beta - (b - a)).$$

But $-\beta \geq 1 \geq b - a$, so $\alpha + \beta > -1$ and then $\alpha + \beta \geq 0$. Therefore $|M'| \geq |S(z)|$ and

$$|y' - y| + |x' - x| \geq a - x + b - y + \alpha + \beta \geq a - x + b - y.$$

So, $|M| \geq |B^{y-b}S(z)A^{x-a}|$.

Now suppose $\alpha \leq 0$. If $\beta \geq 0$, the case is the same as the preceding one, since $z > a(b - 1)$ and (5) imply

$$b(\alpha + \beta) > -a + \beta(-\alpha - (a - b)) \geq -b + \beta(-\alpha - (a - b))$$

and $-\alpha \geq 0 \geq a - b$. Finally, $\beta \leq -1$ is impossible since then $(a + \alpha)(b + \beta) \leq a(b - 1) < z$, which contradicts (5). That completes the proof.

Case I.2.1. By Remark 2.4, we get $d(x, y, z) \geq x + y$. So, we only need to find a word W of length $x + y$ that represents (x, y, z). If $xy = z$, then we take

$$W = A^xB^y.$$

Now, we are left with $xy > z$. The Euclidean division of z by x gives $z = ux + v$ with $v < x$. If $u \geq y$, then $z \geq xy + v \geq xy$, which is impossible. So $u \leq y - 1$ and we get

$$W = B^{y-u-1}A^vBA^{x-v}B^u$$

and $|W| = x + y$.

Case I.2.2. Here, we have $z > 0$. First consider the case where $y = 0$ and $x > z$. There is at least one B^ε ($\varepsilon = \pm 1$) in M as $z > 0$. But $y = 0$, so there is at least one $B^{-\varepsilon}$ also. Therefore $d(x, 0, z) \geq x + 2$, and we can represent $(x, 0, z)$ by

$$B^{-1}A^zBA^{x-z}.$$

So $d(x, 0, z) = x + 2$. Now we are left with $x \leq z$.

We write $\overline{u} = \lceil z/x \rceil$. In the definition of U, we can check that $\overline{u} = u + 1$ when $v > 0$, and $\overline{u} = u$ when $v = 0$. Hence, $\overline{u} \geq y$. By Lemma 2.8, we can take $M' \in \mathcal{M}'_+$ which represents (x', y', z). We define two integers α, β by $x' = x + \alpha$ and $y' = \overline{u} + \beta$. We will prove that $\alpha = \beta = 0$ is possible by showing that any other case is either impossible or the word

$$M = B^{y-y'}M'A^{x-x'}$$

has the same length as U. As $M' \in \mathcal{M}'_+$, we have $|M'| = x + \alpha + \overline{u} + \beta$. By Remark 2.3,

$$(x + \alpha)(\overline{u} + \beta) \geq z. \tag{6}$$

We first suppose $\alpha \geq 1$. If $\beta \geq 0$, then $|M'| > |U'|$, $|y' - y| \geq \overline{u} - y$ and $|x' - x| > 0$. So $|M| > |U|$, which is impossible. Now suppose $\beta \leq -1$. We have $z = ux + v > \overline{u}x - x$, so (6) implies

$$(\alpha + \beta)x > -x - \alpha\beta + \alpha(x - \overline{u}).$$

Using $z \leq x^2$, we get $x \geq \overline{u}$. Moreover $\alpha\beta < 0$, so $\alpha + \beta > -1$, and so $\alpha + \beta \geq 0$. Therefore $|M'| \geq |U'|$, $|y' - y| + |x' - x| \geq \overline{u} - y + \beta + \alpha \geq \overline{u} - y$. Hence $|M| \geq |U|$, so $|M| = |U|$ by definition of $\mathcal{M}$.

Now suppose that $\alpha \leq 0$. The case $\beta \leq -1$ is impossible because then $(x+\alpha)(\overline{u}+\beta) \leq x\overline{u} - x < z$, contrary to (6). If $\beta \geq 0$, then $|y'-y|+|x'-x| = \beta + \overline{u} - y - \alpha$, so $|M| = |U| + 2\beta \geq |U|$. That completes the proof.

Case II.1. By Lemma 2.7, we can take $M' \in \mathcal{M}'_\pm$ and the corresponding $M \in \mathcal{M}'_\pm$. By definition of $\mathcal{M}'_\pm$, M' can be written as $M'_-M'_+$. Then M'_- represents $(-\alpha_2, -\beta_2, z_2)$, and M'_+ represents (α_1, β_1, z_1). By Remark 2.3, we have $z_1 \leq \alpha_1\beta_1$, $z_2 \leq \alpha_2\beta_2$, and so

$$z \leq \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_2\beta_1. \tag{7}$$

Now, we compare $|M|$ and $|T|$ exactly as we compared $|M_2|$ and $|T|$ in the proof of Lemma 2.8. Using (7), we conclude that $|T| \leq |M|$, and so $T \in \mathcal{M}$.

Case II.2. By Lemma 2.8 we can take $M' \in \mathcal{M}'_+$ and the corresponding $M \in \mathcal{M}_+$. The proof is as in case I.2.2, since we did not use $y \geq 0$.

3. Almost connectedness of the spheres. In this section we prove the almost connectedness of the spheres of the Heisenberg group.

Proposition 3.1. *The spheres of $\mathbb{H}_3$ with the word distance associated to the set $\{A^{\pm 1}, B^{\pm 1}\}$ of generators are almost connected with constants $(6,4)$.*

Proof. The idea of the proof is first to split the sphere into several regions according to the expression of the distance (see Theorem 2.2). Then we build a path (equivalent to a 1-path in the discrete setting) from a given point to a particular one in the boundary of its region. As before, we only need to consider points with $z \geq 0$, $x \geq 0$ and $x \geq y \geq -x$. We first suppose that $y \geq 0$. Each point at distance n from the origin has a neighbor at distance $n-1$. Then we just need to prove the theorem for a sphere of radius $2n$, and we will adapt the constants at the end. Let $d(x,y,z) = 2n$; the aim is to build a path from $M = (x,y,z)$ to (n,n,n^2) which belongs to all the regions. Here, the notation $M = (x,y,z)$ is not confusing since we always use Theorem 2.2 to compute the distance.

Let (x,y,z) be in Region I.1. Then $x \leq \sqrt{z}$, and we have

$$2n = 2\lceil 2\sqrt{z}\rceil - x - y \geq 2\lceil 2\sqrt{z}\rceil - 2x \geq \lceil 2\sqrt{z}\rceil.$$

So,

$$x \leq \sqrt{z} \leq \lceil 2\sqrt{z}\rceil/2 \leq n.$$

We will increase x and y up to n. At each step, we will "correct" the value of z by multiplying by C^ε's ($\varepsilon = \pm 1$), where $C = ABA^{-1}B^{-1}$, in order to come back within $B(2n+1) \setminus B(2n-1)$ (multiplication by C changes only the value of z). The reason why we cannot come back exactly on $\partial B(2n)$ in general is the following remark.

Remark 3.2. *When $n \geq 3$, multiplication by C changes the distance by 1 or 2.*

First, we increase y up to the value of x. We have

$$MB = (x, y+1, z+x).$$

As $x \leq \sqrt{z}$,

$$2\sqrt{z+x} \leq 2\sqrt{z+\sqrt{z}+1/4} \leq 2\sqrt{z}+1.$$

So, MB belongs to $B(2n+2) \setminus B(2n-2)$. So, we correct the distance, if necessary, by multiplication by C^ε's until we get MBC^α within $B(2n+1) \setminus B(2n-1)$. Easy computations show that $|\alpha| \leq \sqrt{z}+1/4$. Finally, we have went from M to MBC^α within $B(2n+4)\setminus B(2n-4)$. We repeat this process until we have $x = y$.

Now we may consider $M = (x,x,z)$. As long as $x \leq \sqrt{z}-1$, we do

$$MAB = (x+1, x+1, z+x+1).$$

As $\sqrt{z+x+1} \leq \sqrt{z+\sqrt{z}+1/4} = \sqrt{z}+1/2$, MAB is within $B(2n+1) \setminus B(2n-3)$. So, we increase z, if necessary, by C's until we get $MABC^{\alpha}$ within $B(2n+1) \setminus B(2n-1)$. We get $\alpha \leq \sqrt{z}+1/4$ and we have went from M to $MABC^{\alpha}$ within $B(2n+1) \setminus B(2n-5)$. If $\sqrt{z}-1 < x$, we increase z by C's until $\sqrt{z}$ becomes an integer. Then $x = \sqrt{z}$ and $d(x,x,x^2) = 2x$. So, $x = n$ and we are at (n,n,n^2). The path we have built stays within $B(2n+4) \setminus B(2n-5)$.

Let (x,y,z) be in Region I.2.1. Here, the value of z does not appear in the distance, as long as it satisfies the conditions of this region. So, we can first increase z up to xy, by C's. As $x+y=2n$, we have $y \leq n \leq x$. The idea is to increase y up to n and to decrease x down to n simultaneously (as $n-y = x-n$). At the same time, we "correct" the value of z as before, in order to keep the relation $xy = z$. We have

$$MC^{xy-z}B = (x, y+1, xy+x).$$

We are still in Region I.2.1. Thus, we can decrease the value of z down to $xy+x-y+1$ without changing the distance. Then we reduce x to get

$$MC^{xy-z}BC^{1-y}A^{-1} = (x-1, y+1, xy+x-y+1),$$

which satisfies $xy = z$. The path we have built stays within $B(2n+4) \setminus B(2n-3)$.

Let (x,y,z) be in Region I.2.2. As long as $x \geq y \geq 0$, we do not need to check if we stay in this region, because as soon as we exit it, we know how to reach (n,n,n^2). We have $y \leq \sqrt{z} \leq x$. The idea is again to decrease x and increase y simultaneously. Now,

$$MBA^{-1} = (x-1, y+1, z+x)$$

satisfies $d(x-1,y+1,z+x) = d(x,y,z)$. So, we can iterate this process until $x-y \leq 1$. If $x-y=1$, then we multiply again by B. Finally we enter one of the preceding regions. The path we have built stays within $B(2n+3) \setminus B(2n-2)$.

Finally, we take $y \leq 0$. Let (x,y,z) be in Region II.2. We only need to find a path to some point with $y \geq 0$ in order to connect it to one of the preceding regions. We build

$$MB^2C^{-x} = (x, y+2, z+x),$$

which is still in Region II.2 (unless $y+2 \geq 0$). So $d(x,y+2,z+x) = d(x,y,z)$, and thus we can increase y. The path we have built stays within $B(2n+3) \setminus B(2n-5)$.

Let (x,y,z) be in Region II.1. As long as $x \leq \sqrt{z-xy}$, we do

$$MA = (x+1, y, z).$$

As

$$\sqrt{z-xy-y} \le \sqrt{z-xy+\sqrt{z-xy}+1/4} = \sqrt{z-xy}+1/2,$$

MA belongs to $B(2n+2)\setminus B(2n-2)$. So, we correct z, if necessary, by C's, until we get MAC^α within $B(2n+1)\setminus B(2n-1)$. We get $\alpha \le \sqrt{z-xy}+1/4$, and the path we have built stays within $B(2n+3)\setminus B(2n-4)$. We repeat this process until we reach one of the preceding regions.

Finally, even if we look at a sphere with an odd radius, we get the almost connectedness with constants $(6,4)$. ■

Acknowledgments. The author wishes to thank Laurent Saloff-Coste for his help during the preparation of this manuscript.

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Received 24 July 2001 (4093)