COLLOQUIUM MATHEMATICUM

VOL. 95 2003 NO. 1

A NOTE ON RARE MAXIMAL FUNCTIONS

BY

PAUL ALTON HAGELSTEIN (Princeton, NJ)

Abstract. A necessary and sufficient condition is given on the basis of a rare maximal function M_l such that $M_l f \in L^1([0,1])$ implies $f \in L \log L([0,1])$.

For a locally integrable function $f : \mathbb{R} \to \mathbb{R}$, the Hardy–Littlewood maximal function $M_{\mathrm{HL}} f$ is defined by

$$M_{\mathrm{HL}} f(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)|\, dy,$$

where the supremum is taken over all bounded intervals I containing x.

In 1969, E. M. Stein proved in [2] that if f is supported on the unit interval $\mathbb{I} = [0,1]$, then $M_{\mathrm{HL}} f \in L^1(\mathbb{I})$ if and only if $f \in L \log L(\mathbb{I})$. The proof of this result is based on classical weak-type inequalities for M_{HL}, which in turn strongly depend on the covering properties associated to the set of intervals in $\mathbb{R}$.

In [1], K. Hare and A. Stokolos investigated whether or not Stein's result still holds when the Hardy–Littlewood maximal operator is replaced by a *rare* maximal operator, an operator similar to M_{HL} but where the supremum is taken only over a given subset of the set of intervals on $\mathbb{R}$. For the sake of specificity, we here define a rare maximal operator M_l as follows:

Definition 1. Let $l = \{l_k\}$, where the $l_k \leq 1$, $l_k \downarrow 0$, and let

$$\mathcal{I} = \{\text{intervals } I \subset \mathbb{R} : |I| \in l\}.$$

We define the *rare maximal function* $M_l f$ by

$$M_l f(x) = \sup_{I \in \mathcal{I},\, x \in I} \frac{1}{|I|} \int_I |f(y)|\, dy.$$

In [1], Hare and Stokolos showed that there exists a sequence l and a locally integrable function f supported on $\mathbb{I}$ such that $f \notin L \log L(\mathbb{I})$, but $M_l f \in L^1(\mathbb{I})$. Moreover, they found that if $\{m_k\} \subset \mathbb{N}$ is a strictly increasing

2000 *Mathematics Subject Classification*: Primary 42B25.

sequence satisfying

$$\sup_{k\in\mathbb{N}} \frac{m_k}{k} = \infty \tag{1}$$

and $l = \{2^{-m_k}\}$, then there necessarily exists a locally integrable function f supported in $\mathbb{I}$ such that $f \notin L\log L(\mathbb{I})$, but $\int_0^1 M_l f < \infty$. Having found these results, Hare and Stokolos indicated the question whether or not the condition (1) above is sharp, i.e. if $l = \{2^{-m_k}\}$ and if there exists a locally integrable function f supported in $\mathbb{I}$ such that $f \notin L\log L(\mathbb{I})$ but $\int_0^1 M_l f < \infty$, must then $\sup_{k\in\mathbb{N}} m_k/k = \infty$?

In this paper we indicate the negative answer to the above question, as well as provide a necessary and sufficient condition on $\{m_k\}$ such that there exists $f \notin L\log L(\mathbb{I})$ such that $M_l f \in L^1(\mathbb{I})$.

THEOREM 1. *Let* $l = \{2^{-m_k}\}$, *where* $\{m_k\}$ *is an increasing sequence of nonnegative integers.*

(i) *Suppose that for any positive integer* m, *there exists* $k \in \mathbb{N}$ *such that none of the* m_j *lie in* $A_{k,m} = \{k, k+1, \ldots, k+m\} \subset \mathbb{N}$. *Then there exists a locally integrable function* f *supported in* $\mathbb{I}$ *such that* $f \notin L\log L(\mathbb{I})$, *but* $M_l f \in L^1(\mathbb{I})$.

(ii) *Suppose there exists a positive integer* m *such that any set* $A_{k,m}$ *necessarily contains an element of the sequence* $\{m_k\}$. *Then if* f *is a locally integrable function such that* $M_l f \in L^1(\mathbb{I})$, f *must be in* $L\log L(\mathbb{I})$.

Before we begin the proof, we remark that part (i) of the theorem provides a negative answer to the question of Hare and Stokolos, given the existence of increasing sequences of positive integers $\{m_k\}$ that satisfy the condition in (i) but also such that $\sup_{k\in\mathbb{N}} m_k/k < \infty$.

Proof. (i) We construct a function f such that $M_l f \in L^1(\mathbb{I})$ but $f \notin L\log L(\mathbb{I})$ as follows:

Let $m \in \mathbb{N}$. There exists $k \in \mathbb{N}$ such that none of the m_j lie in $A_{k,m}$. Let E_m be the subset of $\mathbb{I}$ defined by

$$E_m = [0, 2^{-k-m}] \cup [2^{-k}, 2^{-k} + 2^{-k-m}] \cup \ldots \cup [1 - 2^{-k}, 1 - 2^{-k} + 2^{-k-m}].$$

Let

$$l_{m,1} = \{2^{-m_j} : m_j \le k\}, \quad l_{m,2} = \{2^{-m_j} : m_j > k+m\}.$$

Note that for any locally integrable function g,

$$M_l g(x) \le M_{l_{m,1}} g(x) + M_{l_{m,2}} g(x).$$

Now, $M_{l_{m,1}} \chi_{E_m}(x) \le 2 \cdot 2^{-m}$ pointwise, and

$$M_{l_{m,2}} \chi_{E_m}(x) \le \chi_{E_m}(x) + \chi_{E_m}(x - 2^{-k-m}) + \chi_{E_m}(x + 2^{-k-m}).$$

So

$$\int_{\mathbb{I}} M_{l_{m,1}}\chi_{E_m} \le 2\cdot 2^{-m}, \qquad \int_{\mathbb{I}} M_{l_{m,2}}\chi_{E_m} \le 3\cdot |E_m| = 3\cdot 2^{-m}.$$

Therefore

$$\int_{\mathbb{I}} M_l\chi_{E_m} \le 5\cdot 2^{-m}.$$

Note that by direct computation one sees that $\|\chi_{E_m}\|_{L\log L(\mathbb{I})} \sim m\cdot 2^{-m}$, as $\|\chi_{E_m}\|_{L\log L(\mathbb{I})} \sim \|M_{\mathrm{HL}}\chi_{E_m}\|_{L^1(\mathbb{I})}$ by Stein's $L\log L$ result [2].

Let now

$$f = \sum_{j=1}^{\infty} 2^{-j}2^{2^{2j}}\chi_{E_{2^{2j}}}.$$

Clearly

$$\begin{aligned}\|f\|_{L\log L(\mathbb{I})} &\ge \lim_{j\to\infty} 2^{-j}2^{2^{2j}}\|\chi_{E_{2^{2j}}}\|_{L\log L(\mathbb{I})} = \lim_{j\to\infty} 2^{-j}\cdot 2^{2^{2j}}\cdot 2^{2j}\cdot 2^{-2^{2j}}\\ &= \lim_{j\to\infty} 2^{j} = \infty.\end{aligned}$$

However,

$$\int_{\mathbb{I}} M_l f \le \sum_{j=1}^{\infty} 2^{-j}2^{2^{2j}}\int_{\mathbb{I}} M_l\chi_{E_{2^{2j}}} \le 5\sum_{j=1}^{\infty} 2^{-j}2^{2^{2j}}2^{-2^{2j}} = 5\sum_{j=1}^{\infty} 2^{-j} = 5.$$

So $M_l f \in L^1(\mathbb{I})$, but $\|f\|_{L\log L(\mathbb{I})} = \infty$.

(ii) We assume without loss of generality that $1 \in l$. Let I be an interval in $\mathbb{I}$. Let $j \in \mathbb{N}$ be such that $2^{-j} \le |I| \le 2^{-j+1}$.

If $j \le m$, then $|\mathbb{I}| \le 2^m|I|$. If $j > m$, then the sequence

$$\{j-m, j-m+1, \ldots, j-1, j\}$$

contains an element of the sequence $\{m_k\}$. These two cases imply that there exists an interval I' such that $|I'| \in l$, $I \subset I'$, and $|I'| \le 2^{m+2}|I|$. So

$$M_{\mathrm{HL}}f \le 2^{m+2}M_l f.$$

Hence, if $M_l f \in L^1(\mathbb{I})$, then $M_{\mathrm{HL}}f \in L^1(\mathbb{I})$, and thus $f \in L\log L(\mathbb{I})$. ∎

REFERENCES

[1] K. Hare and A. Stokolos, *On weak type inequalities for rare maximal functions*, Colloq. Math. 83 (2000), 173–182.

[2] E. M. Stein, *Note on the class $L\log L$*, Studia Math. 32 (1969), 305–310.

Department of Mathematics
Princeton University
Princeton, NJ 08540, U.S.A.
E-mail: phagelst@math.princeton.edu

Received 21 January 2002 (4157)