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TORSIONS OF CONNECTIONS ON TIME-DEPENDENT WEIL BUNDLES

BY

MIROSLAV DOUPOVEC (Brno)

Abstract. We introduce the concept of a dynamical connection on a time-dependent Weil bundle and we characterize the structure of dynamical connections. Then we describe all torsions of dynamical connections.

1. Introduction. Roughly speaking, non-autonomous Lagrangian dynamics can be considered as an extension of autonomous Lagrangian dynamics by introducing the additional time coordinate. From this point of view, many structures and geometric objects from autonomous Lagrangian dynamics can be naturally extended and introduced also in the non-autonomous case. In this way we can define time-dependent (or dynamical) vector fields, Lagrangians, connections, sprays and other structures. For example, if $\Gamma : FM \to J^1FM$ is a general connection on a natural bundle F, a dynamical connection is a section $\Gamma_d : \mathbb{R} \times FM \to J^1(\mathbb{R} \times FM)$. We remark that the concept of a dynamical connection on the tangent bundle TM was introduced by de León and Rodrigues in [13]. Time-dependent geometrical objects and structures have also been studied e.g. by Anastasiei and Kawaguchi [1], by Crampin *et al.* [2], Krupková [7] and Vondra [15], [16].

The aim of this paper is to describe torsions of dynamical connections on time-dependent Weil bundles. We show that a time-dependent connection has three types of torsion. The first torsion is an extension of the autonomous torsion by means of some difference tensor and the second one is completely determined by the generalized tension of the associated autonomous connection.

All manifolds and maps are assumed to be infinitely differentiable. In what follows we shall use the terminology and notations from the book [6].

2. The general torsion and tension. We recall that the Frölicher–Nijenhuis bracket is a map [,] which transforms a tangent valued p-form

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K and a tangent valued q-form L on a manifold M into a tangent valued $(p+q)$-form $[K,L]$ (cf. [6]). In general, an *affinor* on a manifold M is a linear morphism $TM \to TM$ over the identity of M. Clearly, this is exactly a tangent valued one-form on M, i.e. a section of $TM \otimes T^*M$. If $\Gamma : Y \to J^1Y$ is a general connection on a fibered manifold $Y \to M$, then Γ can be identified with its horizontal projection $TY \to TY$, which is a special affinor on Y. Taking an arbitrary canonical affinor Q on Y, the (general) torsion of Γ is defined as the Frölicher–Nijenhuis bracket $[\Gamma, Q]$ of Γ and Q.

Consider now a natural bundle F on the category $\mathcal{M}f_m$ of m-dimensional manifolds and their local diffeomorphisms and let $\Gamma : FM \to J^1FM$ be a connection on FM.

Definition. A *natural affinor* on the natural bundle F is a system of affinors $Q_M : TFM \to TFM$ for every m-manifold M satisfying $TFf \circ Q_M = Q_N \circ TFf$ for every local diffeomorphism $f : M \to N$.

Definition. Let Q be a non-identical natural affinor on F. The Frölicher–Nijenhuis bracket $[\Gamma, Q]$ is called a (general) *torsion* of the connection Γ.

The above definition of a torsion is due to I. Kolář and M. Modugno [5] and generalizes the classical torsion of a linear connection. In this way all general torsions of Γ are completely determined by the list of all natural affinors on FM. That is why there are numerous papers which classify all natural affinors on some natural bundles (cf. [3], [8], [11], [14]).

Let (x^i, y^p) be some local fibered coordinates on Y. Then a connection $\Gamma : Y \to J^1Y$ has equations

$$dy^p = \Gamma^p_i(x,y)dx^i \tag{1}$$

and an affinor $Q \in C^\infty(TY \otimes T^*Y)$ on Y has the coordinate form

$$(dx^i, dy^p) \mapsto (Q^i_j dx^j + Q^i_p dy^p, Q^p_i dx^i + Q^p_q dy^q). \tag{2}$$

By Kureš [12], the Frölicher–Nijenhuis bracket $[\Gamma, Q]$ is of the form

$$\begin{aligned}(3)\quad & \left(\Gamma^p_i \frac{\partial Q^k_j}{\partial y^p} - Q^k_p \frac{\partial \Gamma^p_j}{\partial x^i}\right)\frac{\partial}{\partial x^k} \otimes dx^i \wedge dx^j \\ &+ \left(\frac{\partial Q^k_i}{\partial y^p} + \Gamma^q_i \frac{\partial Q^k_p}{\partial y^q} + Q^k_q \frac{\partial \Gamma^q_i}{\partial y^p}\right)\frac{\partial}{\partial x^k} \otimes dx^i \wedge dy^p \\ &+ \left(\frac{\partial Q^p_j}{\partial x^i} + Q^k_i \frac{\partial \Gamma^p_j}{\partial x^k} - \Gamma^p_k \frac{\partial Q^k_j}{\partial x^i} + \Gamma^q_i \frac{\partial Q^p_j}{\partial y^q} + Q^q_i \frac{\partial \Gamma^p_j}{\partial y^q}\right. \\ &\left. - Q^p_q \frac{\partial \Gamma^q_j}{\partial x^i}\right)\frac{\partial}{\partial y^p} \otimes dx^i \wedge dx^j\end{aligned}$$

$$+\left(\frac{\partial Q^p_q}{\partial x^i} - Q^j_q\,\frac{\partial \Gamma^p_i}{\partial x^j} - \Gamma^p_j\,\frac{\partial Q^j_q}{\partial x^i} + \Gamma^p_j\,\frac{\partial Q^j_i}{\partial y^q} + \Gamma^r_i\,\frac{\partial Q^p_q}{\partial y^r} - Q^r_q\,\frac{\partial \Gamma^p_i}{\partial y^r}\right.$$
$$\left. + Q^p_r\,\frac{\partial \Gamma^r_i}{\partial y^q}\right)\frac{\partial}{\partial y^p}\otimes dx^i\wedge dy^q.$$

Clearly, $[\Gamma, Q]\in C^\infty(TY\otimes\bigwedge^2 T^*Y)$. An affinor $Q\in C^\infty(TY\otimes T^*Y)$ is called *vertical* if Q has values in the vertical bundle VY, i.e. $Q\in C^\infty(VY\otimes T^*Y)$. Moreover, taking into account the canonical inclusion $T^*M\subset T^*Y$, we can consider vertical affinors of the form $Q\in C^\infty(VY\otimes T^*M)$, called *soldering forms*. The coordinate expression of a soldering form $Q: TM\to VY$ is

$$(dx^i)\mapsto (0, Q^p_i dx^i).$$

Let $J:(dx^i, dy^p)\mapsto(0, dx^i)$ be the canonical almost tangent structure of the tangent bundle TM and let $L = y^p\frac{\partial}{\partial y^p}$ be the classical Liouville vector field. Clearly, J is a natural affinor on TM. Grifone [4] identified connections on TM with vector valued one-forms $\bar\Gamma: TTM\to TTM$ satisfying $J\bar\Gamma = J$, $\bar\Gamma J = -J$ and defined the *weak torsion* t and the *tension* h of a connection $\bar\Gamma$ by

$$t = \tfrac12[J,\bar\Gamma],\qquad h = \tfrac12[L,\bar\Gamma]. \tag{4}$$

Obviously, $\Gamma = \frac12(\mathrm{Id}_{TM} + \bar\Gamma)$ is the horizontal form of a connection $\Gamma: TM\to J^1(TM\to M)$ (denoted by the same symbol Γ). In this way Grifone's formulas (4) can be rewritten in the form

$$t = [J,\Gamma],\qquad h = [L,\Gamma]. \tag{5}$$

Thus the definition of a general torsion $[\Gamma, Q]$ of a connection Γ on a natural bundle F as the Frölicher–Nijenhuis bracket of Γ with an arbitrary natural affinor Q can be viewed as a generalization of Grifone's formula for the weak torsion t on TM. It turns out that it is also useful to study the tension of a connection from a more general point of view. Analogously to the concept of a general torsion, if we replace the Liouville vector field L on TM with an arbitrary natural vector field X on a natural bundle F, we obtain the concept of a general tension.

Definition. A *natural* (or *absolute*) *vector field* on a natural bundle F is a system of vector fields $X_M: FM\to TFM$ for every m-manifold M satisfying $TFf\circ X_M = X_N\circ Ff$ for all local diffeomorphisms $f: M\to N$.

Definition. Let X be a natural vector field on F. The Frölicher–Nijenhuis bracket $\mathcal{H} = [\Gamma, X]$ is called a (general) *tension* of the connection $\Gamma: FM\to J^1FM$.

One finds easily that the tension of Γ is a soldering form on FM, i.e. $\mathcal{H}\in C^\infty(VFM\otimes T^*M)$. For example, the classical tension of a connection

(1) on TM has the coordinate form

$$\mathcal{H} = \left(\Gamma^p_i - \frac{\partial \Gamma^p_i}{\partial y^k}\, y^k \right) \frac{\partial}{\partial y^p} \otimes dx^i. \tag{6}$$

Obviously, $\mathcal{H} = 0$ iff the connection Γ is linear.

3. Time-dependent bundles and connections. Let T^A be a Weil functor corresponding to a Weil algebra A (see [6]). Then T^A is a bundle functor on the category $\mathcal{M}f \supset \mathcal{M}f_m$ of all smooth manifolds and all smooth maps, which transforms every manifold M into a fibered manifold $T^A M \to M$ and every smooth map $f : M \to N$ into a fibered manifold morphism $T^A f : T^A M \to T^A N$. The most important examples are the functors T^r_k of k-dimensional velocities of order r,

$$T^r_k M = J^r_0(\mathbb{R}^k, M)$$

and the tangent functor $T = T^1_1$. By [6], there is a complete description of all product preserving bundle functors on $\mathcal{M}f$ in terms of Weil functors: every product preserving bundle functor F on $\mathcal{M}f$ is a Weil functor $F = T^A$, where the corresponding Weil algebra is of the form $A = F\mathbb{R}$. The well known time-dependent tangent bundle $\mathbb{R}\times TM$ can be generalized as follows:

Definition. The *time-dependent Weil bundle* $T^A_{\mathbb{R}}$ corresponding to the Weil algebra A is defined by $T^A_{\mathbb{R}} M = \mathbb{R} \times T^A M$ for every manifold M and by $T^A_{\mathbb{R}} f = \mathrm{Id}_{\mathbb{R}} \times T^A f : T^A_{\mathbb{R}} M \to T^A_{\mathbb{R}} N$ for every smooth map $f : M \to N$.

Clearly, the restriction of a time-dependent Weil bundle $T^A_{\mathbb{R}}$ to the category $\mathcal{M}f_m$ is a natural bundle over m-manifolds, which will be called the *natural m-bundle* $T^A_{\mathbb{R}}$.

Definition. A connection $\Gamma : \mathbb{R} \times T^A M \to J^1(\mathbb{R} \times T^A M \to \mathbb{R} \times M)$ on a time-dependent Weil bundle is called a *time-dependent connection* (or a *dynamical connection*).

If we denote by (x^i) the local coordinates on M, by (y^p) the additional fiber coordinates on $T^A M$ and by t the coordinate on $\mathbb{R}$, then a time-dependent connection Γ has equations

$$dy^p = \Gamma^p_i(t,x,y)dx^i + \Gamma^p(t,x,y)dt. \tag{7}$$

We have

Lemma 1. *Each connection Δ on $T^A M \to M$ determines a dynamical connection $\Gamma := \widetilde{\Delta}$ on $\mathbb{R} \times T^A M \to \mathbb{R} \times M$.*

Proof. A connection $\Delta : T^A M \to J^1(T^A M \to M)$ is of the form $\Delta(x,y) = j^1_x u$, where $u : M \to T^A M$ is a section. Then the map $s : \mathbb{R}\times M \to \mathbb{R}\times T^A M$ defined by $s = \mathrm{Id}_{\mathbb{R}} \times u$ is another section and we can define a connection $\Gamma := \widetilde{\Delta} : \mathbb{R}\times T^A M \to J^1(\mathbb{R}\times T^A M \to \mathbb{R}\times M)$ by $\Gamma(t,x,y) = j^1_{t,x} s$. ∎

If a connection Δ on $T^A M$ has the coordinate form

$$dy^p = \Delta_i^p(x, y)dx^i,$$

then the equations of the induced connection $\Gamma := \widetilde{\Delta}$ on $\mathbb{R} \times T^A M$ are

$$\Gamma_i^p = \Delta_i^p, \quad \Gamma^p = 0.$$

Quite analogously we can prove

LEMMA 2. *A dynamical connection Γ on $\mathbb{R} \times T^A M$ determines a one-parameter family of autonomous connections $\{\Delta_t;\ t \in \mathbb{R}\}$ on $T^A M$.*

Clearly, each connection Δ_t from this one-parameter family has equations

$$dy^p = \Gamma_i^p(t, x, y)dx^i. \tag{8}$$

LEMMA 3. *For a given $t \in \mathbb{R}$, a dynamical connection Γ on $\mathbb{R} \times T^A M$ can be expressed in the form*

$$\Gamma = \widetilde{\Delta}_t + \Psi_t \tag{9}$$

where $\widetilde{\Delta}_t$ is a dynamical connection on $\mathbb{R} \times T^A M$ induced from a fixed connection Δ_t on $T^A M$ and $\Psi_t \in C^\infty(V(\mathbb{R} \times T^A M) \otimes T^*(\mathbb{R} \times M))$ is an affinor on $\mathbb{R} \times T^A M$.

Proof. Let $\{\Delta_t;\ t \in \mathbb{R}\}$ be the one-parameter family of connections on $T^A M$ from Lemma 2 and denote by $\widetilde{\Delta}_t$ the connection on $\mathbb{R} \times T^A M$ induced by Δ_t. The first jet prolongation $J^1 Y \to Y$ of a fibered manifold $Y \to M$ is an affine bundle with the associated vector bundle $VY \otimes T^*M$, so that $J^1(\mathbb{R} \times T^A M) \to \mathbb{R} \times T^A M$ is an affine bundle with the associated vector bundle $V(\mathbb{R} \times T^A M) \otimes T^*(\mathbb{R} \times M)$. Then the difference $\Psi_t := \Gamma - \widetilde{\Delta}_t$ of connections is a section of the associated vector bundle. ■

Obviously, the connection $\widetilde{\Delta}_t$ on $\mathbb{R} \times T^A M$ has equations (8) and the affinor $\Psi_t : T(\mathbb{R} \times M) \to V(\mathbb{R} \times T^A M)$ is of the form

$$(dt, dx^i) \mapsto (0, 0, \Gamma^p(t, x, y)dt).$$

We can see that Ψ_t is even a soldering form on $\mathbb{R} \times T^A M$.

4. Natural affinors on time-dependent Weil bundles. We first recall the description of all natural affinors on the Weil bundle T^A. Every element $a \in A$ induces a natural affinor $Q(a)$ on the natural m-bundle T^A as follows. Denote by $\mu_M : \mathbb{R} \times TM \to TM$ the multiplication of tangent vectors by reals. Applying the functor T^A we obtain $T^A \mu_M : T^A \mathbb{R} \times T^A TM \to T^A TM$. From the general theory of Weil functors it follows that $T^A \mathbb{R} = A$ and there is a canonical exchange map $T^A TM \approx TT^A M$. Hence $T^A \mu_M$ can be interpreted as a map $A \times TT^A M \to TT^A M$ and its restriction to $a \in A$ defines a natural affinor $Q(a)_M : TT^A M \to TT^A M$. By Kolář and

Modugno [5], all natural affinors on the natural m-bundle T^A are of the form $Q(a)$, $a \in A$.

The natural affinor $Q(a)$ on T^A induces a natural affinor $\widetilde{Q}(a)$ on $T^A_{\mathbb{R}}$ by means of the product structure on $\mathbb{R} \times T^A M$. Analogously, the identity of $T\mathbb{R}$ determines another natural affinor $\widetilde{\mathrm{Id}}_{T\mathbb{R}}$ on $T^A_{\mathbb{R}}$.

Let X be a vector field on $T^A M$ and dt be the canonical one-form on $\mathbb{R}$. Then the tensor product $X \otimes dt$ is an affinor on $\mathbb{R} \times T^A M$. This is a general model of the third type of natural affinors on $\mathbb{R} \times T^A M$, which are tensor products of absolute vector fields on $T^A M$ with the canonical one-form dt on $\mathbb{R}$. Denote by $\mathrm{Der}\, A$ the space of all derivations of the algebra A. By [6], every element $D \in \mathrm{Der}\, A$ determines an absolute vector field $\overline{D}$ on the natural m-bundle T^A in the following way. We have an identification of $\mathrm{Der}\, A$ with the Lie algebra of the Lie group $\mathrm{Aut}\, A$ of all automorphisms of A. Hence $D \in \mathrm{Der}\, A$ is of the form $\frac{d}{dt}\big|_0 \delta(t)$, where $\delta(t)$ is a curve on $\mathrm{Aut}\, A$. By [6], every $\delta(t)$ determines a natural transformation $\overline{\delta}(t)_M : T^A M \to T^A M$ and we can define $\overline{D}_M = \frac{d}{dt}\big|_0 \overline{\delta}(t)_M$. Proposition 42.8 from [6] states that all absolute vector fields on T^A are of the form $\overline{D}$, $D \in \mathrm{Der}\, A$. Then the tensor products $\overline{D}_M \otimes dt$ define a natural affinor $\overline{D} \otimes dt$ on $T^A_{\mathbb{R}}$ for every $D \in \mathrm{Der}\, A$. I. Kolář and the author have proved in [3] the following:

Proposition 1. *All natural affinors on the natural m-bundle $T^A_{\mathbb{R}}$ are linear combinations of*

(i) $\widetilde{Q}(a)$, $a \in A$,

(ii) $\overline{D} \otimes dt$, $D \in \mathrm{Der}\, A$,

(iii) $\widetilde{\mathrm{Id}}_{T\mathbb{R}}$,

the coefficients being arbitrary smooth functions on $\mathbb{R}$.

Recall that (t, x^i, y^p) are the local coordinates on $\mathbb{R} \times T^A M$. Then the natural affinors from Proposition 1 are of the form:

$$\text{(i)} \qquad \begin{aligned} a = e: &\quad \widetilde{Q}(e)(dt, dx^i, dy^p) = (0, dx^i, dy^p), \\ a \in A \text{ nilpotent}: &\quad \widetilde{Q}(a)(dt, dx^i, dy^p) = (0, 0, Q^p_i dx^i + Q^p_q dy^q), \end{aligned}$$

where $e \in A$ denotes the unit element. We can see that for nilpotent $a \in A$, all natural affinors $\widetilde{Q}(a)$, are vertical, i.e. $\widetilde{Q}(a) : T(\mathbb{R} \times T^A M) \to V(\mathbb{R} \times T^A M \to \mathbb{R} \times M)$.

Clearly, every absolute vector field $\overline{D}$ on $T^A M$ is vertical, so that also all affinors $\overline{D} \otimes dt$, $D \in \mathrm{Der}\, A$, are vertical. Moreover, all such affinors are even soldering forms $T(\mathbb{R} \times M) \to V(\mathbb{R} \times T^A M \to \mathbb{R} \times M)$ of the form

$$\text{(ii)} \qquad (\overline{D} \otimes dt)(dt, dx^i) = (0, 0, Q^p dt).$$

Finally

(iii) $$\widetilde{\mathrm{Id}}_{T\mathbb{R}}(dt, dx^i, dy^p) = (dt, 0, 0).$$

From (i) and (iii) we can see that the sum $\widetilde{Q}(e) + \widetilde{\mathrm{Id}}_{T\mathbb{R}}$ is an identical affinor on $\mathbb{R} \times T^A M$.

For example, on the time-dependent tangent bundle $\mathbb{R} \times TM$, all natural affinors are generated by $Q_1(dt, dx^i, dy^p) = (0, dx^i, dy^p)$, $Q_2(dt, dx^i, dy^p) = (0, 0, dx^i)$, $Q_3(dt, dx^i, dy^p) = (0, 0, y^p dt)$, $Q_4(dt, dx^i, dy^p) = (dt, 0, 0)$.

5. Torsions of a time-dependent connection. Let $\Gamma : \mathbb{R} \times T^A M \to J^1(\mathbb{R} \times T^A M \to \mathbb{R} \times M)$ be a time-dependent connection. The Frölicher–Nijenhuis bracket of Γ with the natural affinors $\widetilde{Q}(a)$, $\overline{D} \otimes dt$ and $\widetilde{\mathrm{Id}}_{T\mathbb{R}}$ from Proposition 1 gives rise to three types of torsion of Γ. In what follows we shall discuss these torsions in detail.

I. $\tau_a := [\Gamma, \widetilde{Q}(a)]$, $a \in A$.

By a direct computation we deduce from (3) that $\tau_e = 0$, so that $\tau_{\lambda e} = 0$ for $\lambda \in \mathbb{R}$. Let Δ_t be a fixed connection on $T^A M$ from the one-parameter family of connections induced by Γ (see Lemma 2) and let $Q(a)_M : TT^A M \to TT^A M$ be the affinor on $T^A M$ defined by Kolář and Modugno. Then

$$\tau_{a,t} := [\Delta_t, Q(a)] \quad \text{for nilpotent } a \in A$$

is the torsion of Δ_t on $T^A M$, $\tau_{a,t} \in C^\infty(VT^A M \otimes \bigwedge^2 T^* T^A M)$. Obviously, if $a = \lambda e$, $\lambda \in \mathbb{R}$, then $\tau_{a,t} = 0$. Denote by $\widetilde{\tau}_{a,t}$ the vector-valued two-form on $\mathbb{R} \times T^A M$ induced by $\tau_{a,t}$ by means of the product structure. By Lemma 3, the connection Γ can be written in the form $\Gamma = \widetilde{\Delta}_t + \Psi_t$, where Ψ_t is an affinor on $\mathbb{R} \times T^A M$. Then we can write

$$[\Gamma, \widetilde{Q}(a)] = [\widetilde{\Delta}_t, \widetilde{Q}(a)] + [\Psi_t, \widetilde{Q}(a)] = \widetilde{[\Delta_t, Q(a)]} + [\Psi_t, \widetilde{Q}(a)],$$

so that we have proved

Proposition 2. *Let $a \in A$. The torsion τ_a is of the form*

$$\tau_a = \begin{cases} 0 & \text{for } a = \lambda e,\ \lambda \in \mathbb{R}, \\ \widetilde{\tau}_{a,t} + \tau^*_{a,t} & \text{for nilpotent } a, \end{cases}$$

where $\tau^*_{a,t} = [\Psi_t, \widetilde{Q}(a)]$.

We can see that for fixed $t \in \mathbb{R}$, the torsion τ_a on $\mathbb{R} \times T^A M$ can be expressed as a sum of the extension $\widetilde{\tau}_{a,t}$ of the "autonomous" torsion $\tau_{a,t}$ on $T^A M$ and some difference tensor $\tau^*_{a,t}$. Since both affinors $\widetilde{Q}(a)$ and Ψ_t are vertical, we have $\tau^*_{a,t} \in C^\infty(V(\mathbb{R} \times T^A M) \otimes \bigwedge^2 T^*(\mathbb{R} \times T^A M))$.

Corollary 1. *If a time-dependent connection Γ on $\mathbb{R}\times T^AM$ is induced by a connection Δ on T^AM, then the difference tensor $\tau^*_{a,t}$ vanishes.*

Example 1. Let Γ be a time-dependent connection on $\mathbb{R}\times TM$ with equations (7). The canonical almost tangent structure J is the only natural affinor on TM. Then J induces a natural affinor $\widetilde{J}(dt, dx^i, dy^p) = (0, 0, dx^i)$ on $\mathbb{R}\times TM$. By (3), the corresponding torsion $\tau_1 := [\Gamma, \widetilde{J}]$ on $\mathbb{R}\times TM$ is of the form

$$\tau_1 = \frac{\partial \Gamma^p_i(t,x,y)}{\partial y^j}\frac{\partial}{\partial y^p}\otimes (dx^i\wedge dx^j) + \frac{\partial \Gamma^p(t,x,y)}{\partial y^j}\frac{\partial}{\partial y^p}\otimes (dt\wedge dx^j). \tag{10}$$

The first term of (10) is the torsion of an autonomous connection $\Delta_t : TM \to J^1TM$ on TM, which was geometrically constructed by Kolář and Modugno in [5].

Remark 1. Up till now, geometrical constructions of all torsions on T^AM are known only for some particular Weil functors T^A. For example, Kolář and Modugno [5] constructed all torsions on the tangent bundle TM, on the bundle of k-dimensional 1-velocities T^1_kM, on the bundle T^2_1M and on the frame bundle PM. Further, Kureš described torsions on iterated tangent bundles, on the bundles T^r_1M and on non-holonomic bundles of higher order velocities (see [10], [9] and [11]). But there is no universal geometrical description of all general torsions on T^AM for every Weil functor T^A.

II. $\tau_{\overline{D}} = [\Gamma, \overline{D}\otimes dt]$, where $\overline{D} : T^AM \to TT^AM$ is the absolute vector field determined by $D \in \operatorname{Der} A$.

We first show that one can define the exterior product of an affinor and a one-form as follows. Let $K \in C^\infty(TM\otimes T^*M)$ be an affinor on M and $\omega : M \to T^*M$ a one-form. Then K is locally a sum of $(X\otimes\varphi)$'s, where $X : M \to TM$ is a vector field and $\varphi : M \to T^*M$ is a one-form. We can define $K\wedge\omega \in C^\infty(TM\otimes \bigwedge^2 T^*M)$ by $(X\otimes\varphi)\wedge\omega = X\otimes(\varphi\wedge\omega)$.

Take a fixed connection Δ_t from the one-parameter family of connections on T^AM induced by Γ and denote by

$$\mathcal{H}_{\overline{D},t} := [\Delta_t, \overline{D}\,]$$

the general tension of Δ_t. By Section 2, $\mathcal{H}_{\overline{D},t} : TM \to VT^AM$ is a soldering form on T^AM. Denote further by $\widetilde{\mathcal{H}}_{\overline{D},t} : T(\mathbb{R}\times M) \to V(\mathbb{R}\times T^AM)$ the extension of $\mathcal{H}_{\overline{D},t}$ to an affinor on $\mathbb{R}\times T^AM$ by means of the product structure.

Proposition 3. *Let $\overline{D}$ be an absolute vector field on T^AM and $\mathcal{H}_{\overline{D},t} := [\Delta_t, \overline{D}\,]$ the general tension of an induced connection Δ_t on T^AM. Then*

$$\tau_{\overline{D}} := [\Gamma, \overline{D}\otimes dt] = \widetilde{\mathcal{H}}_{\overline{D},t}\wedge dt.$$

In this way the torsion $\tau_{\overline{D}}$ of a dynamical connection Γ on $\mathbb{R}\times T^AM$ is completely determined by the general tension $\mathcal{H}_{\overline{D},t}$ of an induced connection Δ_t on T^AM. Further, $\tau_{\overline{D}}\in C^\infty(V(\mathbb{R}\times T^AM)\otimes\bigwedge^2 T^*(\mathbb{R}\times M))$ because all affinors $\overline{D}\otimes dt$ are soldering forms on $\mathbb{R}\times T^AM$.

Proof of Proposition 3. Every absolute vector field $\overline{D}$ on T^AM is vertical, so that its coordinate form in local fiber coordinates (x^i,y^p) on T^AM is $\overline{D}=A^p\partial/\partial y^p$. Then the affinor $\overline{D}\otimes dt$ is a soldering form on $\mathbb{R}\times T^AM$ of the form $(dt,dx^i,dy^p)\mapsto(0,0,A^p dt)$. From (3) it follows that

$$\tau_{\overline{D}}=[\Gamma,\overline{D}\otimes dt]=\left(\frac{\partial A^p}{\partial y^q}\,\Gamma^q_i-A^q\,\frac{\partial\Gamma^p_i}{\partial y^q}\right)\frac{\partial}{\partial y^p}\otimes(dx^i\wedge dt).$$

On the other hand, applying the general formula 8.10 from [6] for the coordinate form of the Frölicher–Nijenhuis bracket, we obtain directly

$$\mathcal{H}_{\overline{D},t}=[\Delta_t,\overline{D}\,]=\left(\frac{\partial A^p}{\partial y^q}\,\Gamma^q_i-A^q\,\frac{\partial\Gamma^p_i}{\partial y^q}\right)\frac{\partial}{\partial y^p}\otimes dx^i.\ \blacksquare$$

Example 2. The only absolute vector field on TM is the Liouville vector field $L=y^p\frac{\partial}{\partial y^p}$ and the corresponding affinor $L\otimes dt$ on $\mathbb{R}\times TM$ is of the form $(dt,dx^i,dy^p)\mapsto(0,0,y^p dt)$. By a direct computation we deduce from (3) that

$$\tau_L:=[\Gamma,L\otimes dt]=\left(\Gamma^p_i-\frac{\partial\Gamma^p_i}{\partial y^l}\,y^l\right)\frac{\partial}{\partial y^p}\otimes(dx^i\wedge dt).$$

Clearly, from the formula (6) for the classical tension $\mathcal{H}$ on TM we can see that $\tau_L=\widetilde{\mathcal{H}}\wedge dt$, where $\widetilde{\mathcal{H}}$ is the extension of $\mathcal{H}$ to $\mathbb{R}\times TM$ by means of the product structure.

We remark that dynamical connections on $\mathbb{R}\times TM$ were also studied by Vondra [15]. He called the difference $\tau_L-\tau_1$ a *weak torsion.*

Corollary 2. *Let $\Gamma=\widetilde{\Delta}$ be a connection on $\mathbb{R}\times TM$ induced by a connection Δ on TM. Then $\tau_L=0$ if and only if Δ is linear.*

III. $\tau_t:=[\Gamma,\widetilde{\mathrm{Id}}_{T\mathbb{R}}]$.

Using (3) we compute directly

$$\tau_t=\left(\frac{\partial\Gamma^p_j}{\partial t}\right)\frac{\partial}{\partial y^p}\otimes(dt\wedge dx^j).$$

The torsion τ_t has the following geometric interpretation:

Proposition 4. (a) *If a connection Γ on $\mathbb{R}\times T^AM$ is induced by a connection Δ on T^AM, then $\tau_t=0$.*

(b) *If $\tau_t=0$, then Γ induces a unique connection Δ on T^AM. In this case the expression* (9) *is of the form $\Gamma=\widetilde{\Delta}+\Psi_t$.*

Proof. The equation $\tau_t = 0$ is equivalent to the condition that the Γ_i^p are independent of t. ∎

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Department of Mathematics
Brno University of Technology
FSI VUT
Brno, Technická 2
616 69 Brno, Czech Republic
E-mail: doupovec@mat.fme.vutbr.cz

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