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MINIMAL NONHOMOGENEOUS CONTINUA

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Abstract. We show that there are (1) nonhomogeneous metric continua that admit minimal noninvertible maps but have the fixed point property for homeomorphisms, and (2) nonhomogeneous metric continua that admit both minimal noninvertible maps and minimal homeomorphisms. The former continua are constructed as quotient spaces of the torus or as subsets of the torus, the latter are constructed as subsets of the torus.

1. Introduction. Minimality is a central topic in topological dynamics (see e.g. [Au], [Br] and [deV]). A dynamical system (X, f), where X is a topological space and $f : X \to X$ is continuous, is called (topologically) *minimal* if there is no proper subset $M \subseteq X$ which is nonempty, closed and f-invariant (i.e., $f(M) \subseteq M$). In such a case we also say that the map f itself is *minimal*. Clearly, the system (X, f) is minimal if and only if the (forward) orbit of every point $x \in X$ is dense in X.

We will call a space X *minimal* if it admits a minimal map $f : X \to X$. An important and old question is which compact Hausdorff spaces (compact metric spaces, continua, ...) admit minimal maps.

If a space allows a minimal map, the proof usually builds on a standard example of a minimal homeomorphism (see [E1], [P], [GW]). Such standard examples are the Cantor set and the torus of dimension ≥ 2, which admit both minimal noninvertible maps and minimal homeomorphisms (see [KST]). The circle (see [AK]) admits no minimal noninvertible map, while admitting a minimal homeomorphism. The Klein bottle admits a minimal homeomorphism [E2], [P] (and we conjecture that also a minimal noninvertible map). A general theorem by Fathi & Herman [FH] ties the existence of minimal diffeomorphisms to the existence of locally free diffeomorphisms.

Proofs of nonminimality often rely on the fixed (periodic) point property. For example, any homeomorphism on a compact manifold with nonzero

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Euler characteristic (homotopic to the identity or not) has a periodic point ([Fu] and [DG, p. 138]), hence all compact surfaces except the torus and the Klein bottle do not admit minimal homeomorphisms.

In this paper we give an affirmative answer to a question posed in [KST]:

Are there compact Hausdorff spaces that admit minimal noninvertible maps but do not admit any minimal homeomorphism (1)*?*

More precisely, we construct a continuum with these properties which is a nonhomogeneous quotient space of the torus (see Theorem A) or a nonhomogeneous subset of the torus (see Corollary D). (Recall that a *continuum* is a nondegenerate compact connected Hausdorff space. A space X is called *homogeneous* if for any two points $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$.) It is also worth noticing that our constructions give spaces that have the fixed point property for homeomorphisms but not for general continuous maps (see Corollaries B and D).

The classical examples of minimal maps were given on homogeneous spaces (circle, torus, Cantor set). Since nonhomogeneity of the space increases the likelihood that the space has the fixed (periodic) point (invariant subset) property, it was a natural question to ask whether there exists a nonhomogeneous minimal space. The first examples of nonhomogeneous compact metric spaces admitting minimal homeomorphisms were provided by Floyd in [Fl] (disconnected spaces) and by Jones, see [GH, Theorem 14.24] (metric continuum which is not locally connected). Today many examples of nonhomogeneous (usually disconnected) compact metric spaces admitting minimal homeomorphisms are known.

In this paper we use homeomorphisms of the torus with wandering domains (the analog of Denjoy's example on the circle) to show the existence of nonhomogeneous metric continua (in fact subsets of the torus which are similar to the Sierpiński curve of the sphere, on which however no minimal homeomorphism exists [G2], [AO]), admitting both minimal noninvertible maps and minimal homeomorphisms (see Theorem C). A slight modification of this construction also provides nonhomogeneous metric continua which admit minimal maps but *not* minimal homeomorphisms. Moreover, these continua are locally connected (see Corollary D and cf. Theorem A).

Remarks and open problems. In the case of homeomorphisms, there exist two (in general nonequivalent) definitions of minimality. Since this is sometimes a source of confusion, we repeat that in the present paper minimality means the density of all *forward* orbits regardless of whether the map is in-

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(1) Without the requirement that the space be Hausdorff the answer is trivially affirmative—take $X = \{a, b\}$ with $a \neq b$ and the topology $\mathcal{T} = \{\emptyset, X, \{b\}\}$.

vertible or not (2). For clarity, let us call a homeomorphism *weakly minimal* if all its *full* orbits are dense, i.e., $\overline{\bigcup_{i\in\mathbb{Z}} f^i(x)} = X$ for all $x \in X$. Obviously, minimal homeomorphisms are weakly minimal, but the converse is not true (take $f(n) = n+1$, $n \in \mathbb{Z}$). In compact metric spaces, minimality and weak minimality of a homeomorphism are equivalent [G1]. Note also that a noncompact, locally compact metric space does not admit any minimal map at all [G1].

The present paper deals with *discrete* dynamical systems. Concerning the analogous problem of the existence of minimal *flows* (i.e. with continuous time $t \in \mathbb{R}$; here minimality traditionally means the denseness of all full orbits), let us only mention that the Klein bottle is an example of a space that admits a minimal homeomorphism, but does not admit a minimal flow [Kn]. Ellis [E1] showed that the Cartesian product of the 2-torus and the Klein bottle admits a minimal flow (see [BM] and [I] for more details about the existence of minimal flows).

In the table below, we indicate for some spaces whether they allow a minimal homeomorphism, a minimal noninvertible map, resp. a weakly minimal homeomorphism.

	min. homeo.	min. noninv. map	weakly min. homeo.
Cantor set	yes	yes	yes
$\mathbb{S}^1$	yes	no, [AK]	yes
$\mathbb{S}^2$	no	no	no
$\mathbb{S}^2\setminus$finite set	no, [G1]	no, [G1]	no, [H], [LCY], [Fr]
$\mathbb{T}^2$	yes	yes, [KST]	yes
$\mathbb{T}^2 \setminus \{p\}$	no, [G1]	no, [G1]	yes, [A]
Klein bottle	yes, [E2], [P]	?	yes
closed annulus	no	?	no

(According to [A], a weakly minimal homeomorphism on the punctured torus $\mathbb{T}^2 \setminus \{p\}$ can be constructed as the time-1 map of an irrational flow on $\mathbb{T}^2$ with a rest point p; cf. [O]. Related results were presented in [LT] and [Eg].)

The next table summarizes some examples of metric continua which have prescribed properties: The upper two entries in each column indicate if the space admits or does not admit a minimal noninvertible map resp. a minimal homeomorphism. The lower two entries give examples of such spaces in the class of nonhomogeneous resp. homogeneous metric continua.

min. noninv. map	no	no	yes	yes
min. homeo	no	yes	no	yes
nonhomog. cont.	$[0,1]$	?	Thm. A	Thm. C
homog. cont.	$\mathbb{S}^2$	$\mathbb{S}^1$, [AK]	?	$\mathbb{T}^2$, [KST]

(2) Some authors call this notion *semiminimality* (see e.g. [G1]), while minimality (defined then only for homeomorphisms) is reserved for the denseness of all *full* orbits $\bigcup_{i\in\mathbb{Z}} f^i(x)$.

Of course, many open questions remain. The above tables contain several question marks. There are many more, for example:

(i) If spaces X and Y allow minimal homeomorphisms (maps), does the same hold for $X \times Y$?

(ii) Is the circle the only infinite continuum that admits a minimal homeomorphism but no minimal noninvertible map? The pseudo-circle is possibly a candidate.

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2. The constructions. Recall that an *arc* is any space which is homeomorphic to the closed interval $[0,1]$.

Let $f : X \to X$, $g : Y \to Y$ and $\varphi : X \to Y$ be continuous and surjective. If $\varphi \circ f = g \circ \varphi$, i.e., φ is a *semiconjugacy*, then f is called an *extension* of g and g a *factor* of f. This extension of g (factor of f) is called *almost one-to-one* if the semiconjugacy φ is an almost one-to-one map, i.e., if for every y in a residual subset of Y, $\mathrm{card}(\varphi^{-1}(y)) = 1$.

Let (X, ϱ) be a metric space and $f : X \to X$ be continuous. Then two different points $x, y \in X$ are called *asymptotic* if $\lim_{n\to\infty} \varrho(f^n x, f^n y) = 0$.

A decomposition $\mathcal{D}$ of a topological space X is *upper semicontinuous* (*u.s.c.*) if for each element E in $\mathcal{D}$ and each open set U containing E, there is an open set V such that $E \subset V \subset U$ and V is the union of members of $\mathcal{D}$.

Let us recall the classical *Moore theorem* from [Mo], [RS] and [Bo]. Let M be a surface without boundary (not necessarily orientable). By a *Moore decomposition* of a space M we understand any u.s.c. decomposition G of M such that each element g of G is a continuum having arbitrarily small neighborhoods (in M) homeomorphic to the plane $\mathbb{R}^2$. Obviously, each u.s.c. decomposition G of a surface M into elements which are disks, arcs and individual points is necessarily a Moore decomposition. The Moore theorem says that for every Moore decomposition G of M the quotient space M/G is homeomorphic to M. This construction may be less abstract than it seems. In complex dynamics, the *mating construction* between (the Julia sets of) polynomials on $\mathbb{C}$ is based on the Moore theorem [Mo], and provides extremely nice geometric results (see [T], [R2], [Mi]).

A continuous map S from the 2-torus $\mathbb{T}^2$ into itself is called a *skew product* if it is of the form $S(x,y) = (f(x),\ g(x,y))$. Obviously, if S is minimal then so is the circle map f. But then f is topologically conjugate to an irrational rotation $x \mapsto x+\alpha$ (see [AK]). A set of the form $\{x_0\}\times I \subseteq \mathbb{T}^2$ where I is an interval on the circle is said to be a *vertical interval* on the torus.

Using the Moore theorem, it is proved in [KST] that any minimal skew product homeomorphism of the 2-torus $\mathbb{T}^2$ having an asymptotic pair of points (an example of such a homeomorphism is constructed by Rees [R1]) has an almost one-to-one factor which is a noninvertible minimal map of $\mathbb{T}^2$. We will sketch the proof from [KST], because it will be used later.

Let $S(x,y) = (f(x),\ g(x,y))$ be a minimal skew product homeomorphism of $\mathbb{T}^2$. Further assume that S has an asymptotic pair of points $\{z_1, z_2\}$. Of course, these points lie in one fiber, i.e., they are of the form $z_1 = (x, y_1)$, $z_2 = (x, y_2)$.

Since the homeomorphism S is a skew product, the S-image of a vertical interval is again a vertical interval whose endpoints are the S-images of the endpoints of the original interval. The points z_1 and z_2 are the endpoints of two "complementary" vertical compact intervals. Since z_1 and z_2 are asymptotic and S is uniformly continuous, one of these two vertical intervals, denote it by I_0, is such that for $I_n := S^n(I_0)$ we have $\operatorname{diam} I_n \to 0$ as $n \to \infty$. In fact, Rees' example also gives $\operatorname{diam} I_n \to 0$ as $n \to -\infty$. Because I_0 is not attracted to a periodic orbit, we call I_0 a *wandering arc.*

Let $\mathcal{D}$ be the decomposition of $\mathbb{T}^2$ whose elements are the (pairwise disjoint) compact intervals I_n, $n \geq 0$, and the individual points from $\mathbb{T}^2 \setminus \bigcup_{n=0}^{\infty} I_n$. Consider the quotient space $\mathbb{T}^2/\mathcal{D}$. We can say that we *squeeze* the intervals I_n, $n \geq 0$, *to points.* Then by the Moore theorem the quotient space $\mathbb{T}^2/\mathcal{D}$ is homeomorphic to $\mathbb{T}^2$ (see [KST] for more details).

Since S maps an element of the decomposition $\mathcal{D}$ into an element of $\mathcal{D}$, there is a unique map $\Psi : \mathbb{T}^2/\mathcal{D} \to \mathbb{T}^2/\mathcal{D}$ with $\Psi \circ p = p \circ S$, where the semiconjugacy p is the natural projection $p : \mathbb{T}^2 \to \mathbb{T}^2/\mathcal{D}$ and Ψ is continuous (see [KST] or an analogous argument below, in the proof of Theorem A). Then Ψ is also minimal, being a factor of a minimal map. Obviously, it is an almost one-to-one factor of S. The map Ψ is noninvertible at the point $p(I_0)$. In fact, if we denote by I_{-1} the subset of $\mathbb{T}^2$ with $S(I_{-1}) = I_0$, then $\Psi(p(I_{-1})) = p(I_0)$. Note that the restriction of the natural projection p to the vertical interval I_{-1} is injective and so $p(I_{-1})$ is a homeomorphic image of I_{-1}. Hence it is an arc.

Taking into account that $\mathbb{T}^2/\mathcal{D}$ is homeomorphic to $\mathbb{T}^2$, we conclude that there is a minimal noninvertible map $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ which is an almost one-to-one factor of S and maps an arc on the torus $\mathbb{T}^2$ to a point.

Finally we are ready to prove the main result of this paper:

Theorem A. *Any minimal skew product homeomorphism of the* 2*-torus* $\mathbb{T}^2$ *having an asymptotic pair of points has a factor which is a noninvertible minimal map of a* 2*-dimensional nonhomogeneous metric continuum* X, *such that any homeomorphism of* X *has a fixed point.*

Proof. Let $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ be the map described before the statement of the theorem and let $I \subset \mathbb{T}^2$ be an arc such that $\Phi(I)$ is a point $x_I \in \mathbb{T}^2$.

Let $\varkappa : [0,1] \to I$ be a parametrization of I. Let $\mathcal{D}_I$ be the decomposition of $\mathbb{T}^2$ whose elements are the set $\varkappa(\{0,1\})$ and all the individual points of $\mathbb{T}^2 \setminus \varkappa(\{0,1\})$. We say that the interval I is *pinched*. Consider the quotient space $X := \mathbb{T}^2/\mathcal{D}_I$. Since $\mathcal{D}_I$ is obviously a u.s.c. decomposition of $\mathbb{T}^2$, the space X is a metric continuum (see Theorem 3.10 of [Na]).

Any element of the decomposition $\mathcal{D}_I$ is mapped into an element of $\mathcal{D}_I$ by Φ. Hence there is a unique map $F : X \to X$ with $F \circ p = p \circ \Phi$, where the semiconjugacy is the natural projection $p : \mathbb{T}^2 \to X$. We prove that F is continuous. Since the map $h = p \circ \Phi :\ \mathbb{T}^2 \to X$ is continuous and $h = F \circ p$, for each open set $U \subseteq X$, $h^{-1}(U) = p^{-1}(F^{-1}(U))$ is open in $\mathbb{T}^2$. Then, since p is a quotient map, the set $F^{-1}(U)$ is open in X. Thus F is continuous and, being a factor of a minimal map, also minimal.

The space X is the union of the open set $V = p(\mathbb{T}^2 \setminus \{\varkappa(0), \varkappa(1)\})$ and the closed set $W = p(\{\varkappa(0), \varkappa(1)\})$. Clearly, V is the set of points with Euclidean neighborhoods, and W the set of points without. Hence, both V and W are invariant under homeomorphisms and X is a nonhomogeneous space. But W is a single point. Thus any homeomorphism of X has at least one fixed point at W. ■

Remark. The idea of the pinched torus in the previous proof can be generalized. If f is a noninvertible minimal map on a compact manifold M, then taking two points with the same f-image and collapsing them to one point, we get a factor which is a minimal noninvertible map on a nonhomogeneous "pinched manifold" which has the fixed point property for homeomorphisms.

Corollary B. *There is a* 2*-dimensional nonhomogeneous metric continuum*, *in fact a quotient space of the torus*, *which has the fixed point property for homeomorphisms but does not have the fixed point property for continuous selfmaps.*

Similar constructions can be made on a 1-dimensional space as well. Following [No], we say that a homeomorphism f of the torus $\mathbb{T}^2$ is of *Denjoy type* if it is semiconjugate (but not conjugate) to some irrational rotation of that torus by a map h such that $\{x \in \mathbb{T}^2 : h^{-1}(x)$ is not a singleton$\}$ is at most a countable set. A homeomorphism f of Denjoy type is said to be of *Sierpiński type* if it has a *wandering* domain, i.e. an open set D such that

$f^n(D) \cap f^m(D) = \emptyset$ for all $n \neq m \in \mathbb{Z}$, and $f^n(U)$ does not converge to a periodic orbit as $n \to \pm\infty$. The simplest topological way of constructing a Sierpiński homeomorphism is by "blowing up" the points of one (dense) full orbit of an irrational torus rotation to round disks whose diameters tend to zero. This fact is a folklore (see [No, p. 132]). It seems that the construction has not been described explicitly and with full details in the literature ([3]). Nevertheless, using other methods it was shown in [Ak], [NV], [McS] that even Sierpiński type diffeomorphisms exist. McSwiggen has constructed, for any $\alpha < 1$, a $C^{2+\alpha}$ diffeomorphism f of Sierpiński type (the diameters of the mentioned disks tend to zero).

Let M be a surface and $A \subset M$ a *curve*, i.e., a one-dimensional continuum. Then A is said to be an *S-curve* in M (cf. [W]) if it is locally connected and there exists a sequence $\{D_i\}$ of mutually disjoint closed disks in M such that $A = M \setminus \bigcup_{i=1}^{\infty} \operatorname{int} D_i$. (As A is one-dimensional, $\bigcup_{i=1}^{\infty} \operatorname{int} D_i$ is necessarily dense in M. If M is compact then, due to the local connectivity of A, $\operatorname{diam} D_i \to 0$.)

For simplicity, let f be a Sierpiński type homeomorphism obtained from an irrational rotation by blowing up one orbit to disks. These disks are obviously wandering. The map f has a minimal set $\widetilde{S} \subsetneq \mathbb{T}^2$ which is an S-curve. (It is similar ([4]) to the Sierpiński curve on the sphere, but in contrast to the case of the torus, the Sierpiński curve on the sphere does not admit any minimal homeomorphism [G2], [AO].) The set $\widetilde{S}$ emerges from removing the full orbit of wandering disks from $\mathbb{T}^2$. More precisely, we remove the interiors of these closed disks from the torus, i.e., $\widetilde{S}$ can be written in the form $\widetilde{S} = \mathbb{T}^2 \setminus \operatorname{orb}(\operatorname{int} D)$. It is one-dimensional, because every $p \in \widetilde{S}$ has arbitrarily small neighborhoods U such that $\partial U \cap \widetilde{S}$ is contained in a Cantor set. Secondly, $\widetilde{S}$ is nonhomogeneous, because points in $\bigcup_i f^i(\partial D)$ are not enclosed by topological circles in $\widetilde{S}$ of arbitrarily small diameter, whereas points in $\widetilde{S} \setminus \bigcup_i f^i(\partial D)$ are. Finally, $\widetilde{S}$ is connected and locally connected.

THEOREM C. *There exists a 1-dimensional nonhomogeneous continuum on the 2-torus that admits both a minimal homeomorphism and a minimal noninvertible map.*

Proof. Start with a Sierpiński type homeomorphism f obtained from an irrational rotation of the torus by blowing up one orbit to disks. Take the above-mentioned set $\widetilde{S} \subsetneq \mathbb{T}^2$. The restriction of f to this set is a minimal homeomorphism. We are going to show that this set admits also a minimal

([3]) A similar construction on the 2-sphere can be found in [AO]. It should be noted that the underlying surface is of importance, because the example in the 2-sphere allows no minimal homeomorphism, while examples on the torus do.

([4]) Note that two S-curves A_M on a surface M and A'_M on a surface M' are homeomorphic if and only if the surfaces M and M' are homeomorphic [Bo].

noninvertible map. To this end take an arc I_0 in the boundary of a wandering disk D_0. Then I_0 has asymptotic endpoints, and the diameters of $I_n := f^n(I_0)$ tend to 0 for $n \to \pm\infty$. Write also $D_n = f^n(D_0)$, $n \in \mathbb{Z}$.

Repeat the construction of a quotient space described at the beginning of this section. Let $\mathcal{D}$ be the decomposition of $\mathbb{T}^2$ whose elements are the (pairwise disjoint) compact intervals I_n, $n \geq 0$, and the individual points from $\mathbb{T}^2 \setminus \bigcup_{n=0}^{\infty} I_n$. Then by the Moore theorem the quotient space $\mathbb{T}^2/\mathcal{D}$ is homeomorphic to $\mathbb{T}^2$ (see [KST] for more details). Recall that any two S-curves in a given surface are homeomorphic [Bo]. Since all intervals I_n are contained in $\widetilde{S}$ and $\widetilde{S} = \mathbb{T}^2 \setminus \bigcup_{n\in\mathbb{Z}} \operatorname{int} D_n$, the quotient space $\widetilde{S}/\mathcal{D}$ is an S-curve and therefore is also homeomorphic to $\widetilde{S}$.

Since f maps an element of the decomposition $\mathcal{D}$ into an element of $\mathcal{D}$, there is a unique map $\widetilde{f} : \widetilde{S}/\mathcal{D} \to \widetilde{S}/\mathcal{D}$ with $\widetilde{f} \circ p = p \circ f|_{\widetilde{S}}$, where the semiconjugacy p is the natural projection $p : \widetilde{S} \to \widetilde{S}/\mathcal{D}$ and $\widetilde{f}$ is continuous. The map $\widetilde{f}$ is minimal, being a factor of the minimal map $f|_{\widetilde{S}}$. Obviously, it is an almost one-to-one factor of $f|_{\widetilde{S}}$. The map $\widetilde{f}$ is noninvertible at the point $p(I_0)$.

Taking into account that $\widetilde{S}/\mathcal{D}$ is homeomorphic to $\widetilde{S}$, we deduce that there is a minimal noninvertible map $\Phi : \widetilde{S} \to \widetilde{S}$ which is an almost one-to-one factor of $f|_{\widetilde{S}}$ and maps an arc on $\widetilde{S}$ to a point. ∎

This example on $\widetilde{S}$ can easily be adapted (cf. the proof of Theorem A) to yield the following

Corollary D. *There are* 1*-dimensional nonhomogeneous continua on the* 2*-torus that admit minimal noninvertible maps but have the fixed point property for homeomorphisms.*

Proof. Let $\Phi : \widetilde{S} \to \widetilde{S}$ be the minimal noninvertible map constructed in the proof of Theorem C. Recall that on the boundary of the open disk D_{-1} which is complementary to $\widetilde{S}$ there is an arc which is mapped by Φ to a point. Take two points a, b from that arc. Collapse a and b to one point and denote this point by z. We are going to prove that what we get is (up to homeomorphism) the required continuum X with a minimal noninvertible selfmap.

First note that X is homeomorphic to a subset of the torus. To see this, imagine an arc through the interior of D_{-1} connecting a and b, and collapse this arc to a point. The resulting space contains X, and is homeomorphic to a torus (use the Moore theorem).

Clearly Φ induces a minimal noninvertible map on X. On the other hand, X has the fixed point property for homeomorphisms, because z is the only point in X that has a neighborhood U in X such that $U \setminus \{z\}$ is not connected. ∎

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