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NOTE ON THE ISOMORPHISM PROBLEM FOR WEIGHTED UNITARY OPERATORS ASSOCIATED WITH A NONSINGULAR AUTOMORPHISM

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Abstract. We give a negative answer to a question put by Nadkarni: Let S be an ergodic, conservative and nonsingular automorphism on $(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m)$. Consider the associated unitary operators on $L^2(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m)$ given by $\widetilde{U}_S f = \sqrt{d(m \circ S)/dm} \cdot (f \circ S)$ and $\varphi \cdot \widetilde{U}_S$, where φ is a cocycle of modulus one. Does spectral isomorphism of these two operators imply that φ is a coboundary? To answer it negatively, we give an example which arises from an infinite measure-preserving transformation with countable Lebesgue spectrum.

1. Question. Let $(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m)$ be a standard probability Borel space and let $S : (\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m) \to (\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m)$ be an automorphism nonsingular with respect to the measure m (i.e. $m \circ S \equiv m$). Given a cocycle $\varphi : \widetilde{X} \to \mathbb{S}^1$ we define two unitary operators acting on $L^2(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m)$:

$$\widetilde{U}_S f = \sqrt{d(m \circ S)/dm} \cdot (f \circ S), \qquad \widetilde{V}_{\varphi,S} f = \varphi \cdot \widetilde{U}_S f.$$

If φ is a coboundary then $\widetilde{U}_S$ and $\widetilde{V}_{\varphi,S}$ are spectrally isomorphic. Conversely, if S is additionally ergodic and preserves the measure m then spectral isomorphism of $\widetilde{U}_S$ and $\widetilde{V}_{\varphi,S}$ implies that φ is a coboundary.

The following question was put by M. G. Nadkarni in [4]: Let S be an ergodic and conservative transformation, nonsingular with respect to m. Does spectral isomorphism of $\widetilde{U}_S$ and $\widetilde{V}_{\varphi,S}$ imply that φ is a coboundary?

The answer to this question is negative. First notice that it is sufficient to give a counterexample in the family of infinite measure-preserving automorphisms. Namely, take an ergodic conservative infinite measure-preserving automorphism $S : (\widetilde{X}, \mathcal{B}_{\widetilde{X}}, \varrho) \to (\widetilde{X}, \mathcal{B}_{\widetilde{X}}, \varrho)$ (ϱ is σ-finite) with countable Lebesgue spectrum and a cocycle $\varphi : \widetilde{X} \to \mathbb{S}^1$ which is not a coboundary and for which the Koopman operator $U_S f = f \circ S$, $f \in L^2(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, \varrho)$, and the cor-

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responding weighted operator $V_{\varphi,S} = \varphi \cdot U_S$ are spectrally isomorphic. Then take an arbitrary finite measure m on $\mathcal{B}_{\widetilde{X}}$ equivalent to ϱ, i.e. $dm = Fd\varrho$ for a positive function $F \in L^1(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, \varrho)$. Then S is ergodic, conservative and nonsingular with respect to m and obviously φ is still not a coboundary. Moreover, spectral isomorphism of U_S and $V_{\varphi,S}$ implies spectral isomorphism of $\widetilde{U}_S$ and $\widetilde{V}_{\varphi,S}$. Indeed, the operator $W : L^2(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m) \to L^2(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, \varrho)$ given by $Wf = \sqrt{F} \cdot f$ establishes spectral isomorphism between $\widetilde{U}_S$ and U_S, as well as between $\widetilde{V}_{\varphi,S}$ and $V_{\varphi,S}$. Hence, any S and φ satisfying the above assumptions give a negative answer to the question considered.

Recall that some examples of ergodic, conservative and infinite measure-preserving transformations with countable Lebesgue spectrum are provided by infinite K-automorphisms (see [5]). An example of such an automorphism was given in [1] (see Example 1). We next show that for any ergodic and conservative S with countable Lebesgue spectrum we can find a cocycle φ with the required properties.

We will need some facts about L^∞-eigenvalues of a nonsingular automorphism. Recall that a complex number γ is said to be an *L^∞-eigenvalue* of an ergodic and nonsingular automorphism S if there exists a nonzero function $g \in L^\infty(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m)$ such that $g(Sx) = \gamma g(x)$ m-a.e. The group of all L^∞-eigenvalues, denoted by $e(S)$, is a subgroup of the circle group $\mathbb{S}^1$. Moreover, it was proved in [2] that if S is conservative and ergodic then $e(S) \subsetneq \mathbb{S}^1$. Therefore if $S : (\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m) \to (\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m)$ is conservative and ergodic then the constant cocycle $\varphi : \widetilde{X} \to \mathbb{S}^1$ given by $\varphi \equiv a \in \mathbb{S}^1 \setminus e(S)$ is not a coboundary. Indeed, if $a = \xi(Sx)/\xi(x)$ for a measurable function $\xi : \widetilde{X} \to \mathbb{S}^1$ then $a \in e(S)$ (since $\xi \in L^\infty(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m)$). Moreover, if the operator U_S has countable Lebesgue spectrum then the spectrum of $V_{a,S}$ is also countable Lebesgue, hence U_S and $V_{a,S}$ are spectrally isomorphic. Indeed, for an arbitrary $g \in L^2(\widetilde{X}, \mathcal{B}_{\widetilde{X}}, m)$ we get the following connection between its spectral measures with respect to both operators considered: $\sigma_{g,V_{a,S}} = \delta_a * \sigma_{g,S}$, and the cyclic subspaces generated by g with respect to both operators coincide.

Now the question is whether it is possible to find a nonconstant, for instance with integral zero, cocycle φ with the required properties.

2. Remarks on examples arising from cylindrical transformations. We now show another way of finding infinite measure-preserving transformations with countable Lebesgue spectrum. This approach gives some examples of such systems which have zero entropy, unlike K-automorphisms. Namely, we will study the so-called cylindrical transformations. For a given automorphism T of a standard probability Borel space $(X, \mathcal{B}, \mu)$ and a real cocycle $f : X \to \mathbb{R}$ we define a *cylindrical automorphism* of

$(X \times \mathbb{R}, \mathcal{B} \otimes \mathcal{B}_{\mathbb{R}}, \mu \otimes \lambda_{\mathbb{R}})$ ($\lambda_{\mathbb{R}}$ stands for Lebesgue measure) by

$$T_f(x,r) = (Tx, r + f(x)).$$

Such an automorphism preserves the (infinite) measure $\mu \otimes \lambda_{\mathbb{R}}$. Moreover, since $\mu \otimes \lambda_{\mathbb{R}}$ is nonatomic, the ergodicity of T_f implies its conservativity. The maximal spectral type of T_f is strictly connected with the maximal spectral types $\sigma_{V_{e^{2\pi icf},T}}$ of the operators $V_{e^{2\pi icf},T} : L^2(X,\mathcal{B},\mu) \to L^2(X,\mathcal{B},\mu)$, $c \neq 0$, and with the maximal spectral type σ_{Σ} of the translation $\Sigma = (\Sigma_t)_{t\in\mathbb{R}}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda_{\mathbb{R}})$ given by $\Sigma_t r = r + t$. Recall that the $\mathbb{R}$-action Σ has Lebesgue spectrum on $L^2(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda_{\mathbb{R}})$. Moreover, similarly to [3] (see also Lemmas 1 and 2 in [6]), the following result holds.

LEMMA 1. *The maximal spectral type of T_f on $L^2(X \times \mathbb{R}, \mathcal{B} \otimes \mathcal{B}_{\mathbb{R}}, \mu \otimes \lambda_{\mathbb{R}})$ is given by*

$$\sigma_{T_f} \equiv \int_{\mathbb{R}} \sigma_{V_{e^{2\pi icf},T}}\, d\sigma_{\Sigma}(c),$$

and for an arbitrary $h \in L^2(\mathbb{R}, \lambda_{\mathbb{R}})$ the maximal spectral type of T_f on $L^2(X,\mu) \otimes Z(h)$, where $Z(h) = \operatorname{span}\{h \circ \Sigma_t : t \in \mathbb{R}\}$, is given by

$$\sigma_{T_f|L^2(X,\mu)\otimes Z(h)} \equiv \int_{\mathbb{R}} \sigma_{V_{e^{2\pi icf},T}}\, d\sigma_{h,\Sigma}(c).$$

Now, if we could find f and T such that each $V_{e^{2\pi icf},T}$, $c \neq 0$, has Lebesgue spectrum on $L^2(X,\mathcal{B},\mu)$ then by Lemma 1, T_f has countable Lebesgue spectrum on $L^2(X \times \mathbb{R}, \mathcal{B} \otimes \mathcal{B}_{\mathbb{R}}, \mu \otimes \lambda_{\mathbb{R}})$.

Suppose that $Tx = x + \alpha$ is an irrational rotation on $(\mathbb{R}/\mathbb{Z}, \mu)$, where μ stands for Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ and the sequence of arithmetical means of partial quotients of α is bounded. Consider cocycles $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ with piecewise continuous second derivative such that f' has finitely many discontinuity points for which the one-sided limits exist with at least one equal to infinity and $f'(x) > 0$ for all $x \in \mathbb{R}/\mathbb{Z}$. It was shown in [6] that for such functions f the operators $V_{e^{2\pi icf},T}$, $c \neq 0$, have Lebesgue spectrum. However, we do not know if T_f is ergodic for this class of cocycles.

Nevertheless, we are able to give an example of ergodic cylindrical transformation with countable Lebesgue spectrum. Namely, take as T a Gaussian automorphism on $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}_{\mathbb{R}^{\mathbb{Z}}}, \mu_{\mathcal{G}})$. Assume that the process $(\Pi_n)_{n\in\mathbb{Z}}$ of projections on the nth coordinate is independent and the distribution ν of Π_0 is a centered Gaussian distribution on $\mathbb{R}$ (so in fact T is a Bernoulli system). Put $f = \Pi_0 : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$. Then T_{Π_0} preserves the measure $\mu_{\mathcal{G}} \otimes \lambda_{\mathbb{R}}$ and is ergodic, because T_{Π_0} is a random walk on $\mathbb{R}$ with transition probability determined by ν. Moreover, for each $c \neq 0$ the operator $V_{e^{2\pi ic\Pi_0},T}$ has Lebesgue spectrum on $L^2(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}_{\mathbb{R}^{\mathbb{Z}}}, \mu_{\mathcal{G}})$.

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