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POISSON SUSPENSIONS OF COMPACTLY REGENERATIVE TRANSFORMATIONS

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Abstract. For infinite measure preserving transformations with a compact regeneration property we establish a central limit theorem for visits to good sets of finite measure by points from Poissonian ensembles. This extends classical results about (noninteracting) infinite particle systems driven by Markov chains to the realm of systems driven by weakly dependent processes generated by certain measure preserving transformations.

1. Introduction. On a first encounter with infinite ergodic theory one is immediately led to ask what an infinite invariant measure can possibly tell us about the dynamics of a transformation. Consider a conservative ergodic nonsingular map T on some σ-finite measure space $(X, \mathcal{A}, m)$.

In the standard situation where T has an invariant *probability measure* $\mu \ll m$, the pointwise ergodic theorem shows that for any $A \in \mathcal{A}$,

$$\frac{1}{n} S_n(A) := \frac{1}{n} \sum_{k=0}^{n-1} 1_A \circ T^k \to \mu(A) \quad \mu\text{-a.e. on } X \text{ as } n \to \infty, \tag{1}$$

meaning that the invariant measure $\mu(A)$ of the set A asymptotically represents the frequency of visits of a μ-typical single orbit to A. Under additional assumptions on the (mixing) behaviour of the map T and on the set A (satisfied by various nontrivial and interesting examples), it is in fact possible to establish a central limit theorem (CLT) asserting that

$$\mu\left[\frac{S_n(A) - n\mu(A)}{\sigma(A)\sqrt{n}} \le t\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2}\, ds \quad \begin{array}{l}\text{for every } t \in \mathbb{R}\\ \text{as } n \to \infty,\end{array} \tag{2}$$

which provides us with detailed information about the convergence in (1) by clarifying the asymptotic form of the distribution of the T-*occupation times* $S_n(A)$.

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In contrast, if T preserves an *infinite* (yet σ-finite) *measure* $\mu \ll m$, then

$$(3) \qquad \frac{1}{n}\, S_n(A) = \frac{1}{n} \sum_{k=0}^{n-1} 1_A \circ T^k \to 0 \qquad \mu\text{-a.e. on } X \text{ as } n \to \infty,$$

for every $A \in \mathcal{A}$ with $\mu(A) < \infty$. While Hopf's ratio ergodic theorem (e.g. [A0], [H], or [Z3]) shows that the ratios $S_n(A)/S_n(B)$ of occupation times converge a.e. to the ratios $\mu(A)/\mu(B)$ of the respective measures, it does not identify the asymptotic order of magnitude of the $S_n(A)$. In fact, according to Aaronson's ergodic theorems (§2.4 of [A0]), no such order exists for a.e. convergence. Precise information in terms of the distributions of the $S_n(A)$ is available under certain additional assumptions (cf. §3.6 of [A0] and [TZ], [Z4]; some information on the complicated pointwise behaviour of $S_n(A)$ for sets A of infinite measure can be found in [ATZ]).

In what follows, we take a different point of view, which enables us to recover the interpretation of μ as giving the asymptotic frequency of visits also in situations with $\mu(X) = \infty$, which we assume from now on. The trivial limiting behaviour in (3) means that the orbit of a typical *single* point which T attempts to distribute over the infinite space is hardly ever visible in a reference set of finite measure. Why not replace the randomly chosen single point, which works well in a probability space, by some *randomly chosen countable ensemble of points*, distributed over the space $(X, \mathcal{A}, \mu)$ (which is a countable disjoint union of probability spaces) in such a way that we expect one point per unit measure?

This, in essence, is what the *Poisson suspension* does: it describes the simultaneous action of T on (suitable) countable collections of points. Roughly speaking, T acts on ensembles $\mathbf{x} = \{x_i\}_{i\geq 1} \subseteq X$ via $\mathbf{T}\mathbf{x} := \{Tx_i\}_{i\geq 0}$, and it turns out that there is a natural invariant *probability* measure $\boldsymbol{\mu}$ for $\mathbf{T}$, making precise a natural random choice of $\mathbf{x}$. Under suitable assumptions, $\mathbf{T}$ turns out to be ergodic for $\boldsymbol{\mu}$, which immediately yields an ergodic theorem for the orbits of countable ensembles, ensuring that for any $A \in \mathcal{A}$,

$$(4) \qquad \frac{1}{n}\, \mathbf{S}_n(A) := \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{N}_A \circ \mathbf{T}^k \to \mu(A) \qquad \boldsymbol{\mu}\text{-a.e. as } n \to \infty.$$

Here $\mathbf{N}_A(\mathbf{x}) = \sum_{i\geq 0} 1_A(x_i)$ denotes the number of points from $\mathbf{x}$ in A, and we will call $\mathbf{S}_n(A) := \sum_{k=0}^{n-1} \mathbf{N}_A \circ \mathbf{T}^k$ the $\mathbf{T}$-*occupation time* (up to time n) of A. We thus recover the interpretation of $\mu(A)$ as the average number of visits of orbits to A if we start with $\boldsymbol{\mu}$-typical countable ensembles $\mathbf{x}$ rather than single points. The present note is devoted to the study of Poisson suspensions of certain infinite measure preserving transformations, and provides sufficient conditions for a CLT of the form

$$\boldsymbol{\mu}\left[\frac{\mathbf{S}_n(A)-n\mu(A)}{\boldsymbol{\sigma}_n(A)}\le t\right]\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-s^2/2}\,ds \qquad \begin{array}{l}\text{for every } t\in\mathbb{R}\\ \text{as } n\to\infty,\end{array}$$

to accompany the strong law (4).

Here is a formal definition of the *Poisson suspension* $(\mathbf{X},\mathfrak{A},\boldsymbol{\mu},\mathbf{T})$ of the measure preserving system $(X,\mathcal{A},\mu,T)$, where $(X,\mathcal{A},\mu)$ is σ-finite and T need not be invertible: We let $\mathbf{X}$ denote the set of *counting measures* on $(X,\mathcal{A})$, i.e. of all measures $\mathbf{x}:\mathcal{A}\to\overline{\mathbb{N}}_0=\{0,1,\dots,\infty\}$, which we interpret as countable *ensembles* of points. For any $A\in\mathcal{A}$ the function $\mathbf{N}_A:X\to\overline{\mathbb{N}}_0$ evaluates counting measures at A, that is, $\mathbf{N}_A(\mathbf{x}):=\mathbf{x}(A)$, $\mathbf{x}\in\mathbf{X}$. Naturally, we want each $\mathbf{N}_A$ to be measurable, and hence equip $\mathbf{X}$ with the σ-field $\mathfrak{A}:=\sigma(\mathbf{N}_A:A\in\mathcal{A})$ generated by them. Next, we define $\mathbf{T}:\mathbf{X}\to\mathbf{X}$ by letting $\mathbf{T}\mathbf{x}:=\mathbf{x}\circ T^{-1}$, the image of the measure $\mathbf{x}$ under T. Then $\mathbf{T}$ is easily seen to be measurable with respect to $\mathfrak{A}$, since by measurability of T each $\mathbf{N}_A\circ\mathbf{T}=\mathbf{N}_{T^{-1}A}$, $A\in\mathcal{A}$, is $\mathfrak{A}$-measurable.

There exists a unique probability measure $\boldsymbol{\mu}$ on $(\mathbf{X},\mathfrak{A})$, called the *Poisson random measure with intensity* μ, such that for any finite collection of pairwise disjoint sets $A_1,\dots,A_l\in\mathcal{A}$ the corresponding $\mathbf{N}_{A_1},\dots,\mathbf{N}_{A_l}$ are independent random variables on $(\mathbf{X},\mathfrak{A},\boldsymbol{\mu})$, and each $\mathbf{N}_A$ has a *Poisson distribution* P_λ with expectation $\lambda=\mathbb{E}_{\boldsymbol{\mu}}[\mathbf{N}_A]=\int_{\mathbf{X}}\mathbf{N}_A\,d\boldsymbol{\mu}=\mu(A)$. It is easy to see that $\mu\circ T^{-1}=\mu$ implies $\boldsymbol{\mu}\circ\mathbf{T}^{-1}=\boldsymbol{\mu}$: the distribution of $(\mathbf{N}_{A_1},\dots,\mathbf{N}_{A_l})$ under $\boldsymbol{\mu}\circ\mathbf{T}^{-1}$ is $\boldsymbol{\mu}\circ\mathbf{T}^{-1}\circ(\mathbf{N}_{A_1},\dots,\mathbf{N}_{A_l})^{-1}=\boldsymbol{\mu}\circ(\mathbf{N}_{A_1}\circ\mathbf{T},\dots,\mathbf{N}_{A_l}\circ\mathbf{T})=\boldsymbol{\mu}\circ(\mathbf{N}_{T^{-1}A_1},\dots,\mathbf{N}_{T^{-1}A_l})$, the $\boldsymbol{\mu}$-distribution of $(\mathbf{N}_{T^{-1}A_1},\dots,\mathbf{N}_{T^{-1}A_l})$, which are independent (since the $T^{-1}A_i$ are pairwise disjoint) Poisson variables on $(\mathbf{X},\mathfrak{A},\boldsymbol{\mu})$ with respective expectations $\mu(T^{-1}A_i)=\mu(A_i)$.

In a more probabilistic language, $(\mathbf{X},\mathfrak{A},\boldsymbol{\mu},\mathbf{T})$ is a (*noninteracting*) *infinite particle system driven by the dynamical system* $(X,\mathcal{A},\mu,T)$. In an ergodic-theoretical context, [CFS] introduces Poisson suspensions as abstract versions of *ideal gas* models. Situations in which the underlying system is (the shift-space representation of) some Markov process have been studied earlier: see e.g. [P1], [P2] (discrete time) or §VIII.5 of [D] (continuous time). In particular, the results of [P2] contain most of Proposition 1 and Theorem 1 below for the special case of null-recurrent Markov shifts on a countable alphabet. [P2] also covers certain Markov chains with general state space, but depends on conditions not necessarily satisfied for the dynamical systems we are interested in. Our aim is to go beyond these processes with a clear-cut dependence structure by extending a CLT which is known in that classical setup to the family of transformations considered in [Z4].

2. Main result. Let T be a conservative ergodic measure preserving transformation (c.e.m.p.t.) on the σ-finite space $(X,\mathcal{A},\mu)$ with $\mu(X)=\infty$.

(We refer to [A0] for a wealth of information on such situations.) In terms of its *transfer operator* $\widehat{T} : L_1(\mu) \to L_1(\mu)$, characterized by

$$\int_X (g \circ T) \cdot u \, d\mu = \int_X g \cdot \widehat{T} u \, d\mu$$

for all $u \in L_1(\mu)$ and $g \in L_\infty(\mu)$, this means that $\widehat{T}1 = 1$ ($\widehat{T}$ naturally extends to $\{u : X \to [0,\infty)$ measurable $\mathcal{A}\}$), and $\sum_{k\geq 0} \widehat{T}^k u = \infty$ a.e. for all $u \in L_1(\mu)$ with $\int u \, d\mu > 0$. Let $\mathfrak{H}$ be a collection of probability densities with respect to μ. If there is some $K \in \mathbb{N}_0$ such that $\inf_{u\in\mathfrak{H}} \inf_Y \sum_{k=0}^{K} \widehat{T}^k u > 0$, we call $\mathfrak{H}$ *uniformly sweeping* (*in* K *steps*) for Y. The measure space is *standard* if $\mathcal{A}$ is the Borel σ-field of some complete separable metric on X.

Important quantitative characteristics of T are given in terms of its return-time distributions to suitable fixed reference sets: For $Y \in \mathcal{A}$ with $\mu(Y) > 0$ the *first return* (*entrance*) *time* of Y is

$$\varphi(x) = \varphi_Y(x) := \min\{n \geq 1 : T^n x \in Y\}, \qquad x \in X,$$

and we let $T_Y x := T^{\varphi(x)} x$, $x \in X$. The restricted measure $\mu|_{Y\cap\mathcal{A}}$ is invariant under the *first return map*, T_Y restricted to Y, that is, $1_Y = \sum_{k\geq 1} \widehat{T}^k 1_{Y\cap\{\varphi=k\}}$ a.e. If $\mu(Y) < \infty$, then φ is a random variable on the probability space $(Y, Y\cap\mathcal{A}, \mu_Y)$, $\mu_Y(E) := \mu(Y)^{-1}\mu(Y\cap E)$. Under additional assumptions making Y a suitable reference set, the asymptotics of the *tail probabilities* $q_n(Y) := \mu_Y(Y \cap \{\varphi_Y > n\})$ of its *return distribution*, or the *wandering rate* of Y given by $w_N(Y) := \mu(Y)\sum_{n=0}^{N-1} q_n(Y) = \mu(Y^N)$, where $Y^N := \bigcup_{n=0}^{N-1} T^{-n}Y$, $N \geq 1$, is decisive. We shall follow the convention of [TZ] and [Z4] to denote $Y_0 := Y$ and $Y_n := Y^{\mathrm{c}} \cap \{\varphi = n\}$, $n \geq 1$ (which are disjoint and satisfy $Y^N = \bigcup_{n=0}^{N-1} Y_n$ and $\mu(Y_n) = \mu(Y)\, q_n(Y)$).

As a warm-up, we point out how to interpret the wandering rate in terms of the Poisson suspension. Note that for the following probabilistic laws to hold, no special assumptions on the system or the sets are required: Poisson suspensions *a priori* come with a lot of inbuilt independence. Note that (as Y^n is the set of points which visit Y within time $\{0,\dots,n-1\}$), $\mathbf{N}_{Y^n}(\mathbf{x})$ can be interpreted as the number of distinct points from the ensemble $\mathbf{x} \in \mathbf{X}$ which visit Y at least once before time n. Similarly, $\boldsymbol{\tau}_Y(\mathbf{x}) := \min\{j \geq 0 : \mathbf{T}^j\mathbf{x}(Y) > 0\}$ represents the first time at which some point from $\mathbf{x} \in \mathbf{X}$ visits Y.

PROPOSITION 1 (Number of distinct visitors and waiting time for the first). *Let T be a c.e.m.p.t. on the σ-finite infinite measure space $(X, \mathcal{A}, \mu)$, and let $(\mathbf{X}, \mathfrak{A}, \boldsymbol{\mu}, \mathbf{T})$ be its Poisson suspension. For every $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$, the variables $\mathbf{N}_{Y^n}$ satisfy a strong law,*

$$\mathbb{E}_{\boldsymbol{\mu}}[\mathbf{N}_{Y^n}] = w_n(Y) \quad \textit{and} \quad \frac{\mathbf{N}_{Y^n}}{w_n(Y)} \to 1 \quad \boldsymbol{\mu}\textit{-a.e.}, \tag{5}$$

and a central limit theorem,

$$(6)\qquad \boldsymbol{\mu}\left[\frac{\mathbf{N}_{Y^n}-w_n(Y)}{\sqrt{w_n(Y)}}\le t\right]\to\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{t}e^{-s^2/2}ds \qquad \textit{for every } t\in\mathbb{R} \textit{ as } n\to\infty.$$

Moreover,

$$(7)\qquad \boldsymbol{\mu}[\boldsymbol{\tau}_Y\ge n]=e^{-w_n(Y)},\quad n\ge 1.$$

Proof. This is easy if we observe that $\mathbf{N}_{Y^n}=\mathbf{N}_{Y_0}+\cdots+\mathbf{N}_{Y_{n-1}}$ is a sum of n independent Poisson variables with expectations $\mathbb{E}_{\boldsymbol{\mu}}[\mathbf{N}_{Y_k}]=\mu(Y_k)$ satisfying $\mathbb{E}_{\boldsymbol{\mu}}[\mathbf{N}_{Y_0}]+\cdots+\mathbb{E}_{\boldsymbol{\mu}}[\mathbf{N}_{Y_{n-1}}]=w_n(Y)\to\infty$. We can now use standard facts about Poissonian variables:

For example, (5) is equivalent to saying that the image Q of $\boldsymbol{\mu}$ under the map $\mathbf{x}\mapsto(\sum_{j=0}^{n-1}\mathbf{N}_{Y_j}(\mathbf{x}))_{n\ge1}$ gives full measure to the event $\{s_n/w_n(Y)\to 1\}$ in the sequence space $\mathfrak{S}:=\{s=(s_j)_{j\ge1}:s_j\in\overline{\mathbb{N}}_0\}$ (with product σ-field). But Q coincides with the distribution of $\omega\mapsto(N_{w_n(Y)}(\omega))_{n\ge1}\in\mathfrak{S}$ where $(N_t)_{t\ge0}$ is a Poisson process with $\mathbb{E}[N_1]=1$ on $(\Omega,\mathcal{F},P)$. Now it is well known that $(N_t)_{t\ge0}$ satisfies the strong law $N_t/t\to1$ a.s. and *a fortiori* $N_{w_n(Y)}/w_n(Y)\to 1$ a.s. We therefore see that indeed $Q[s_n/w_n(Y)\to1]=P[N_{w_n(Y)}/w_n(Y)\to1]=1$.

Checking the CLT (6) is a routine probability exercise (cf. Problem 27.3 of [B]), using the characteristic function of the Poisson distribution P_λ,

$$(8)\qquad \widehat{\mathsf{P}}_\lambda(t)=\exp[-\lambda(1-e^{it})],\qquad t\in\mathbb{R}.$$

Finally, (7) is clear from $\{\boldsymbol{\tau}_Y\ge n\}=\{\mathbf{N}_{Y^n}=0\}$ and $\mu(Y^n)=w_n(Y)$. ∎

Recall that a measurable function $a:(L,\infty)\to(0,\infty)$ is *regularly varying of index* $\varrho\in\mathbb{R}$ *at infinity*, written $a\in\mathcal{R}_\varrho$, if $a(ct)/a(t)\to c^\varrho$ as $t\to\infty$ for all $c>0$. We shall tacitly interpret sequences $(a_n)_{n\ge0}$ as functions on $\mathbb{R}_+$ via $t\mapsto a_{[t]}$. Furthermore, $\mathcal{R}_\varrho(0)$ is the family of functions $r:(0,\varepsilon)\to\mathbb{R}_+$ *regularly varying of index* ϱ *at zero* (same condition as above, but for $t\searrow0$). For background information we refer to Chapter 1 of [BGT]. We write $a(t)\sim b(t)$ as $t\to\infty$ if $a(t)/b(t)\to1$, and $a(t)\asymp b(t)$ as $t\to\infty$ to indicate that the ratio $a(t)/b(t)$ is bounded away from 0 and ∞ for $t\ge t_0$.

Remark 1 (Minimal wandering rates). The asymptotics of the wandering rate $(w_N(Y))$ does depend on the set Y, and there never are sets maximizing this rate for a given system (cf. Proposition 3.8.2 in [A0]). Still, some transformations do have distinguished sets Y with *minimal wandering rate*, meaning that $\underline{\lim}_{N\to\infty}w_N(Z)/w_N(Y)\ge1$ for all $Z\in\mathcal{A}$, $0<\mu(Z)<\infty$. Equivalently, $w_N(Y)\sim w_N(Z)$ provided $\mu(Z)>0$ and $Z\subseteq Y$. This common rate is then an asymptotic characteristic of the measure preserving system, the *wandering rate of* T, $(w_N(T))$.

We are now ready to state the main result of the present paper. It provides us with a CLT for $\mathbf{T}$-occupation times of a large family of sets—those contained in some distinguished reference set Y having a *compact regeneration property* (and hence minimal wandering rate, cf. Proposition 3.2 and Remark 3.6 of [TZ]). We refer to [TZ], [Z4] for further information on this type of condition, which (in a slightly stronger form) was first used in [T3]. The assumption (10) on the set Y in the result below is exactly the structural condition of Theorem 2.1 in [Z4] (which in addition requires regular variation of $(w_N(Y))$).

THEOREM 1 ($\mathbf{T}$-occupation times inside compactly regenerative sets). *Let T be a c.e.m.p.t. on the σ-finite infinite standard measure space $(X,\mathcal{A},\mu)$, and let $(\mathbf{X},\mathfrak{A},\boldsymbol{\mu},\mathbf{T})$ be its Poisson suspension. Then $\mathbf{T}$ is ergodic, and for any $Y\in\mathcal{A}$, $0<\mu(Y)<\infty$,*

$$\frac{1}{n}\,\mathbf{S}_n(Y)\to\mu(Y)\qquad \boldsymbol{\mu}\text{-a.e. as } n\to\infty. \tag{9}$$

Suppose, in addition, that Y is such that

$$\mathfrak{H}_Y := \left\{\frac{1}{w_N(Y)}\sum_{n=0}^{N-1}\widehat{T}^n 1_{Y_n}\right\}_{N\geq 1} \quad \begin{array}{l}\text{is precompact in } L_\infty(\mu)\\ \text{and uniformly sweeping.}\end{array} \tag{10}$$

Then, for every $E\in Y\cap\mathcal{A}$ with $\mu(E)>0$, and every probability measure $\boldsymbol{\nu}\ll\boldsymbol{\mu}$,

$$\boldsymbol{\nu}\left[\frac{\mathbf{S}_n(E)-n\mu(E)}{\boldsymbol{\sigma}_n(E)}\leq t\right]\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-s^2/2}\,ds \qquad \begin{array}{l}\text{for every } t\in\mathbb{R}\\ \text{as } n\to\infty,\end{array} \tag{11}$$

where

$$\boldsymbol{\sigma}_n^2(E):=\mathrm{Var}_{\boldsymbol{\mu}}[\mathbf{S}_n(E)]\asymp\frac{n^2}{w_n(Y)}\qquad \text{as } n\to\infty. \tag{12}$$

If, moreover, $(w_N(Y))\in\mathcal{R}_{1-\alpha}$ for some $\alpha\in[0,1]$, then

$$\boldsymbol{\sigma}_n^2(E)\sim\frac{2\mu(E)^2}{\Gamma(2-\alpha)\Gamma(2+\alpha)}\cdot\frac{n^2}{w_n(Y)}\qquad \text{as } n\to\infty. \tag{13}$$

We briefly indicate a few situations in which the conditions of [Z4], and hence our present results, apply. In each case we identify a large family $\mathcal{E}(T)$ of good sets E, i.e. each $E\in\mathcal{E}(T)$ is contained in some Y satisfying (10).

EXAMPLE 1 (Random walks driven by Gibbs–Markov maps). Assume that $(M,\mathcal{B},\nu,R,\xi)$ is an ergodic probability preserving fibred system given by a *Gibbs–Markov map* (cf. [A0], [AD]) with finite image partition, $\#R\xi<\infty$. Let $\phi: M\to\mathbb{Z}$ be a ξ-measurable function and assume (see [AD] for definitions) that ϕ is aperiodic, and either that $\phi\in L_2(\mu)$ with $\int_X\phi\,d\mu=0$, or that the μ-distribution of ϕ is in the strict domain of attraction of a nondegenerate stable distribution of order $p \in (1,2)$. Then the $\mathbb{Z}$-*extension* $T = R_\phi$ of R, that is, the map on the σ-finite infinite measure space $(X, \mathcal{A}, \mu) := (M \times \mathbb{Z}, \mathcal{B} \otimes \mathfrak{P}(\mathbb{Z}), \nu \otimes \nu_{\mathbb{Z}})$, with $\nu_{\mathbb{Z}}$ denoting counting measure on $\mathbb{Z}$, given by $T(x,g) := (Rx, g + \phi(x))$, is a c.e.m.p.t. Any set of the form $Y := M \times D$ with $D \subseteq \mathbb{Z}$ finite satisfies (10), and we have $(w_N(Y)) \in \mathcal{R}_{1-\alpha}$ with $\alpha := 1/2$ or $\alpha := 1 - 1/p \in (0, 1/2)$, respectively (cf. §7.3 of [Z4]). Theorem 1 therefore applies to any positive measure set from $\mathcal{E}(T) := \{E : \pi(E)$ bounded$\}$, where $\pi(x,g) := g$.

Example 2 (Interval maps with indifferent fixed points). A large class of infinite measure preserving piecewise monotonic interval maps $(X, \mathcal{A}, \mu, T, \xi)$, called *AFN-maps*, has been studied in [Z1], generalizing earlier results from [A0], [A3], [T1]. We refer to [Z1] or [TZ] for definitions and notation. Their ergodic behaviour is determined by a finite set $\zeta \subseteq \xi$ of cylinders Z having an *indifferent fixed point* x_Z at the boundary. The considerations of §8 of [TZ] show that any set E from $\mathcal{E}(T) := \{F \in \mathcal{A}$: there is some $\varepsilon > 0$ such that $F \cap (x_Z - \varepsilon, x_Z + \varepsilon) \cap Z = \emptyset$ for all $Z \in \zeta\}$ is contained in some Y satisfying (10). Regular variation of $(w_N(Y))$ depends on details of the local behaviour of T at the x_Z (see e.g. §4 of [T2]).

Example 3 (S-unimodal Misiurewicz maps with flat tops). Further examples with dynamics governed by some distinct *indifferent orbits* are maps T on the interval with *flat critical points*, i.e. points c at which all derivatives of T vanish. [Z2] was devoted to flat S-unimodal maps T on an interval $X := [a,b]$ satisfying the *Misiurewicz condition*, meaning that there is some open subinterval Y around c (without loss of generality, a union of two cylinders) to which the orbit of c does not return, $c_n := T^n c \notin Y$ for $n \geq 1$. As pointed out in §7.2 of [Z4], this set Y satisfies (10), and we take $\mathcal{E}(T)$ containing all measurable sets inside a sufficiently small neighbourhood of c. Such a map T always has an absolutely continuous conservative ergodic invariant measure μ which is infinite iff $\int \log |T'x| \, dx = -\infty$. Regular variation of $(w_N(Y))$ depends on the local behaviour of T at c and on the existence of the postcritical Lyapunov exponent of T (cf. Theorem 5 of [Z2]).

3. Proof of Theorem 1. We follow the strategy used in [P2], adapting it to our setup. The specific difficulties are dealt with in the following auxiliary proposition, which exploits information obtained in the proof of Theorem 2.1 of [Z4]. The latter result states that under the assumptions on $(X, \mathcal{A}, \mu, T)$ and Y in Theorem 1 above, plus $(w_N(Y)) \in \mathcal{R}_{1-\alpha}$ for some $\alpha \in [0,1]$, one has, for every $f \in L_1(\mu)$ with $\mu(f) \neq 0$, distributional convergence

$$\frac{1}{a_n} S_n(f) \overset{\nu}{\Rightarrow} \mu(f) \mathcal{M}_\alpha, \tag{14}$$

with respect to any probability measure $\nu \ll \mu$, where $\mathcal{M}_\alpha$ is a random variable distributed according to the *normalized Mittag-Leffler law of order* α, which is uniquely characterized by its moments $\mathbb{E}[\mathcal{M}_\alpha^r] = r!\,(\Gamma(1+\alpha))^r/\Gamma(1+r\alpha)$, $r \geq 1$, and

$$a_n := \frac{1}{\mu(Y)} \int_Y S_n(Y)\, d\mu_Y \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \cdot \frac{n}{w_n(Y)} \qquad \text{as } n \to \infty. \tag{15}$$

This is established by proving that the moments of $S_n(Y)$ with respect to μ_Y converge, i.e.

$$\int_Y \left(\frac{S_n(Y)}{\mu(Y)}\right)^r d\mu_Y \sim \mathbb{E}[\mathcal{M}_\alpha^r] \cdot a_n^r \quad \text{as } n \to \infty. \tag{16}$$

Here we obtain further information in this direction:

PROPOSITION 2 (The μ-moments of T-occupation times). *Let T be a c.e.m.p.t. on the σ-finite infinite measure space $(X, \mathcal{A}, \mu)$, and suppose that $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, is such that*

$$\mathfrak{H}_Y = \left\{ \frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \right\}_{N \geq 1} \quad \begin{array}{l} \textit{is precompact in } L_\infty(\mu) \\ \textit{and uniformly sweeping.} \end{array} \tag{17}$$

Then, for every $E \in Y \cap \mathcal{A}$ with $\mu(E) > 0$, and every integer $r \geq 1$,

$$\int_X S_n^r(E)\, d\mu \asymp w_n(Y) \int_Y S_n^r(E)\, d\mu \asymp w_n(Y) \left(\frac{n}{w_n(Y)}\right)^r \quad \textit{as } n \to \infty. \tag{18}$$

If, moreover, $(w_N(Y)) \in \mathcal{R}_{1-\alpha}$ for some $\alpha \in [0,1]$, then

$$\int_X \left(\frac{S_n(E)}{\mu(E)}\right)^r d\mu \sim \frac{r!\,(\Gamma(2-\alpha))^{1-r}}{\Gamma(2+(r-1)\alpha)} \cdot \mu(Y) w_n(Y) \left(\frac{n}{w_n(Y)}\right)^r \quad \textit{as } n \to \infty. \tag{19}$$

Before applying this to the Poisson suspension, we record a straightforward consequence: Recall (cf. [A1] or §3.3 of [A0]) that a c.e.m.p.t. T on $(X, \mathcal{A}, \mu)$ is called *rationally ergodic* if there exists some $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, satisfying a *Rényi inequality*, i.e. there is some $M \in (0, \infty)$ such that

$$\int_Y S_n^2(Y)\, d\mu \leq M \cdot \left(\int_Y S_n(Y)\, d\mu\right)^2 \quad \text{for all } n \geq 1.$$

COROLLARY 1 (Rational ergodicity). *Let T be a c.e.m.p.t. on the σ-finite infinite measure space $(X, \mathcal{A}, \mu)$ and $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, with* (17). *Then Y satisfies a Rényi inequality.*

Proof. Immediate from (18) in Proposition 2. ∎

Assuming Proposition 2, we can now argue as follows:

Proof of Theorem 1. (i) Ergodicity of Poisson suspensions of infinite measure preserving conservative ergodic *automorphisms* is established in [R, Proposition 2.6.2]. According to Theorems 3.1.5 and 3.1.7 of [A0], our system $(X, \mathcal{A}, \mu, T)$ has an (invertible) conservative ergodic natural extension $(X', \mathcal{A}', \mu', T')$, i.e. there is a measurable factor map $\pi : X' \to X$ with $\pi \circ T' = T \circ \pi$ and $\mu' \circ \pi^{-1} = \mu$. The Poisson suspension $(\mathbf{X}', \mathfrak{A}', \boldsymbol{\mu}', \mathbf{T}')$ of the latter is ergodic by Roy's result. Therefore, ergodicity of $(\mathbf{X}, \mathfrak{A}, \boldsymbol{\mu}, \mathbf{T})$ follows if we check that (parallel to Theorem 2.4.4 of [R] for automorphisms) generally

(20) the suspension of an extension is an extension of the suspension.

To see this, consider any extension $(X', \mathcal{A}', \mu', T')$ (not necessarily invertible, with factor map π) of $(X, \mathcal{A}, \mu, T)$, and define $\boldsymbol{\pi} : \mathbf{X}' \to \mathbf{X}$ by $\boldsymbol{\pi}\mathbf{x}' := \mathbf{x}' \circ \boldsymbol{\pi}^{-1}$. As the σ-fields $\mathfrak{A}, \mathfrak{A}'$ are generated by the evaluations $\mathbf{N}_A$, $A \in \mathcal{A}$, and $\mathbf{N}'_{A'}$, $A' \in \mathcal{A}'$, respectively, measurability of $\boldsymbol{\pi}$ follows from that of the compositions $\mathbf{N}_A \circ \boldsymbol{\pi}$, $A \in \mathcal{A}$, which is clear since $\mathbf{N}_A \circ \boldsymbol{\pi}(\mathbf{x}') = \mathbf{N}'_{A'}(\mathbf{x}')$ where $A' := \pi^{-1}A \in \mathcal{A}'$. Next, observe that $\mathbf{T} \circ \boldsymbol{\pi} = \boldsymbol{\pi} \circ \mathbf{T}'$ since

$$\begin{aligned}(\mathbf{T} \circ \boldsymbol{\pi}(\mathbf{x}'))(A) &= \mathbf{x}' \circ \pi^{-1} \circ T^{-1}A = \mathbf{x}' \circ (T')^{-1} \circ \pi^{-1}A \\ &= (\boldsymbol{\pi} \circ \mathbf{T}'(\mathbf{x}'))(A)\end{aligned}$$

for $A \in \mathcal{A}$. Analogous manipulations show that $\boldsymbol{\mu}' \circ \boldsymbol{\pi}^{-1}$ is the Poisson random measure with intensity μ, and hence equals $\boldsymbol{\mu}$: for any $A \in \mathcal{A}$, the distribution $\boldsymbol{\mu}' \circ \boldsymbol{\pi}^{-1} \circ \mathbf{N}_A^{-1}$ of $\mathbf{N}_A$ equals $\boldsymbol{\mu}' \circ (\mathbf{N}'_{A'})^{-1}$ with $A' := \pi^{-1}A \in \mathcal{A}'$, and hence $\mathsf{P}_{\mu'(A')} = \mathsf{P}_{\mu(A)}$. The independence condition follows since π^{-1} preserves disjointness. This completes the proof of (20).

Statement (9) is just the ergodic theorem for the suspension.

(ii) For the proof of the CLT (11) we let $S_n := S_n(E)$, $n \geq 1$. For $n \in \mathbb{N}$ and $r \in \mathbb{N}_0$ the number of points from an ensemble $\mathbf{x}$ which visit E exactly r times by time n is $\mathbf{N}_{\{S_n = r\}}(\mathbf{x})$, and therefore

$$\mathbf{S}_n(E) := \sum_{k=0}^{n-1} \mathbf{N}_E \circ \mathbf{T}^k = \sum_{r=1}^{n-1} r\, \mathbf{N}_{\{S_n = r\}}.$$

Observe that for fixed n the sets $\{S_n = r\}$, $r \in \{1, \ldots, n-1\}$, are pairwise disjoint, so that $\mathbf{N}_{\{S_n = r\}}$, $r \in \{1, \ldots, n-1\}$, are independent Poisson random variables on $(\mathbf{X}, \mathfrak{A}, \boldsymbol{\mu})$ with $\mathbb{E}_{\boldsymbol{\mu}}[\mathbf{N}_{\{S_n = r\}}] = \mu(\{S_n = r\})$. Consequently,

$$\mathrm{Var}_{\boldsymbol{\mu}}[\mathbf{S}_n(E)] = \sum_{r=1}^{n-1} r^2 \mu(\{S_n = r\}) = \int_X S_n^2 \, d\mu, \tag{21}$$

so that (13) immediately follows from (19). For the same reason the characteristic function of $\mathbf{S}_n(E)$ is

$$
\begin{aligned}
\text{(22)}\qquad \mathbb{E}_{\boldsymbol{\mu}}[\exp(i\theta\mathbf{S}_n(E))] &= \prod_{r=1}^{n-1}\mathbb{E}_{\boldsymbol{\mu}}[\exp(i\theta r\mathbf{N}_{\{S_n=r\}})] \\
&= \exp\Big[\sum_{r=1}^{n-1}(e^{i\theta r}-1)\mu(\{S_n=r\})\Big] \\
&= \exp\Big[\int_X (e^{i\theta S_n}-1)\,d\mu\Big],
\end{aligned}
$$

where $\theta\in\mathbb{R}$. Abbreviating $\boldsymbol{\sigma}_n := \sqrt{\mathrm{Var}_{\boldsymbol{\mu}}[\mathbf{S}_n(E)]}$, $n\in\mathbb{N}$, we find that

$$
\log \mathbb{E}_{\boldsymbol{\mu}}\Big[\exp\Big(i\theta\,\frac{\mathbf{S}_n(E)-n\mu(E)}{\boldsymbol{\sigma}_n}\Big)\Big] = -\frac{\theta^2}{2}+R_n(\theta),
$$

where

$$
R_n(\theta) := \int_X \Big[\exp\Big(i\theta\frac{S_n}{\boldsymbol{\sigma}_n}\Big) - \Big(1+i\theta\,\frac{S_n}{\boldsymbol{\sigma}_n}-\frac{1}{2}\Big(\theta\,\frac{S_n}{\boldsymbol{\sigma}_n}\Big)^2\Big)\Big]\,d\mu,\qquad \theta\in\mathbb{R}.
$$

Therefore the CLT with respect to $\boldsymbol{\mu}$,

$$
\boldsymbol{\mu}\Big[\frac{\mathbf{S}_n(E)-n\mu(E)}{\boldsymbol{\sigma}_n(E)}\le t\Big] \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-s^2/2}\,ds \qquad \begin{array}{l}\text{for every } t\in\mathbb{R}\\ \text{as } n\to\infty,\end{array}
$$

follows once we verify that

$$
\lim_{n\to\infty} R_n(\theta)=0 \qquad \text{for all } \theta\in\mathbb{R}.
$$

But by an easy standard estimate, we have

$$
|R_n(\theta)| \le \frac{|\theta|^3}{6}\cdot\int_X \frac{S_n^3}{\boldsymbol{\sigma}_n^3}\,d\mu,
$$

and the integral on the right-hand side tends to zero, since the $r=2$ and $r=3$ cases of (18) in Proposition 2 ensure that

$$
\Big(\int_X S_n^3(E)\,d\mu\Big)^2 = o\Big(\int_X S_n^2(E)\,d\mu\Big)^3 \qquad \text{as } n\to\infty.
$$

Finally, the extension to other measures $\boldsymbol{\nu}\ll\boldsymbol{\mu}$ is immediate from Eagleson's theorem (cf. [E] or Corollary 1 of [Z5]). ■

4. Proof of Proposition 2. The proof of this crucial proposition exploits a number of facts established in the proof of Theorem 2.1 of [Z4]. Hardly surprising, the argument for the regularly varying case will depend on Karamata's Tauberian theorem (KTT) and the Monotone Density theorem for regularly varying functions (see [BGT] or Proposition 4.2 and Lemma 4.1 of [TZ]), that is, LEMMA 1 (Karamata's Tauberian theorem, Monotone Density theorem). *Let* (b_n) *be a sequence in* $[0,\infty)$ *such that* $B(s) := \sum_{n\geq 0} b_n\, e^{-ns} < \infty$ *for all* $s > 0$. *Suppose that* $\ell \in \mathcal{R}_0$ *and* $\varrho \in [0,\infty)$. *Then*

(i) $B(s) \sim (1/s)^{\varrho}\ell(1/s)$ *as* $s \searrow 0$ *if and only if*
(ii) $\sum_{k=0}^{n-1} b_k \sim n^{\varrho}\ell(n)/\Gamma(\varrho+1)$ *as* $n \to \infty$.

If (b_n) *is eventually monotone and* $\varrho > 0$, *then both are equivalent to*

(iii) $b_n \sim \varrho n^{\varrho-1}\ell(n)/\Gamma(\varrho+1)$ *as* $n \to \infty$.

In order to deal with non-regularly varying situations, we need to supply a few less familiar tools from Karamata theory: A measurable function $a : (L,\infty) \to (0,\infty)$ is *O-regularly varying at infinity*, written $a \in \mathcal{OR}$, if for all $c > 0$,

$$0 < \varliminf_{t\to\infty} \frac{a(ct)}{a(t)} \leq \varlimsup_{t\to\infty} \frac{a(ct)}{a(t)} < \infty.$$

$\mathcal{OR}(0)$ will denote the class of functions *O-regular varying at zero* (same condition, but for $t \searrow 0$). This is just one of several useful concepts generalizing regular variation which still enable a meaningful asymptotic theory. For the reader's convenience we explicitly state a few facts which we are going to use below. The first observation is due to Feller (cf. Corollary 2.0.6 of [BGT]).

LEMMA 2 (O-regular variation of monotone sequences). *If* $(w_N)_{N\geq 0}$ *is a nondecreasing sequence in* $(0,\infty)$ *with* $\varlimsup_{N\to\infty} w_{c_0 N}/w_N < \infty$ *for some* $c_0 > 1$, *then* $w \in \mathcal{OR}$.

The argument to follow hinges on two Tauberian results for O-regular variation. For the first, see Theorem 2.10.2 of [BGT], or [dHS].

LEMMA 3 (de Haan–Stadtmüller O-Tauberian theorem). *If* $(u_n)_{n\geq 0}$ *is a sequence in* $(0,\infty)$, *then the following are equivalent*:

(i) $(v_N)_{N\geq 0} := (\sum_{n=0}^{N-1} u_n)_{N\geq 1}$ *is O-regularly varying at infinity*,
(ii) $U(s) := \sum_{n\geq 0} u_n e^{-ns}$, $s > 0$, *is O-regularly varying at zero*,
(iii) $U(1/N) \asymp v(N)$ *as* $N \to \infty$.

LEMMA 4 (O-Monotone Density theorem). *If* $(u_n)_{n\geq 0}$ *is a non decreasing sequence in* $(0,\infty)$ *with* $(v_N)_{N\geq 0} := (\sum_{n=0}^{N-1} u_n)_{N\geq 1}$ *O-regularly varying at infinity*, *then* $U(s) := \sum_{n\geq 0} u_n e^{-ns}$, $s > 0$, *satisfies*

(iv) $U(1/N) \asymp N u(N)$ *as* $N \to \infty$.

Proof. This is a variant of Exercise 2.12.26 of [BGT]. Simply observe that $v_N \leq N u_N \leq \sum_{n=N}^{2N} u_n \leq v_{2N} \leq \text{const} \cdot v_N$ and apply the preceding lemma. ∎

We are now ready for

Proof of Proposition 2. (i) Since $w_N(Y) = \mu(Y) \sum_{n=0}^{N-1} q_n(Y)$ with $q_n(Y) \searrow 0$, we have $w_{2N}(Y) \le 2w_N(Y)$ for all $N \ge 0$, which by Lemma 2 implies $(w_N(Y)) \in \mathcal{OR}$. According to Lemma 3 this entails O-regular variation at zero of $Q_Y(s) := \sum_{n\ge 0} q_n(Y)e^{-ns}$, $s > 0$, and hence also of $s \mapsto (sQ_Y(s))^{-r}/s$ for any $r \ge 1$. Moreover,

$$w_N(Y) \asymp Q_Y\left(\frac{1}{N}\right) \quad \text{as } N \to \infty. \tag{23}$$

The proof of Theorem 2.1 of [Z4] shows, without using regular variation, that for any $r \ge 1$,

$$A_{Y,r}(s) := \sum_{n\ge 0}\Big(\int_Y S_n^r(Y)\, d\mu_Y\Big)e^{-ns} \asymp \frac{1}{s}\left(\frac{1}{sQ_Y(s)}\right)^r \quad \text{as } s \searrow 0. \tag{$\Diamond_r$}$$

As the right-hand side belongs to $\mathcal{OR}(0)$, we conclude that for any $r \ge 1$, the same is true for $A_{Y,r}$. Using Lemma 3 again, we thus see that

$$\sum_{n=0}^{N}\int_Y S_n^r(Y)\, d\mu \asymp A_{Y,r}\left(\frac{1}{N}\right) \asymp N\left(\frac{N}{Q_Y(1/N)}\right)^r \quad \text{as } N \to \infty, \tag{24}$$

with all three sequences O-regularly varying. In particular, as the leftmost sum is in $\mathcal{OR}$ and $\int_Y S_n^r(Y)\, d\mu$ is nondecreasing in n, we can appeal to Lemma 4 to obtain

$$\int_Y S_n^r(Y)\, d\mu \asymp \left(\frac{n}{Q_Y(1/n)}\right)^r \asymp \left(\frac{n}{w_n(Y)}\right)^r \asymp a_n^r \quad \text{as } n \to \infty, \tag{25}$$

with $a_n := \mu(Y)^{-1}\int_Y S_n(Y)\, d\mu_Y$. We thus have, for each $r \ge 1$, boundedness of the moment sequence $(\int_Y (S_n(Y)/a_n)^r\, d\mu)_{n\ge 1}$, and hence also of $(\int_Y (S_n(E)/a_n)^r\, d\mu)_{n\ge 1}$ for any fixed $E \in Y \cap \mathcal{A}$. Moreover, we also see that $\underline{\lim}_{n\to\infty} \int_Y (S_n(Y)/a_n)^r\, d\mu > 0$. Combining these two facts with Hopf's ratio ergodic theorem we conclude (using uniform integrability of the sequence $((S_n(E)/a_n)^r)_{n\ge 1}$) that for any $r \ge 1$ and $E \in Y \cap \mathcal{A}$,

$$\int_Y S_n^r(E)\, d\mu \sim \left(\frac{\mu(E)}{\mu(Y)}\right)^r \int_Y S_n^r(Y)\, d\mu \asymp a_n^r \quad \text{as } n \to \infty. \tag{26}$$

Together with (25) this gives the second part of (18).

Define $R_n := S_n(E)/a_n$, $n \ge 1$. It is also shown in [Z4] that for any $r \ge 1$, and $E = Y$,

(27) $(R_n^r(E))_{n\ge 1}$ satisfies the assumptions of Proposition 3.2 of [Z4],

which implies that for $E = Y$ we have

$$\int_Y S_n^r(E) \cdot h\, d\mu \sim \int_Y S_n^r(E)\, d\mu_Y \quad \text{as } n \to \infty, \text{ uniformly in } h \in \mathfrak{H}_Y. \tag{28}$$

We claim that (27), and hence also (28) hold for every fixed $E \in Y \cap \mathcal{A}$ with $\mu(E) > 0$: Note that

$$S_n(E) = S_{n-k}(E) \circ T^k \quad \text{on } Y_k,\ k \in \{0, \dots, n\}, \tag{29}$$

and the previously observed boundedness of all moment sequences gives weak precompactness of $(1_Y R_n^r(E))_{n \geq 1}$. Finally, we need to verify that

$$\|(R_n^r(E) \circ T - R_n^r(E)) \cdot u\|_1 \to 0 \quad \text{for all } u \in L_\infty(\mu) \text{ supported on some } Y^M.$$

But, precisely as in the case $E = Y$ considered in [Z4] (cf. equation (4.9) there), this follows from (26) via the mean-value theorem.

(ii) Now fix some $E \in Y \cap \mathcal{A}$ with $\mu(E) > 0$, and observe that due to (29),

$$\int_X S_n^r(E)\, d\mu = \sum_{k=0}^{n} \int_X S_{n-k}^r(E) \circ T^k \cdot 1_{Y_k}\, d\mu = \sum_{k=0}^{n} \int_Y S_{n-k}^r(E) \cdot \widehat{T}^k 1_{Y_k}\, d\mu.$$

As an immediate consequence, we see that

$$\sum_{n=0}^{N} \int_X S_n^r(E)\, d\mu = \sum_{n=0}^{N} w_{n+1}(Y) \cdot \int_Y S_{N-n}^r(E) \cdot h_n\, d\mu,$$

where $h_N := w_{N+1}(Y)^{-1} \sum_{n=0}^{N} \widehat{T}^n 1_{Y_n} \in \mathfrak{H}_Y$. Since $\int_X S_n^r(E)\, d\mu \to \infty$ as $n \to \infty$, (28) enables us to conclude that

$$\sum_{n=0}^{N} \int_X S_n^r(E)\, d\mu \sim \sum_{n=0}^{N} w_{n+1}(Y) \cdot \int_Y S_{N-n}^r(E)\, d\mu \quad \text{as } N \to \infty,$$

and

$$\begin{aligned} \sum_{n \geq 0} \Big(\int_X S_n^r(E)\, d\mu \Big) e^{-ns} \tag{30} \\ &= (1 - e^{-s}) \sum_{N \geq 0} \Big(\sum_{n=0}^{N} \int_X S_n^r(E)\, d\mu \Big) e^{-Ns} \\ &\sim (1 - e^{-s}) \sum_{N \geq 0} \Big(\sum_{n=0}^{N} w_{n+1}(Y) \cdot \int_Y S_{N-n}^r(E)\, d\mu \Big) e^{-Ns} \\ &= \mu(Y) Q_Y(s) \cdot \sum_{n \geq 0} \Big(\int_Y S_n^r(E)\, d\mu \Big) e^{-ns} \\ &\sim \mu(Y)^2 \left(\frac{\mu(E)}{\mu(Y)} \right)^r Q_Y(s) \cdot \sum_{n \geq 0} \Big(\int_Y S_n^r(Y)\, d\mu_Y \Big) e^{-ns} \end{aligned}$$

as $s \searrow 0$, where the last step uses (26). We know that each factor in the rightmost expression is O-regularly varying, and so the same is true for the leftmost expression. Hence, by Lemma 3, the sequence $(\sum_{n=0}^{N} \int_X S_n^r(E)\, d\mu)_{N \geq 0}$

belongs to $\mathcal{OR}$, and by the obvious monotonicity of the individual terms in these sums we can appeal to Lemma 4 to obtain, for any $r \geq 1$,

$$\int_X S_n^r(E)\, d\mu \asymp \frac{1}{n}\, Q_Y\Big(\frac{1}{n}\Big) A_{Y,r}\Big(\frac{1}{n}\Big) \asymp w_n(Y)\Big(\frac{n}{w_n(Y)}\Big)^r \quad \text{as } n \to \infty, \tag{31}$$

where the second relation comes from (23) and (24).

(iii) Finally, assume that $(w_N(Y)) \in \mathcal{R}_{1-\alpha}$ for some $\alpha \in [0,1]$. By KTT this means that there is some function ℓ, slowly varying at infinity, such that

$$Q_Y(s) = \Big(\frac{1}{s}\Big)^{1-\alpha} \ell\Big(\frac{1}{s}\Big) \quad \text{for } s > 0, \qquad w_n(Y) \sim \mu(Y)\frac{n^{1-\alpha}\ell(n)}{\Gamma(2-\alpha)} \quad \text{as } n \to \infty.$$

In the proof of Theorem 2.1 of [Z4], it is shown that in this case, for any $r \geq 1$,

$$\sum_{n\geq 0}\Big(\int_Y S_n^r(Y)\, d\mu_Y\Big)e^{-ns} \sim \frac{r!}{s}\Big(\frac{1}{s\, Q_Y(s)}\Big)^r \quad \text{as } s \searrow 0. \tag{32}$$

Therefore, (30) becomes

$$\sum_{n\geq 0}\Big(\int_X S_n^r(E)\, d\mu\Big)e^{-ns} \sim r!\, \mu(Y)^2\Big(\frac{\mu(E)}{\mu(Y)}\Big)^r \Big(\frac{1}{s}\Big)^{2+(r-1)\alpha} \ell\Big(\frac{1}{s}\Big)^{-(r-1)}$$

as $s \searrow 0$. Applying KTT once again (and monotonicity of $(\int_X S_n^r(E)\, d\mu)_{n\geq 1}$), we thus obtain

$$\int_X S_n^r(E)\, d\mu \sim \mu(Y)^2\Big(\frac{\mu(E)}{\mu(Y)}\Big)^r \frac{r!}{\Gamma(2+(r-1)\alpha)} \cdot n^{1+(r-1)\alpha}\ell(n)^{-(r-1)}$$

as $n \to \infty$, and hence (19). ∎

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