ON THE GLOBAL EXISTENCE FOR A REGULARIZED MODEL OF VISCOELASTIC NON-NEWTONIAN FLUID

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Abstract. We study the generalized Oldroyd model with viscosity depending on the shear stress behaving like $\mu(\mathbf{D}) \sim |\mathbf{D}|^{p-2}$ ($p > 6/5$), regularized by a nonlinear stress diffusion. Using the Lipschitz truncation method we prove global existence of a weak solution to the corresponding system of partial differential equations.

1. Introduction. The well known Oldroyd model describing the flow of incompressible viscoelastic fluid consists of the following system of partial differential equations:

$$\operatorname{div} \mathbf{v} = 0,$$
$$(1.1) \qquad \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = -\nabla \pi + \mu \Delta \mathbf{v} + \operatorname{div} \mathbf{T},$$
$$\mathbf{T} + \partial_t \mathbf{T} + \mathbf{v} \cdot \nabla \mathbf{T} = 2\mu_0 \mathbf{D} + \mathbf{W}\mathbf{T} - \mathbf{T}\mathbf{W} + a(\mathbf{D}\mathbf{T} + \mathbf{T}\mathbf{D}).$$

Here the unknowns are the velocity vector $\mathbf{v}$, the pressure π and the symmetric extra stress tensor $\mathbf{T}$. The tensor $\mathbf{D}$ is the symmetric part of the velocity gradient, $\mathbf{D} = \mathbf{D}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$, and $\mathbf{W}$ denotes its skew-symmetric part, $\mathbf{W} = \mathbf{W}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^T)$; μ and μ_0 are positive constants and $a \in [-1, 1]$ is a real parameter. The special choices $a = -1, 0, 1$ yield respectively the lower convected (Oldroyd-A), corotational, and upper convected (Oldroyd-B) models.

Apart from the work of Lions and Masmoudi [10], where the authors proved global existence of weak solutions for the corotational model, the global existence theory for the Oldroyd models is still an open problem. Existence of weak solutions to (1.1) for general a is proved only under some smallness assumptions, either on the time interval or on the initial data (see e.g. [5], [8], [9], [11]).

It is well known that some fluids, e.g. blood, exhibit both a viscoelastic and shear-thinning behavior. Therefore it is important to consider models which can describe these properties. In this paper we propose a generalized,

and regularized, version of the Oldroyd system (1.1). Namely, instead of a constant viscosity coefficient μ in $(1.1)_2$ we introduce a shear dependent viscosity $\mu(\mathbf{D})$ with properties specified later. This enables the model to better describe the shear thinning behavior of the fluid (or shear thickening, if needed). As even the model with constant viscosity is (except for the special case discussed above) not known to possess a weak solution, it cannot be expected for a more complex (and less regular) model. Hence we regularize equation $(1.1)_3$ for the extra stress by introducing a (nonlinear) stress diffusion. Denoting

$$(1.2) \qquad \mathbf{B}(\mathbf{v}, \mathbf{T}) := \mathbf{WT} - \mathbf{TW} + a(\mathbf{DT} + \mathbf{TD})$$

the system we study is

$$(1.3) \qquad \begin{aligned} &\operatorname{div} \mathbf{v} = 0, \\ &\partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla \pi - \operatorname{div}(\mu(\mathbf{D})\mathbf{D}) = \operatorname{div} \mathbf{T}, \\ &\partial_t \mathbf{T} + \mathbf{v} \cdot \nabla \mathbf{T} - \varepsilon \operatorname{div}(\gamma(\nabla\mathbf{T})\nabla\mathbf{T}) + \mathbf{T} = 2\mu_0 \mathbf{D} + \mathbf{B}(\mathbf{v}, \mathbf{T}). \end{aligned}$$

Here ε is a positive constant and the properties of the functions $\mu(\mathbf{D})$ and $\gamma(\nabla\mathbf{T})$ are stated later.

We consider our system (1.3) on a space-time cylinder $\Omega \times [0, T)$ where $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary, and we add the initial conditions

$$(1.4) \qquad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \quad \text{and} \quad \mathbf{T}(\mathbf{x}, 0) = \mathbf{T}_0(\mathbf{x}) \quad \text{in } \Omega$$

and the boundary conditions

$$(1.5) \qquad \mathbf{v} = \mathbf{0} \quad \text{and} \quad \frac{\partial \mathbf{T}}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on } \partial\Omega \times [0, T).$$

We assume that the function $\mu : \mathbb{R}^{3 \times 3} \to \mathbb{R}^+$ satisfies the following conditions. For some $p > 1$:

(i) $\mu(\mathbf{D})\mathbf{D}$ is *p-coercive*, i.e.

$$(1.6) \qquad \exists c > 0 \; \exists \varphi_1 \in L^1(\Omega \times (0, T)) \; \forall \mathbf{D} \in \mathbb{R}^{3 \times 3}_{\text{sym}} : \; \mu(\mathbf{D})|\mathbf{D}|^2 \geq c|\mathbf{D}|^p - \varphi_1,$$

(ii) $\mu(\mathbf{D})\mathbf{D}$ has $(p-1)$-*growth*, i.e.

$$(1.7) \qquad \exists c > 0 \; \exists \varphi_2 \in L^{p'}(\Omega \times (0, T)) \; \forall \mathbf{D} \in \mathbb{R}^{3 \times 3}_{\text{sym}} : \; \mu(\mathbf{D})|\mathbf{D}| \leq c|\mathbf{D}|^{p-1} + \varphi_2,$$

(iii) $\mu(\mathbf{D})\mathbf{D}$ is *strictly monotone*, i.e.

$$(1.8) \qquad \forall \mathbf{D}_1, \mathbf{D}_2 \in \mathbb{R}^{3 \times 3}_{\text{sym}}, \; \mathbf{D}_1 \neq \mathbf{D}_2 : \; (\mu(\mathbf{D}_1)\mathbf{D}_1 - \mu(\mathbf{D}_2)\mathbf{D}_2) : (\mathbf{D}_1 - \mathbf{D}_2) > 0,$$

and similarly $\gamma : \mathbb{R}^{3 \times 3 \times 3} \to \mathbb{R}^+$ satisfies, for some $q > 1$,

(iv) $\gamma(\nabla\mathbf{T})\nabla\mathbf{T}$ is *q-coercive*, i.e.

$$(1.9) \qquad \exists c > 0 \; \exists \varphi_3 \in L^1(\Omega \times (0, T)) \; \forall \mathbf{T} \in \mathbb{R}^{3 \times 3}_{\text{sym}} :$$
$$\gamma(\nabla\mathbf{T})|\nabla\mathbf{T}|^2 \geq c|\nabla\mathbf{T}|^q - \varphi_3,$$

(v) $\gamma(\nabla\mathbf{T})\nabla\mathbf{T}$ has $(q-1)$-*growth*, i.e.

(1.10) $\quad \exists c > 0 \ \exists\varphi_4 \in L^{q'}(\Omega \times (0,T)) \ \forall\mathbf{T} \in \mathbb{R}^{3\times 3}_{\text{sym}} :$

$$\gamma(\nabla\mathbf{T})|\nabla\mathbf{T}| \leq c|\nabla\mathbf{T}|^{q-1} + \varphi_4,$$

(vi) $\gamma(\nabla\mathbf{T})\nabla\mathbf{T}$ is *monotone*, i.e.

(1.11) $\quad \forall\mathbf{T}_1, \mathbf{T}_2 \in \mathbb{R}^{3\times 3}_{\text{sym}} : \ \big(\gamma(\nabla\mathbf{T}_1)\nabla\mathbf{T}_1 - \gamma(\nabla\mathbf{T}_2)\nabla\mathbf{T}_2\big) : (\nabla\mathbf{T}_1 - \nabla\mathbf{T}_2) \geq 0.$

This model was first introduced in [7] where existence of weak solutions for $p > 8/5$ and q sufficiently large was proven for either periodic boundary conditions or complete slip boundary conditions for the velocity $\mathbf{v}$. The proof in [7] is based on the L^∞ test functions technique developed by Frehse, Málek and Steinhauer [6]. The case of Dirichlet boundary conditions for problem (1.3) was studied in [12], where existence of weak solutions is proved for $p > 6/5$ and $q > 2p'$. The proof is based on the Lipschitz truncation method and the construction of local pressure from [4]. Moreover, in both cases, an additional lower order nonlinear term was used in order to obtain suitable a priori estimates. In this paper we further improve the condition on q, under which the existence of weak solutions is proved, remove the additional lower-order term, and based on the recent result for the Lipschitz truncation to the solenoidal functions (see [1]) we also significantly shorten the proof.

A similar model was studied in [2] where the authors consider the classical Oldroyd-B model with constant viscosity μ and linear stress diffusion, and prove global regularity of solutions in 2D.

In the whole text we denote vectors by small bold letters and tensors by capital bold letters. We introduce the following function spaces:

$$\mathcal{C}^\infty_{0,\text{div}}(\Omega) := \{\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3 \mid \mathbf{v} \in \mathcal{C}^\infty_0(\Omega), \text{div}\,\mathbf{v} = 0\},$$

$$L^2_{0,\text{div}}(\Omega) := \overline{\{\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3 \mid \mathbf{v} \in \mathcal{C}^\infty_{0,\text{div}}(\Omega)\}}^{\|\cdot\|_{L^2(\Omega)}},$$

$$W^{1,p}_{0,\text{div}}(\Omega) := \overline{\{\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3 \mid \mathbf{v} \in \mathcal{C}^\infty_{0,\text{div}}(\Omega)\}}^{\|\cdot\|_{W^{1,p}(\Omega)}},$$

$$L^2_{\text{sym}}(\Omega) := \overline{\{\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3} \mid \mathbf{T} \in \mathcal{C}^\infty(\bar{\Omega}), \ \mathbf{T} \text{ symmetric}\}}^{\|\cdot\|_{L^2(\Omega)}},$$

$$W^{1,q}_{\text{sym}}(\Omega) := \overline{\{\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3} \mid \mathbf{T} \in \mathcal{C}^\infty(\bar{\Omega}), \ \mathbf{T} \text{ symmetric}\}}^{\|\cdot\|_{W^{1,q}(\Omega)}}.$$

Moreover, we denote by X^* the dual space to X, and by $\langle T, \varphi\rangle_{k,r}$ the duality between $W^{k,r}(\Omega)$ and $(W^{k,r}(\Omega))^*$; similarly the duality between the Sobolev space of solenoidal functions with zero trace (or symmetric tensors) and its dual is denoted by $\langle T, \varphi\rangle_{k,r,\text{div}}$ (or $\langle T, \varphi\rangle_{k,r,\text{sym}}$ respectively). For $t > 0$ we denote by $Q_t = \Omega \times (0,t)$ the space-time cylinder. For $s \in [1,\infty]$ we denote by s' its dual exponent, i.e. $1/s + 1/s' = 1$.

Before defining weak solutions of (1.3) we denote

(1.12) $\qquad \mathbf{A}(\mathbf{T}, \boldsymbol{\psi}) = -\mathbf{T}\boldsymbol{\psi} + (\mathbf{T}\boldsymbol{\psi})^T + a(\mathbf{T}\boldsymbol{\psi} + (\mathbf{T}\boldsymbol{\psi})^T)$

and observe that integrating by parts and using the boundary condition $(1.5)_1$ we obtain

$$(1.13) \qquad \int_\Omega \mathbf{B}(\mathbf{v}, \mathbf{T}) : \boldsymbol{\psi}\, d\mathbf{x}\, dt = -\int_\Omega \mathbf{v} \cdot \operatorname{div} \mathbf{A}(\mathbf{T}, \boldsymbol{\psi})\, d\mathbf{x}\, dt$$

for all $\boldsymbol{\psi} \in \mathcal{C}^\infty(\overline{\Omega})$. Note moreover that in our case $\boldsymbol{\psi}$ and $\mathbf{T}$ are symmetric, therefore

$$\mathbf{A}(\mathbf{T}, \boldsymbol{\psi}) = \boldsymbol{\psi}\mathbf{T} - \mathbf{T}\boldsymbol{\psi} + a(\mathbf{T}\boldsymbol{\psi} + \boldsymbol{\psi}\mathbf{T}).$$

DEFINITION 1.1. Let $\mathbf{v}_0 \in L^2_{0,\mathrm{div}}(\Omega)$ and $\mathbf{T}_0 \in L^2_{\mathrm{sym}}(\Omega)$, and let ε, μ_0, T be positive constants. Let $\mu : \mathbb{R}^{3\times3} \to \mathbb{R}^+$ be a continuous function satisfying (1.6)–(1.8) with some $p > 6/5$, and let $\gamma : \mathbb{R}^{3\times3\times3} \to \mathbb{R}^+$ be a continuous function satisfying (1.9)–(1.11) with some $q > 1$. We say that a couple $(\mathbf{v}, \mathbf{T})$ is a *weak solution of system* (1.3) with initial conditions (1.4) and boundary conditions (1.5) if

$$\mathbf{v} \in L^\infty(0, T; L^2_{0,\mathrm{div}}(\Omega)) \cap L^p(0, T; W^{1,p}_{0,\mathrm{div}}(\Omega)),$$

$$\mathbf{T} \in L^\infty(0, T; L^2_{\mathrm{sym}}(\Omega)) \cap L^q(0, T; W^{1,q}_{\mathrm{sym}}(\Omega)),$$

$$\partial_t \mathbf{v} \in L^\sigma(0, T; (W^{1,\sigma'}_{0,\mathrm{div}}(\Omega))^*) \quad \text{for some } 1 \le \sigma \le \tfrac{5}{6}p,$$

$$\partial_t \mathbf{T} \in L^{q'}(0, T; (W^{1,q}_{\mathrm{sym}}(\Omega))^*),$$

and for almost all $t \in (0, T)$ we have

$$(1.14) \qquad \langle \partial_t \mathbf{v}(t), \boldsymbol{\varphi}\rangle_{1,\sigma',\mathrm{div}} + \int_\Omega \big(\mu(\mathbf{D}(t))\mathbf{D}(t) - \mathbf{v} \otimes \mathbf{v} \big) : \nabla\boldsymbol{\varphi}\, d\mathbf{x}$$
$$= \int_\Omega \operatorname{div} \mathbf{T}(t) \cdot \boldsymbol{\varphi}\, d\mathbf{x}$$

for all $\boldsymbol{\varphi} \in \mathcal{C}^\infty_{0,\mathrm{div}}(\Omega)$,

$$(1.15) \qquad \langle \partial_t \mathbf{T}(t), \boldsymbol{\psi}\rangle_{1,q,\mathrm{sym}} + \int_\Omega (\mathbf{v}(t) \cdot \nabla \mathbf{T}(t)) : \boldsymbol{\psi}\, d\mathbf{x}$$
$$+\varepsilon \int_\Omega \gamma(\nabla\mathbf{T}(t))\nabla\mathbf{T}(t) : \nabla\boldsymbol{\psi}\, d\mathbf{x} + \int_\Omega \mathbf{T} : \boldsymbol{\psi}\, d\mathbf{x}$$
$$= 2\mu_0 \int_\Omega \mathbf{D}(t) : \boldsymbol{\psi}\, d\mathbf{x} - \int_\Omega \mathbf{v}(t) \cdot \operatorname{div} \mathbf{A}(\mathbf{T}(t), \boldsymbol{\psi})\, d\mathbf{x}$$

for all symmetric $\boldsymbol{\psi} \in \mathcal{C}^\infty(\overline{\Omega})$,

$$(1.16) \qquad \forall \boldsymbol{\varphi} \in \mathcal{C}^\infty_{0,\mathrm{div}}(\Omega) : \lim_{t\to 0_+} \int_\Omega \mathbf{v}(t) \cdot \boldsymbol{\varphi}\, d\mathbf{x} = \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\varphi}\, d\mathbf{x},$$

$$(1.17) \qquad \forall \boldsymbol{\psi} \in \mathcal{C}^\infty(\overline{\Omega}) : \lim_{t\to 0_+} \int_\Omega \mathbf{T}(t) : \boldsymbol{\psi}\, d\mathbf{x} = \int_\Omega \mathbf{T}_0 : \boldsymbol{\psi}\, d\mathbf{x},$$

and $(1.5)_1$ is fulfilled for almost all $t \in (0, T)$ in the sense of traces.

We are now ready to state the main theorem of this paper.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. Let $\mathbf{v}_0 \in L^2_{0,\mathrm{div}}(\Omega)$, $\mathbf{T}_0 \in L^2_{\mathrm{sym}}(\Omega)$ and let ε, μ_0, T be given positive constants. Let $\mu : \mathbb{R}^{3\times 3} \to \mathbb{R}^+$ be a continuous function satisfying (1.6)–(1.8) and let $\gamma : \mathbb{R}^{3\times 3\times 3} \to \mathbb{R}^+$ be a continuous function satisfying (1.9)–(1.11). Moreover, let*

(1.18)
$$6/5 < p \le 2, \quad q \ge 4,$$
$$2 < p, \qquad q > \frac{2p}{p-1}.$$

Then there exists a weak solution to system (1.3) with initial conditions (1.4) and boundary conditions (1.5).

The rest of the paper is devoted to the proof of Theorem 1.2. The proof is organized as follows. In Section 2 we present an approximate system, show the existence of weak solutions to it and derive a priori estimates. In Section 3 we perform the first limiting procedure, pass to the limit in most of the terms in the equations, and identify the main problem which we have to solve. In Section 4 we recall some recently proved properties of the Lipschitz truncation operator which is a key tool in the final step of the proof. Then we complete the proof.

2. Approximation and a priori estimates. Let $\xi \in \mathcal{C}^\infty([0,\infty))$ be a nonincreasing function such that $\xi \equiv 1$ in $[0,1]$, $\xi \equiv 0$ in $[2,\infty)$ and $0 \ge \xi' \ge -2$. For $m \in \mathbb{N}$ we define

$$\xi^m(s) := \xi(s/m), \quad s \in [0,\infty).$$

We consider the approximate system (denoting $\mathbf{D}^m = \mathbf{D}(\mathbf{v}^m)$, $\mathbf{T}^m_M(t) = |\Omega|^{-1} \int_\Omega \mathbf{T}^m(t,\cdot)\,d\mathbf{x}$)

(2.1)
$$\mathrm{div}\,\mathbf{v}^m = 0,$$
$$\partial_t \mathbf{v}^m + \mathrm{div}(\mathbf{v}^m \otimes \mathbf{v}^m \xi^m(|\mathbf{v}^m|)) + \nabla \pi^m - \mathrm{div}(\mu(\mathbf{D}^m)\mathbf{D}^m) = \mathrm{div}\,\mathbf{T}^m,$$
$$\partial_t \mathbf{T}^m + \mathbf{v}^m \cdot \nabla \mathbf{T}^m - \varepsilon\,\mathrm{div}(\gamma(\nabla\mathbf{T}^m)\nabla\mathbf{T}^m)$$
$$+ \tfrac{1}{m}|\mathbf{T}^m_M|^{q-2}\mathbf{T}^m + \mathbf{T}^m = 2\mu_0 \mathbf{D}^m + \mathbf{B}(\mathbf{v}^m, \mathbf{T}^m)$$

with initial conditions (1.4) and boundary conditions (1.5).

DEFINITION 2.1. By a *weak solution to system* (2.1) we mean a couple $(\mathbf{v}^m, \mathbf{T}^m)$ such that

$$\mathbf{v}^m \in \mathcal{C}([0,T]; L^2_{0,\mathrm{div}}(\Omega)) \cap L^p(0,T; W^{1,p}_{0,\mathrm{div}}(\Omega)),$$
$$\mathbf{T}^m \in \mathcal{C}([0,T]; L^2_{\mathrm{sym}}(\Omega)) \cap L^q(0,T; W^{1,q}_{\mathrm{sym}}(\Omega)),$$
$$\partial_t \mathbf{v}^m \in L^{p'}(0,T; (W^{1,p}_{0,\mathrm{div}}(\Omega))^*),$$
$$\partial_t \mathbf{T}^m \in L^{q'}(0,T; (W^{1,q}_{\mathrm{sym}}(\Omega))^*)$$

satisfying

$$(2.2) \quad \int\limits_0^T \langle \partial_t \mathbf{v}^m, \boldsymbol{\varphi} \rangle_{1,p,\mathrm{div}} \, dt - \int\limits_0^T \int\limits_\Omega (\mathbf{v}^m \otimes \mathbf{v}^m \xi^m(|\mathbf{v}^m|)) : \nabla \boldsymbol{\varphi} \, d\mathbf{x} \, dt$$

$$+ \int\limits_0^T \int\limits_\Omega \mu(\mathbf{D}^m)\mathbf{D}^m : \nabla \boldsymbol{\varphi} \, d\mathbf{x} \, dt = \int\limits_0^T \int\limits_\Omega \mathrm{div} \, \mathbf{T}^m \cdot \boldsymbol{\varphi} \, d\mathbf{x} \, dt$$

for all $\boldsymbol{\varphi} \in L^p(0, T; W^{1,p}_{0,\mathrm{div}}(\Omega))$, and

$$(2.3) \quad \int\limits_0^T \langle \partial_t \mathbf{T}^m, \boldsymbol{\psi} \rangle_{1,q} \, dt + \int\limits_0^T \int\limits_\Omega (\mathbf{v}^m \cdot \nabla \mathbf{T}^m) : \boldsymbol{\psi} \, d\mathbf{x} \, dt$$

$$+ \varepsilon \int\limits_0^T \int\limits_\Omega \gamma(\nabla \mathbf{T}^m)\nabla \mathbf{T}^m : \nabla \boldsymbol{\psi} \, d\mathbf{x} \, dt$$

$$+ \int\limits_0^T \int\limits_\Omega \left(\frac{1}{m}|\mathbf{T}^m_M|^{q-2}\mathbf{T}^m + \mathbf{T}^m \right) : \boldsymbol{\psi} \, d\mathbf{x} \, dt$$

$$= 2\mu_0 \int\limits_0^T \int\limits_\Omega \mathbf{D}^m : \boldsymbol{\psi} \, d\mathbf{x} \, dt - \int\limits_0^T \int\limits_\Omega \mathbf{v}^m \cdot \mathrm{div} \, \mathbf{A}(\mathbf{T}^m, \boldsymbol{\psi}) \, d\mathbf{x} \, dt$$

for all $\boldsymbol{\psi} \in L^q(0, T; W^{1,q}_{\mathrm{sym}}(\Omega))$.

THEOREM 2.2. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with Lipschitz boundary. Let* $\mathbf{v}_0 \in L^2_{0,\mathrm{div}}(\Omega)$, $\mathbf{T}_0 \in L^2_{\mathrm{sym}}(\Omega)$, *let* ε, μ_0, T *be positive constants, and* $m \in \mathbb{N}$. *Let* $\mu : \mathbb{R}^{3\times3} \to \mathbb{R}^+$ *be a continuous function satisfying* (1.6)–(1.8) *and let* $\gamma : \mathbb{R}^{3\times3\times3} \to \mathbb{R}^+$ *be a continuous function satisfying* (1.9)–(1.11). *Moreover, let* (1.18) *hold. Then there exists a weak solution of system* (2.1) *with initial conditions* (1.4) *and boundary conditions* (1.5).

Proof. The existence is proved using the standard Galerkin method. We look for approximations in the form

$$\mathbf{v}^{m,n}(t, \mathbf{x}) = \sum_{i=1}^n c_i^n(t)\mathbf{w}^i(\mathbf{x}),$$

$$\mathbf{T}^{m,n}(t, \mathbf{x}) = \sum_{i=1}^n d_i^n(t)\mathbf{W}^i(\mathbf{x}),$$

where $\{\mathbf{w}^i(x)\}_{i=1}^\infty$ is an orthogonal system in $W^{1,2}_{0,\mathrm{div}}(\Omega)$, orthonormal in $L^2(\Omega)$ formed by smooth functions, and $\{\mathbf{W}^i(x)\}_{i=1}^\infty$ is an orthogonal system in $W^{1,2}_{\mathrm{sym}}(\Omega)$, orthonormal in $L^2(\Omega)$ formed by smooth functions such that $\mathbf{W}^1(\mathbf{x}) = (3|\Omega|)^{-1/2}\mathbf{1}_\Omega(\mathbf{x})\mathbf{I}$, where $\mathbf{1}_\Omega(\mathbf{x})$ is the characteristic function

of the domain Ω and $\mathbf{I}$ denotes the 3×3 identity matrix. The definition of the Galerkin approximation is standard. Local in time existence of a solution to the Galerkin approximation is a direct consequence of the theory of systems of ODEs. The solutions are global as soon as we are able to prove certain a priori estimates. If we "test" the Galerkin approximation for $\mathbf{v}^{m,n}$ with $2\mu_0 \mathbf{v}^{m,n}$ and the approximation for $\mathbf{T}^{m,n}$ with $\mathbf{T}^{m,n}$, and add the formulation for $\mathbf{T}^{m,n}$ "tested" with $\mathbf{W}^1$ (i.e. integrated over Ω) and multiplied by $\mathbf{T}_M^{m,n}$, we get

$$(2.4) \quad \frac{1}{2}\frac{d}{dt}\left(2\mu_0\|\mathbf{v}^{m,n}\|_2^2 + \|\mathbf{T}^{m,n}\|_2^2 + \sqrt{\frac{|\Omega|}{3}}|\mathbf{T}_M^{m,n}|^2\right)$$
$$+ 2\mu_0 c\|\nabla\mathbf{v}^{m,n}\|_p^p + \varepsilon c\|\nabla\mathbf{T}^{m,n}\|_q^q + \frac{c}{m}|\mathbf{T}_M^{m,n}|^q$$
$$\leq C(t) + c\int_\Omega |\mathbf{v}^{m,n}|\,|\nabla\mathbf{T}^{m,n}|(|\mathbf{T}^{m,n}| + |\mathbf{T}_M^{m,n}|)\,d\mathbf{x}.$$

Under the assumptions of the main theorem, it is easy to estimate the integral on the right-hand side, and thus we get control of norms coming from the left-hand side. The procedure is similar to estimates in Theorem 2.3, only slightly easier. Note, however, that the control depends on m; indeed, it is independent of n. Next, by a duality argument, we also estimate the time derivatives of $\mathbf{v}^{m,n}$ and $\mathbf{T}^{m,n}$ as stated in Definition 2.1, and using the Aubin–Lions lemma we get strong convergence of $\mathbf{v}^{m,n}$ and $\mathbf{T}^{m,n}$ in $L^2((0,T)\times\Omega)$. The proof of Theorem 2.2 is completed by the standard monotonicity argument (the Minty trick, see also Section 3) and the density argument to extend the class of test functions. ∎

THEOREM 2.3. *Under the assumptions of Theorem 2.2, for any $m \in \mathbb{N}$,*

$$(2.5) \quad \|\mathbf{v}^m\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{T}^m\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\mathbf{v}^m\|_{L^p(0,T;L^p(\Omega))}$$
$$+ \|\nabla\mathbf{T}^m\|_{L^q(0,T;L^q(\Omega))} \leq c$$

with c independent of m.

Proof. For fixed m, we proceed in much the same way as above. However, we cannot use the information from the term $m^{-1}|\mathbf{T}_M^m|^{q-2}\mathbf{T}^m$, hence the estimates are slightly more complex. Unlike the Galerkin approximation, we do not sum the estimates of $\mathbf{v}^m$ and $\mathbf{T}^m$. We test equation $(2.1)_2$ with $\mathbf{v}^m$ and get

$$(2.6) \quad \frac{1}{2}\int_0^s \frac{d}{dt}\|\mathbf{v}^m\|_2^2\,dt + c\int_0^s \|\nabla\mathbf{v}^m\|_p^p\,dt$$
$$\leq \int_0^s\int_\Omega \varphi_1\,d\mathbf{x}\,dt + \int_0^s\int_\Omega |\mathbf{v}^m|\,|\nabla\mathbf{T}^m|\,d\mathbf{x}\,dt.$$

Next we test equation $(2.1)_3$ with $\mathbf{T}^m$ and obtain

$$(2.7) \quad \frac{1}{2}\int_0^s \frac{d}{dt}\|\mathbf{T}^m\|_2^2\, dt + \varepsilon \int_0^s \|\nabla\mathbf{T}^m\|_q^q\, dt + \int_0^s \|\mathbf{T}^m\|_2^2\, dt$$

$$\leq \int_0^s\int_\Omega \varphi_3\, d\mathbf{x}\, dt + \int_0^s\int_\Omega |\mathbf{v}^m|\,|\nabla\mathbf{T}^m|\, d\mathbf{x}\, dt + \int_0^s\int_\Omega |\mathbf{v}^m|\,|\mathbf{T}^m|\,|\nabla\mathbf{T}^m|\, d\mathbf{x}\, dt.$$

The information about $\mathbf{T}^m$ on the left-hand side is not sufficient, as it is in the second power. We therefore use the same trick as before; we integrate $(2.1)_3$ over Ω and multiply the result by $\mathbf{T}_M^m$. Hence

$$(2.8) \quad \frac{1}{2}\int_0^s \frac{d}{dt}|\mathbf{T}_M^m|^2\, dt + \int_0^s |\mathbf{T}_M^m|^2\, dt \leq c\int_0^s |\mathbf{T}_M^m| \int_\Omega |\mathbf{v}^m|\,|\nabla\mathbf{T}^m|\, d\mathbf{x}\, dt.$$

We first take $p \leq 2$. Then from (2.8) we have

$$\|\mathbf{T}_M^m\|_{L^\infty(0,s)} \leq C\Big(1 + \int_0^s \|\mathbf{v}^m\|_{L^2(\Omega)}\|\nabla\mathbf{T}^m\|_{L^q(\Omega)}\, dt\Big).$$

Hence, writing the last term in (2.7) as

$$\int_0^s\int_\Omega |\mathbf{v}^m|\,|\mathbf{T}^m|\,|\nabla\mathbf{T}^m|\, d\mathbf{x}\, dt$$

$$\leq \int_0^s\int_\Omega |\mathbf{v}^m|\,|\mathbf{T}^m - \mathbf{T}_M^m|\,|\nabla\mathbf{T}^m|\, d\mathbf{x}\, dt + \int_0^s |\mathbf{T}_M^m| \int_\Omega |\mathbf{v}^m||\nabla\mathbf{T}^m|\, d\mathbf{x}\, dt,$$

we get, using Poincaré's inequality,

$$\|\nabla\mathbf{T}^m\|_{L^q(0,s;L^q(\Omega))}^q \leq C\Big(1 + \Big(\int_0^s \|\mathbf{v}^m\|_{L^2(\Omega)}^2\, dt\Big)^{1/2}\Big(\int_0^s \|\nabla\mathbf{T}^m\|_{L^q(\Omega)}^q\, dt\Big)^{1/q}$$

$$+ \Big(\int_0^s \|\mathbf{v}^m\|_{L^2(\Omega)}^2\, dt\Big)^{1/2}\Big(\int_0^s \|\nabla\mathbf{T}^m\|_{L^q(\Omega)}^q\, dt\Big)^{2/q}$$

$$+ \Big(\int_0^s \|\mathbf{v}^m\|_{L^2(\Omega)}^2\, dt\Big)\Big(\int_0^s \|\nabla\mathbf{T}^m\|_{L^q(\Omega)}^q\, dt\Big)^{2/q}\Big).$$

Therefore

$$\|\nabla\mathbf{T}^m\|_{L^q(0,T;L^q(\Omega))} \leq C\Big(1 + \Big(\int_0^s \|\mathbf{v}^m\|_{L^2(\Omega)}^2\, dt\Big)^{1/(q-2)}\Big).$$

Finally, from (2.6),

$$\|\mathbf{v}^m(s,\cdot)\|_{L^2(\Omega)}^2 \leq C\Big(1 + \Big(\int_0^s \|\mathbf{v}^m\|_{L^2(\Omega)}^2\, dt\Big)^{1/2+1/(q-2)}\Big),$$

which leads to the estimate by virtue of the Gronwall lemma provided $q \geq 4$.

For $p > 2$ we use the L^p-norm of the velocity gradient. Proceeding as above we get

$$\|\nabla \mathbf{v}^m\|^p_{L^p(0,T;L^p(\Omega))} \leq C\big(1 + (\|\nabla \mathbf{v}^m\|_{L^p(0,T;L^p(\Omega))})^{1+2/(q-2)}\big),$$

which gives the required a priori estimates provided $q > 2p/(p-1)$. ∎

3. Limiting procedure I. In what follows, we consider only the more interesting case $p \leq 2$. The other case can be proved similarly, we only need to work with different spaces corresponding to the a priori estimates.

LEMMA 3.1. *Let $\{(\mathbf{v}^m, \mathbf{T}^m)\}_{m=1}^{\infty}$ be a sequence of weak solutions of* (2.1). *Then there exists a subsequence (not relabeled) such that*

$$(3.1) \qquad \mathbf{v}^m \rightharpoonup \mathbf{v} \qquad weakly\ in\ L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega)),$$

$$(3.2) \qquad \mathbf{v}^m \rightharpoonup^* \mathbf{v} \qquad weakly^*\ in\ L^{\infty}(0,T;L^2(\Omega)),$$

$$(3.3) \qquad \mathbf{T}^m \rightharpoonup \mathbf{T} \qquad weakly\ in\ L^q(0,T;W^{1,q}_{\mathrm{sym}}(\Omega)),$$

$$(3.4) \qquad \mathbf{T}^m \rightharpoonup^* \mathbf{T} \qquad weakly^*\ in\ L^{\infty}(0,T;L^2(\Omega)),$$

$$(3.5) \qquad \mu(\mathbf{D}^m)\mathbf{D}^m \rightharpoonup \mathbf{S} \qquad weakly\ in\ L^{p'}(0,T;L^{p'}(\Omega)),$$

$$(3.6) \qquad \gamma(\nabla \mathbf{T}^m)\nabla \mathbf{T}^m \rightharpoonup \mathbf{U} \qquad weakly\ in\ L^{q'}(0,T;L^{q'}(\Omega)),$$

$$(3.7) \qquad \mathbf{v}^m \rightharpoonup \mathbf{v} \qquad weakly\ in\ L^{5p/3}\big(0,T;L^{5p/3}(\Omega)\big),$$

$$(3.8) \qquad \partial_t \mathbf{v}^m \rightharpoonup \partial_t \mathbf{v} \qquad weakly\ in\ L^{\sigma}(0,T;(W^{1,\sigma'}_{0,\mathrm{div}}(\Omega))^*),\ 1 \leq \sigma \leq 5p/6,$$

$$(3.9) \qquad \partial_t \mathbf{T}^m \rightharpoonup \partial_t \mathbf{T} \qquad weakly\ in\ L^{q'}(0,T;(W^{1,q}_{\mathrm{sym}}(\Omega))^*),$$

$$(3.10) \qquad \mathbf{v}^m \to \mathbf{v} \qquad strongly\ in\ L^p(0,T;L^r(\Omega)),\ 1 \leq r < \frac{3p}{3-p},$$

$$(3.11) \qquad \mathbf{T}^m \to \mathbf{T} \qquad strongly\ in\ L^q(0,T;L^{\bar{r}}(\Omega)),\ 1 \leq \bar{r} \leq \infty,$$

$$(3.12) \qquad \mathbf{v}^m \to \mathbf{v} \qquad strongly\ in\ L^{2\sigma}(0,T;L^{2\sigma}(\Omega)),\ 1 \leq 2\sigma < 5p/3,$$

$$(3.13) \qquad \mathbf{v}^m \otimes \mathbf{v}^m \xi^m(|\mathbf{v}^m|) \to \mathbf{v} \otimes \mathbf{v} \qquad strongly\ in\ L^{\sigma}(0,T;L^{\sigma}(\Omega)),$$
$$1 \leq \sigma < 5p/6,$$

$$(3.14) \qquad \tfrac{1}{m}|\mathbf{T}^m_M|^{q-2}\mathbf{T}^m \to \mathbf{0} \qquad strongly\ in\ L^1(0,T;L^1(\Omega)).$$

Proof. Convergences (3.1)–(3.4) are direct consequences of the a priori estimates (2.5). Convergences (3.5) and (3.6) are achieved by combining the a priori estimate (2.5) with (1.7) and (1.10) respectively. Interpolating between $L^{\infty}(0,T;L^2(\Omega))$ and $L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega))$ yields (3.7).

Next we prove (3.8). To this end it is enough to prove an a priori bound for the time derivative of $\mathbf{v}^m$ in $L^{\sigma}(0,T;(W^{1,\sigma'}_{0,\mathrm{div}})^*)$. We denote

$$\mathcal{B}_1 := \{\boldsymbol{\varphi} \in L^{\sigma'}(0,T;W^{1,\sigma'}_{0,\mathrm{div}}) \mid \|\boldsymbol{\varphi}\|_{L^{\sigma'}(0,T;W^{1,\sigma'}_{0,\mathrm{div}})} \leq 1\},$$

and using (2.2) we estimate

$$\|\partial_t \mathbf{v}^m\|_{L^\sigma(0,T;(W_{0,\mathrm{div}}^{1,\sigma'})^*)} = \sup_{\boldsymbol{\varphi}\in\mathcal{B}_1}\left|\int_0^T \langle\partial_t\mathbf{v}^m,\boldsymbol{\varphi}\rangle_{1,\sigma',\mathrm{div}}\,dt\right|$$

$$= \sup_{\boldsymbol{\psi}\in\mathcal{B}_1}\left|\int_{Q_T}\left(\mathbf{v}^m\otimes\mathbf{v}^m\xi^m(|\mathbf{v}^m|) - \mu(\mathbf{D}^m)\mathbf{D}^m - \mathbf{T}^m\right):\nabla\boldsymbol{\varphi}\;d\mathbf{x}\,dt\right|$$

$$\leq \sup_{\boldsymbol{\psi}\in\mathcal{B}_1}\left(\|\mathbf{v}^m\|_{L^{2\sigma}(Q_T)}^2 + c\|\mathbf{D}^m\|_{L^{(p-1)\sigma}(Q_T)}^{p-1} + \|\varphi_2\|_{L^\sigma(Q_T)} + \|\mathbf{T}^m\|_{L^\sigma(Q_T)}\right)$$
$$\times\|\nabla\boldsymbol{\varphi}\|_{L^{\sigma'}(Q_T)}$$

$$\leq c\left(1 + \|\mathbf{v}^m\|_{L^{5p/3}(0,T;L^{5p/3}(\Omega))}^2 + \|\mathbf{D}^m\|_{L^{(p-1)\sigma}(0,T;L^{(p-1)\sigma}(\Omega))}^{p-1}\right.$$
$$\left. + \|\mathbf{T}^m\|_{L^2(0,T;L^2(\Omega))}\right)$$

$$\leq c\left(1 + \|\mathbf{v}^m\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\nabla\mathbf{v}^m\|_{L^p(0,T;L^p(\Omega))}^2 + \|\mathbf{D}^m\|_{L^p(0,T;L^p(\Omega))}^{p-1}\right.$$
$$\left. + \|\mathbf{T}^m\|_{L^q(0,T;L^q(\Omega))}\right) \leq c;$$

we used $\sigma \leq 5p/6 < 2$ when we estimated $\|\varphi_2\|$ and $\|\mathbf{T}^m\|$ in $L^\sigma(0,T;L^\sigma(\Omega))$ and $\|\mathbf{D}^m\|$ in $L^{(p-1)\sigma}(0,T;L^{(p-1)\sigma}(\Omega))$. Convergence (3.8) now follows easily. In the same way we also derive (3.9) using (2.3). Once we have estimates for time derivatives, convergences (3.10) and (3.11) are direct consequences of the Aubin–Lions lemma. Convergence (3.12) follows from (3.7) and (3.10) with $r = p$, by interpolation. Further, (3.12) implies that $\mathbf{v}^m(\mathbf{x},t) \to \mathbf{v}(\mathbf{x},t)$ a.e. in Q_T, which together with the uniform estimate

$$\int_{Q_T}|\mathbf{v}^m\otimes\mathbf{v}^m\xi^m(|\mathbf{v}^m|)|^\sigma\,d\mathbf{x}\,dt \leq \int_{Q_T}|\mathbf{v}^m|^{2\sigma}\,d\mathbf{x}\,dt \leq c$$

and a combination of Lebesgue's and Vitali's theorems implies the strong convergence (3.13). Finally, (3.14) is a direct consequence of the bound $m^{-1}\|\mathbf{T}_M^m\|_{L^\infty(0,T)}^{q-1} \leq C$ and the estimates above. ∎

Now we pass to the limit in equation (2.3) for the extra stress tensor $\mathbf{T}$. For this, fix a test function $\boldsymbol{\psi}\in L^q(0,T;W_{\mathrm{sym}}^{1,q}(\Omega))$. Using Lemma 3.1 we claim that we can pass to the limit in all terms in (2.3). For example in the convective term

$$\int_{Q_T}\mathbf{v}^m\cdot\nabla\mathbf{T}^m:\boldsymbol{\psi}\,d\mathbf{x}\,dt$$

we use strong convergence (3.12) and weak convergence $\nabla\mathbf{T}^m \rightharpoonup \nabla\mathbf{T}$ in $L^q(Q_T)$ keeping in mind that $1/(2\sigma) + 2/q \leq 1$ as $q \geq 4$. Exactly the same argument applies to the term

$$\int_{Q_T}\mathbf{v}^m\cdot\mathrm{div}\,\mathbf{A}(\mathbf{T}^m,\boldsymbol{\psi})\,d\mathbf{x}\,dt,$$

and thus the limit equation is

$$(3.15) \quad \int_0^T \langle \partial_t \mathbf{T}, \boldsymbol{\psi} \rangle_{1,q} \, dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{T}) : \boldsymbol{\psi} \, d\mathbf{x} \, dt$$

$$+ \varepsilon \int_0^T \int_\Omega \mathbf{U} : \nabla \boldsymbol{\psi} \, d\mathbf{x} \, dt + \int_0^T \int_\Omega \mathbf{T} : \boldsymbol{\psi} \, d\mathbf{x} \, dt$$

$$= 2\mu_0 \int_0^T \int_\Omega \mathbf{D} : \boldsymbol{\psi} \, d\mathbf{x} \, dt - \int_0^T \int_\Omega \mathbf{v} \cdot \operatorname{div} \mathbf{A}(\mathbf{T}, \boldsymbol{\psi}) \, d\mathbf{x} \, dt$$

for all $\boldsymbol{\psi} \in L^q(0, T; W^{1,q}_{\text{sym}}(\Omega))$. Finally, we have to show that

$$(3.16) \qquad\qquad \mathbf{U} = \gamma(\nabla \mathbf{T}) \nabla \mathbf{T}.$$

To this end we use $\mathbf{T}^m$ as a test function in (2.3) and $\mathbf{T}$ as a test function in (3.15). Note that these are both suitable test functions in these equations. Comparing the results we get

$$\limsup_{m \to \infty} \int_{Q_T} \gamma(\nabla \mathbf{T}^m) \nabla \mathbf{T}^m : \nabla \mathbf{T}^m \, d\mathbf{x} \, dt = \int_{Q_T} \mathbf{U} : \nabla \mathbf{T} \, d\mathbf{x} \, dt.$$

Using the monotonicity assumption (1.11) for γ we get

$$0 \leq \int_{Q_T} \left(\gamma(\nabla \mathbf{T}^m) \nabla \mathbf{T}^m - \gamma(\nabla \mathbf{C}) \nabla \mathbf{C} \right) : (\nabla \mathbf{T}^m - \nabla \mathbf{C}) \, d\mathbf{x} \, dt,$$

and taking the lim sup also

$$0 \leq \int_{Q_T} (\mathbf{U} - \gamma(\nabla \mathbf{C}) \nabla \mathbf{C}) : (\nabla \mathbf{T} - \nabla \mathbf{C}) \, d\mathbf{x} \, dt,$$

which holds for all $\mathbf{C} \in L^q(0, T; W^{1,q}_{\text{sym}}(\Omega))$. Thus we can plug in $\mathbf{C} = \mathbf{T} + \beta \mathbf{E}$ for $\beta \in \mathbb{R}$ and any $\mathbf{E} \in L^q(0, T; W^{1,q}_{\text{sym}}(\Omega))$. Then letting $\beta \to 0_+$ and $\beta \to 0_-$ and using the continuity of the function γ we finally arrive at

$$0 = \int_{Q_T} (\mathbf{U} - \gamma(\nabla \mathbf{T}) \nabla \mathbf{T}) : \mathbf{E} \, d\mathbf{x} \, dt,$$

which yields (3.16).

Next we pass to the limit in the momentum equation (2.2). Here we fix a test function $\boldsymbol{\varphi} \in L^{\sigma'}(0, T; W^{1,\sigma'}_{0,\text{div}}(\Omega))$ with σ' being a dual exponent to $\sigma \in (1, 5p/6]$. Using convergences stated in Lemma 3.1 we can pass to the limit in all terms of (2.2) and arrive at

$$(3.17) \quad \int_0^T \langle \partial_t \mathbf{v}, \boldsymbol{\varphi} \rangle_{1,p,\text{div}} \, dt - \int_0^T \int_\Omega (\mathbf{v} \otimes \mathbf{v}) : \nabla \boldsymbol{\varphi} \, d\mathbf{x} \, dt$$

$$+ \int_0^T \int_\Omega \mathbf{S} : \nabla \boldsymbol{\varphi} \, d\mathbf{x} \, dt = \int_0^T \int_\Omega \operatorname{div} \mathbf{T} \cdot \boldsymbol{\varphi} \, d\mathbf{x} \, dt$$

for all $\boldsymbol{\varphi} \in L^{\sigma'}(0,T; W^{1,\sigma'}_{0,\mathrm{div}}(\Omega))$. It remains to show that

$$(3.18) \qquad\qquad \mathbf{S} = \mu(\mathbf{D})\mathbf{D}.$$

4. Lipschitz truncation and limiting procedure II. To prove (3.18) is not as easy as showing (3.16). Recall that in this case $\mathbf{v}$ is not a suitable test function in the limit equation (3.17). The rest of the paper is thus devoted to the proof of (3.18).

In order to show the convergence

$$\mu(\mathbf{D}^m)\mathbf{D}^m \to \mu(\mathbf{D})\mathbf{D} \qquad \text{a.e. in } Q_T$$

(i.e. $\mathbf{S} = \mu(\mathbf{D})\mathbf{D}$), we will use Theorem 2.2 and Corollary 2.4 from [1]. We first introduce some notation. For $\alpha > 0$ we say that $Q = I \times B \subset \mathbb{R} \times \mathbb{R}^3$ is an α-parabolic cylinder if $r_I = \alpha r_B^2$, where r_I is the radius of the interval I and r_B the radius of the ball B. By $\mathcal{Q}^\alpha$ we denote the set of all α-parabolic cylinders. For $\kappa > 0$ we denote by κQ the scaled cylinder $\kappa Q = (\kappa I) \times (\kappa B)$, where κB is the scaled ball with the same center, and similarly for κI. Then the α-parabolic maximal operators $\mathcal{M}^\alpha$ and $\mathcal{M}^\alpha_s$, $s \in [1,\infty)$, are defined by

$$(\mathcal{M}^\alpha f)(t,\mathbf{x}) := \sup_{Q' \in \mathcal{Q}^\alpha; (t,\mathbf{x}) \in Q'} \frac{1}{|Q'|} \int_{Q'} |f(s,\mathbf{y})|\, ds\, d\mathbf{y},$$

$$(\mathcal{M}^\alpha_s f)(t,\mathbf{x}) := ((\mathcal{M}^\alpha |f|^s)(t,\mathbf{x}))^{1/s}.$$

For $\lambda, \alpha > 0$ and $\sigma > 1$ we define

$$\mathcal{O}^\alpha_\lambda(\mathbf{z}) := \{(t,\mathbf{x}) \mid \mathcal{M}^\alpha_\sigma(\xi_{\frac{1}{3}Q_0}|\nabla^2 \mathbf{z}|) > \lambda \wedge \mathcal{M}^\alpha_\sigma(\xi_{\frac{1}{3}Q_0}|\partial_t \mathbf{z}|) > \lambda\}.$$

Note that $\mathbf{z} \sim \nabla^{-1}\mathbf{u}$; for a more precise definition of $\mathbf{z}$ see [1, proof of Theorem 2.2].

We have (see [1, Theorem 2.2 and Corollary 2.4])

THEOREM 4.1. *Let $1 < p < \infty$, $p, p' > \sigma$. Let $\mathbf{u}_m$ and $\mathbf{G}_m$ satisfy*

$$\langle \partial_t \mathbf{u}_m, \boldsymbol{\varphi} \rangle = \langle \mathrm{div}\, \mathbf{G}_m, \boldsymbol{\varphi} \rangle$$

for all $\boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(Q_0)$, $Q_0 = I_0 \times B_0 \subset \mathbb{R} \times \mathbb{R}^3$. Assume that $\mathbf{u}_m$ is a weak null sequence in $L^p(I_0; W^{1,p}(B_0))$ and a strong null sequence in $L^\sigma(Q_0)$ and bounded in $L^\infty(I_0; L^\sigma(B_0))$. Further assume that $\mathbf{G}_m = \mathbf{G}_{1,m} + \mathbf{G}_{2,m}$ where $\mathbf{G}_{1,m}$ is a weak null sequence in $L^{p'}(Q_0)$ and $\mathbf{G}_{2,m}$ converges strongly to zero in $L^\sigma(Q_0)$. Then there exists a double sequence $\{\lambda_{m,k}\} \subset \mathbb{R}^+$ and $k_0 \in \mathbb{N}$ with

(a) $2^{2^k} \leq \lambda_{m,k} \leq 2^{2^{k+1}}$

such that the double sequence $\mathbf{u}_{m,k} := \mathbf{u}^{\alpha_{m,k}}_{\lambda_{m,k}} \in L^1(Q_0)$, $\alpha_{m,k} := \lambda^{2-p}_{m,k}$ and $\mathcal{O}_{m,k} := \mathcal{O}^{\alpha_{m,k}}_{\lambda_{m,k}}$ defined above satisfy for all $k \geq k_0$,

(b) $\mathbf{u}_{m,k} \in L^s\big(\frac{1}{4}I_0; W^{1,s}_{0,\mathrm{div}}\big(\frac{1}{6}B_0\big)\big)$ *for all* $s < \infty$ *and* $\mathrm{supp}\,\mathbf{u}_{m,k} \subset \frac{1}{6}Q_0$,

(c) $\mathbf{u}_{m,k} = \mathbf{u}_m$ *a.e. on* $\frac{1}{8}Q_0 \setminus \mathcal{O}_{m,k}$,

(d) $\|\nabla \mathbf{u}_{m,k}\|_{L^\infty(\frac{1}{4}(Q_0))} \le c\lambda_{m,k}$,

(e) $\mathbf{u}_{m,k} \to 0$ *in* $L^\infty\big(\frac{1}{4}Q_0\big)$ *for* $m \to \infty$ *and* k *fixed*,

(f) $\nabla \mathbf{u}_{m,k} \rightharpoonup^* 0$ *in* $L^\infty\big(\frac{1}{4}Q_0\big)$ *for* $m \to \infty$ *and* k *fixed*,

(g) $\limsup_{m\to\infty} \lambda^p_{m,k}|\mathcal{O}_{m,k}| \le c2^{-k}$,

(h) $\limsup_{m\to\infty} |\int_{Q_0} \mathbf{G}_m : \nabla \mathbf{u}_{m,k}\,d\mathbf{x}\,dt| \le c\lambda^p_{m,k}|\mathcal{O}_{m,k}|$.

(i) *Additionally, let* $\zeta \in C_0^\infty\big(\frac{1}{6}Q_0\big)$ *with* $\chi_{\frac{1}{8}Q_0} \le \zeta \le \chi_{\frac{1}{6}Q_0}$. *Let* $\mathbf{u}_m$ *be uniformly bounded in* $L^\infty(I_0; L^\sigma(B_0))$. *Then for every* $\mathbf{K} \in L^{p'}\big(\frac{1}{6}Q_0\big)$,

$$\limsup_{m\to\infty} \left|\left(\int_{Q_0}(\mathbf{G}_{1,m} + \mathbf{K}) : \nabla \mathbf{u}_m\right)\zeta\chi_{\mathcal{O}^C_{m,k}}\,d\mathbf{x}\,dt\right| \le c2^{-k/p}.$$

We apply this theorem to our problem (2.1) (cf. [1, Theorem 3.1]). We denote $\mathbf{u}_m = \mathbf{v}_m - \mathbf{v}$. Then

$$\mathbf{u}_m \rightharpoonup \mathbf{0} \qquad \text{in } L^p(0,T; W^{1,p}_{0,\mathrm{div}}(\varOmega)),$$

$$\mathbf{u}_m \to \mathbf{0} \qquad \text{in } L^{2\sigma}(Q_T),$$

$$\mathbf{u}_m \rightharpoonup^* \mathbf{0} \qquad \text{in } L^\infty(0,T; L^2(\varOmega))$$

(see (3.1), (3.12) and (3.2)). Further

$$\int_0^T \int_\varOmega \mathbf{u}_m \cdot \partial_t\boldsymbol{\varphi}\,d\mathbf{x}\,dt = \int_0^T \int_\varOmega \mathbf{G}_m : \nabla\boldsymbol{\varphi}\,d\mathbf{x}\,dt$$

for all $\boldsymbol{\varphi} \in C^\infty_{0,\infty}(Q_T)$, where $\mathbf{G}_m = \mathbf{G}_{1,m} + \mathbf{G}_{2,m}$ with

$$\mathbf{G}_{1,m} = \mu(\mathbf{D}_m)\mathbf{D}_m - \mathbf{S}, \qquad \mathbf{G}_{2,m} = -\mathbf{v}_m \otimes \mathbf{v}_m\xi(|\mathbf{v}_m|) + \mathbf{v} \otimes \mathbf{v} + \mathbf{T}^m - \mathbf{T}.$$

We have $\|\mathbf{G}_{1,m}\|_{L^{p'}(0,T;L^{p'}(\varOmega))} \le C$ and $\mathbf{G}_{2,m} \to \mathbf{0}$ in $L^{\sigma_1}(Q_T)$ with $\sigma_1 = \min\{2\sigma, \overline{r}\}$ (see (2.5), (3.11) and (3.13)).

Take now $Q \subset\subset (0,T) \times \varOmega$. Due to the properties mentioned above, the assumptions of Theorem 4.1(i) are fulfilled. Hence, plugging in $\mathbf{K} = \mathbf{S} - \mu(\mathbf{D})\mathbf{D}$ we get, for $\zeta \in C_0^\infty(\frac{1}{6}Q_0)$,

$$\limsup_{m\to\infty} \left|\int_0^T \int_\varOmega (\mathbf{G}_{1,m} + \mathbf{S} - \mu(\mathbf{D})\mathbf{D}) : \nabla \mathbf{u}_m\zeta\chi_{\mathcal{O}^C_{m,k}}\,d\mathbf{x}\,dt\right| \le C2^{-k/p}.$$

Therefore

$$\limsup_{m\to\infty} \left|\int_0^T \int_\varOmega (\mu(\mathbf{D}_m)\mathbf{D}_m - \mu(\mathbf{D})\mathbf{D}) : \nabla \mathbf{u}_m\zeta\chi_{\mathcal{O}^C_{m,k}}\,d\mathbf{x}\,dt\right| \le C2^{-k/p}.$$

Take $\theta \in (0, 1)$. By Hölder's inequality and Theorem 4.1(g),

$$\limsup_{m \to \infty} \left| \int_0^T \int_\Omega \left((\mu(\mathbf{D}_m)\mathbf{D}_m - \mu(\mathbf{D})\mathbf{D}) : \nabla \mathbf{u}_m \right)^\theta \zeta \chi_{\mathcal{O}_{m,k}} \, d\mathbf{x} \, dt \right|$$
$$\leq C \limsup_{m \to \infty} |\mathcal{O}_{k,m}|^{1-\theta} \leq C 2^{-(1-\theta)k/p}.$$

Thus, both estimates imply

$$\limsup_{m \to \infty} \left| \int_0^T \int_\Omega \left((\mu(\mathbf{D}_m)\mathbf{D}_m - \mu(\mathbf{D})\mathbf{D}) : \nabla \mathbf{u}_m \right)^\theta \zeta \, d\mathbf{x} \, dt \right| \leq C 2^{-(1-\theta)k/p}.$$

As $k \to \infty$, the right-hand side tends to zero. Using now the standard approach from [3] (see also [6], or [7] with more details), due to the strict monotonicity of μ (see (1.8)), we get $\mu(\mathbf{D}_m)\mathbf{D}_m \to \mu(\mathbf{D})\mathbf{D}$ a.e. in $\frac{1}{8}Q_T$, which implies that $\mathbf{S} = \mu(\mathbf{D})\mathbf{D}$. The proof of the main theorem is finished.

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