COLLOQUIUM MATHEMATICUM

VOL. 139 2015 NO. 2

DEFORMED COMMUTATORS ON COMODULE ALGEBRAS OVER COQUASITRIANGULAR HOPF ALGEBRAS

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Abstract. We construct quantum commutators on comodule algebras over coquasitriangular Hopf algebras, so that they are quantum group coinvariant and have the generalized antisymmetry and Leibniz properties. If the coquasitriangular Hopf algebra is additionally cotriangular, then the quantum commutators satisfy a generalized Jacobi identity, and turn the comodule algebra into a quantum Lie algebra. Moreover, we investigate the projective and injective dimensions of some Doi-Hopf modules over a quantum commutative comodule algebra.

Introduction. The notion of quasitriangular Hopf algebras or strict quantum groups, introduced by Drinfeld [10], plays an important role in quantum group theory, particularly in knot theory (see [3], [13]). It is well known that the R-matrix in a quasitriangular Hopf algebra can generate a solution for the quantum Yang–Baxter equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

which frequently appears in many contexts of mathematical physics. From the categorical point of view, quasitriangular Hopf algebras are characterized by the fact that their representation categories are braided rigid tensor categories, which naturally relates them to low dimensional topology. Furthermore, it was shown by Drinfeld that any finite-dimensional Hopf algebra can be embedded in a finite-dimensional quasitriangular Hopf algebra, known now as its Drinfeld double or quantum double.

The dual concept, namely that of coquasitriangular Hopf algebras, was introduced by Majid [15] and independently by Larson and Towber [14]. Similarly to quasitriangular Hopf algebras, the corepresentation category of coquasitriangular Hopf algebras can also give rise to a braided monoidal category, but in a different way: if M, N are corepresentations of a coquasitriangular Hopf algebra (H, σ), then the canonical map, called braiding,

$$\chi_{M,N} : M \otimes N \to N \otimes M, \qquad \chi(m \otimes n) = \sigma(m_{(1)}, n_{(1)})n_{(0)} \otimes m_{(0)},$$

2010 *Mathematics Subject Classification*: Primary 16T05; Secondary 81R50.

Key words and phrases: Hopf algebra, comodule algebra, commutator, quantum Lie algebra, homology dimension.

DOI: 10.4064/cm139-2-2

is also a solution for the quantum Yang–Baxter equation (see [16]). However, what makes coquasitriangular Hopf algebras so important is the fact that the converse of the above statement is also true. Namely, by the celebrated FRT theorem, for any solution R of the quantum Yang–Baxter equation there exists a coquasitriangular bialgebra $(H(R),\sigma)$ such that $R=\chi$ (see [6]).

It is well known that Lie algebras play an important role in the description of many classical physical theories. The theory of algebraic deformation has become an important branch of Hopf algebras and Lie algebras. Multiplicative deformations were applied in algebras and modules of quantum phenomena. These led to the appearance of quantum Lie algebras, which have a quantum parameter built into their structure. The quantum Lie bracket satisfies a generalized antisymmetry, and representations of quantum Lie algebras are defined in terms of a generalized commutator.

Recently, several papers on constructing Lie (co)algebra structures from a quasitriangular Hopf algebra (H,R) have appeared. For example, an (H,R,T)-Lie coalgebra Γ^T was constructed in [22] through a right cocycle T for a given (H,R)-Lie coalgebra Γ, when (H,R) is triangular. Moreover, if $(A,\triangleright)$ is a left H-module algebra, then in [11], a quantum commutator $[\cdot,\cdot]_\chi : A\otimes A\to A\otimes A$ is constructed. It is shown that $[\cdot,\cdot]_\chi$ is quantum group covariant and has the generalized antisymmetry and Leibniz properties. Moreover, if the Hopf algebra H is triangular, then $[\cdot,\cdot]_\chi$ satisfies a generalized Jacobi identity.

The main purpose of this paper is to construct a quantum commutator $[\cdot,\cdot]_\chi : A\otimes A\to A\otimes A$ (see Section 2) related to the above braiding χ on any right H-comodule algebra A over a coquasitriangular Hopf algebra (H,σ) by a q-deformation, so that it is H-covariant and has a q-antisymmetry and a q-Leibniz property. Furthermore, we show that it also satisfies a generalized Jacobi identity, as a result of turning A into a quantum Lie algebra (see Theorem 2.6). In Section 3, we investigate the projective and injective dimensions of an (A,H)-Hopf module over a quantum commutative comodule algebra.

1. Preliminaries. Throughout this paper, we freely use the Hopf algebras and coalgebras terminology introduced in [20] and [16]. All (co)algebras and (co)modules are defined over a fixed field k. We use Sweedler's notation for coalgebras and comodules. For a coalgebra C, we write its comultiplication $\Delta(c)=c_1\otimes c_2$ for any $c\in C$; for a right C-comodule M, we denote its coaction by $\rho(m)=m_{(0)}\otimes m_{(1)}$ for any $m\in M$. Any unexplained definitions and notations may be found in [19], [8] and [17].

DEFINITION 1.1. A Hopf algebra H is called *coquasitriangular* (see [16]) if there exists a k-bilinear form $\sigma : H\otimes H\to k$ which is invertible with respect to convolution product in $\mathrm{Hom}(H\otimes H,k)$, such that for all $x,y,z\in H$,

$$\sigma(x_1, y_1)y_2x_2 = x_1y_1\sigma(x_2, y_2), \tag{1.1}$$
$$\sigma(x, yz) = \sigma(x_1, z)\sigma(x_2, y), \tag{1.2}$$
$$\sigma(xy, z) = \sigma(x, z_1)\sigma(y, z_2). \tag{1.3}$$

Furthermore, a coquasitriangular Hopf algebra (H, σ) is called *cotriangular* if the inverse map σ^{-1} of σ equals $\sigma \circ \tau$.

Let (H, σ) be a coquasitriangular Hopf algebra. Then so are $(H^{\mathrm{op}}, \sigma^{-1})$ and $(H^{\mathrm{cop}}, \sigma^{-1})$. Moreover, $\sigma^{-1} = \sigma\circ(S\otimes 1) = \sigma\circ(1\otimes S^{-1})$ and $(H, \sigma^{-1}\circ\tau)$ is another coquasitriangular structure for H by [2]. At the same time, σ satisfies a 2-cocycle condition in the sense that

$$\sigma(x_1, z_1)\sigma(y, x_2z_2) = \sigma(y_1, x_1)\sigma(y_2x_2, z) \tag{1.4}$$

for all $x, y, z \in H$ (see [9] and [23]).

Definition 1.2. A *quantum Lie algebra Γ* (see [4]) is generated by elements χ_i $(i = 1, \dots, N)$ subject to relations

$$\chi_i\chi_j - \sigma_{ij}^{mk}\chi_m\chi_k = C_{ij}^k\chi_k, \tag{1.5}$$

where the structure constants C_{ij}^k obey

$$C_{ni}^pC_{pj}^l = \sigma_{ij}^{mk}C_{nm}^pC_{pk}^l + C_{ij}^pC_{np}^l, \tag{1.6}$$
$$C_{ni}^k\sigma_{kq}^{pm} = \sigma_{iq}^{sj}\sigma_{ns}^{pk}C_{kj}^m, \tag{1.7}$$
$$(\sigma_{im}^{pj}C_{qp}^n + \delta_q^nC_{im}^j)\sigma_{nj}^{ks} = \sigma_{qi}^{jn}(\sigma_{nm}^{ps}C_{jp}^k + \delta_j^kC_{nm}^s), \tag{1.8}$$

and the matrix σ_{ij}^{mk} satisfies the Yang–Baxter equation

$$\sigma_{i_1i_2}^{j_1j_2}\sigma_{j_2i_3}^{n_2k_3}\sigma_{j_1n_2}^{k_1k_2} = \sigma_{i_2i_3}^{j_2j_3}\sigma_{i_1j_2}^{k_1n_2}\sigma_{n_2j_3}^{k_2k_3}. \tag{1.9}$$

Definition 1.3. A *$\star$-Hopf algebra* (see [7]) is a Hopf algebra $(H, 1, \Delta, \varepsilon, S)$ over the complex numbers $\mathbb{C}$ together with an antilinear involution $\star$ such that

(1) $(H, \star)$ is a $\star$-algebra, that is,

$$(x^\star)^\star = x, \quad (\lambda x)^\star = \overline{\lambda}x^\star, \quad (xy)^\star = y^\star x^\star, \quad \lambda \in \mathbb{C}, \tag{1.10}$$

where $\overline{\lambda}$ is the conjugate complex number of λ;

(2) Δ is a $\star$-algebra map.

Let H be a bialgebra. An algebra A is a *right H-comodule algebra* if A is a right H-comodule with the comodule structure map ρ being an algebra map. A is said to be an *H-module algebra* if A is an H-module satisfying the compatibility condition $h\cdot(ab) = (h_1\cdot a)(h_2\cdot b)$ and $h\cdot 1_A = \varepsilon(h)1_A$, for $h \in H, a, b \in A$. The smash product algebra $A \# H$ of an H-module algebra A and H is defined to be the k-space $A \otimes H$ with multiplication $(a \# h)(b \# g) = a(h_1\cdot b) \# h_2g$ for $h, g \in H$ and $a, b \in A$.

Definition 1.4. Let H be a Hopf algebra, and A a right H-comodule algebra. A left A-module M is called a *left-right* (A,H)*-Hopf module* if M is also a right H-comodule satisfying the compatibility condition

$$\rho_M(a\cdot m)=a_{(0)}\cdot m_{(0)}\otimes a_{(1)}m_{(1)}. \tag{1.11}$$

Denote by ${}_A\mathcal{M}^H$ the Hopf module category, and write ${}_A\mathrm{Hom}^H(M,N)$ for the set of all Hopf morphisms from M to N which are both left A-module morphisms and right H-comodule morphisms. In a similar way, we can define the right (A,H)-Hopf module category $\mathcal{M}_A^H$.

Definition 1.5. An object M of the category ${}_A\mathcal{M}_A^H$ of *two-sided* (A,H)*-Hopf modules* is an A-bimodule and a right H-comodule such that

(1) M is a left-right (A,H)-Hopf module, and
(2) M is a right (A,H)-Hopf module.

2. Construction of a quantum commutator of A**.** Let (H,σ) be a coquasitriangular Hopf algebra. By [16], the category of right H-comodules over a coquasitriangular Hopf algebra (H,σ) is braided monoidal: for any $M,N\in\mathcal{M}^H$, we have $M\otimes N\in\mathcal{M}^H$ via

$$\rho_{M\otimes N}:M\otimes N\to M\otimes N\otimes H,\ \rho_{M\otimes N}(m\otimes n)=m_{(0)}\otimes n_{(0)}\otimes m_{(1)}n_{(1)}. \tag{2.1}$$

The braiding is given by

$$\chi_{M,N}:M\otimes N\to N\otimes M,\qquad \chi(m\otimes n)=\sigma(m_{(1)},n_{(1)})n_{(0)}\otimes m_{(0)},$$

with inverse

$$\chi^{-1}_{N,M}:N\otimes M\to M\otimes N,\qquad \chi^{-1}(n\otimes m)=\sigma^{-1}(m_{(1)},n_{(1)})m_{(0)}\otimes n_{(0)}.$$

Moreover, the category is symmetric (i.e. $\chi_{M,N}\circ\chi_{N,M}=\mathrm{id}$) if (H,σ) is cotriangular.

Definition 2.1. Let (H,σ) be a coquasitriangular Hopf algebra, and (A,ρ) a right H-comodule algebra. Let $m:A\otimes A\to A$ be the multiplication of A and let $\chi=\chi_{A,A}$ be the corresponding braiding. The map

$$[\cdot,\cdot]_\chi:A\otimes A\to A,\qquad [a,b]_\chi=m\circ(\mathrm{id}_{A\otimes A}-\chi)(a\otimes b), \tag{2.2}$$

is defined to be the *quantum commutator* of A, or the *q-commutator*, for short.

Lemma 2.2. *The map* $[\cdot,\cdot]_\chi$ *has the q-Leibniz property in the first variable:*

$$[\cdot,\cdot]_\chi\circ(m\otimes\mathrm{id}_A)=m\circ(\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)+m\circ([\cdot,\cdot]_\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi), \tag{2.3}$$

or equivalently

$$\chi\circ(m\otimes\mathrm{id}_A)=(\mathrm{id}_A\otimes m)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi). \tag{2.4}$$

The corresponding equations for the Leibniz rule in the second variable are

$$[\cdot,\cdot]_\chi \circ (\mathrm{id}_A \otimes m) = m \circ ([\cdot,\cdot]_\chi \otimes \mathrm{id}_A) + m \circ (\mathrm{id}_A \otimes [\cdot,\cdot]_\chi) \circ (\chi \otimes \mathrm{id}_A), \tag{2.5}$$

or equivalently

$$\chi \circ (\mathrm{id}_A \otimes m) = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi) \circ (\chi \otimes \mathrm{id}_A). \tag{2.6}$$

Proof. We only write down the proof of (2.3) and (2.4), since (2.5) and (2.6) can be checked similarly. For any $a, b, c \in A$, we have the equality (2.4):

$$\begin{aligned}
\chi \circ (m \otimes \mathrm{id}_A)(a \otimes b \otimes c) &= \sigma(a_{(1)}b_{(1)}, c_{(1)})c_{(0)} \otimes a_{(0)}b_{(0)} \\
&\overset{(1.3)}{=} (\mathrm{id}_A \otimes m)(c_{(0)} \otimes a_{(0)} \otimes b_{(0)}\sigma(a_{(1)}, c_{(1)})\sigma(b_{(1)}, c_{(2)})) \\
&= (\mathrm{id}_A \otimes m) \circ (\chi \otimes 1)(a \otimes c_{(0)} \otimes b_{(0)}\sigma(b_{(1)}, c_{(1)})) \\
&= (\mathrm{id}_A \otimes m) \circ (\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi)(a \otimes b \otimes c).
\end{aligned}$$

Hence, we obtain (2.3):

$$\begin{aligned}
&[\cdot,\cdot]_\chi \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_{A\otimes A} - \chi) \circ (m \otimes \mathrm{id}_A) = m \circ (m \otimes \mathrm{id}_A) - m \circ \chi \circ (m \otimes \mathrm{id}_A) \\
&\overset{(2.4)}{=} m \circ (m \otimes \mathrm{id}_A) - m \circ (\mathrm{id}_A \otimes m) \circ (\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi) \\
&= [m \circ (\mathrm{id}_A \otimes m) - m \circ (\mathrm{id}_A \otimes m) \circ (\mathrm{id}_A \otimes \chi)] \\
&\quad + [m \circ (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi) - m \circ (m \otimes \mathrm{id}_A) \circ (\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi)] \\
&= m \circ (\mathrm{id}_A \otimes m) \circ [\mathrm{id}_A \otimes (\mathrm{id}_{A\otimes A} - \chi)] + m \circ (m \otimes \mathrm{id}_A) \\
&\quad \circ [(\mathrm{id}_{A\otimes A} - \chi) \otimes \mathrm{id}_A)] \circ (\mathrm{id}_A \otimes \chi) \\
&= m \circ [\mathrm{id}_A \otimes m \circ (\mathrm{id}_{A\otimes A} - \chi)] + m \circ [m \circ (\mathrm{id}_{A\otimes A} - \chi) \otimes \mathrm{id}_A)] \circ (\mathrm{id}_A \otimes \chi) \\
&= m \circ (\mathrm{id}_A \otimes [\cdot,\cdot]_\chi) + m \circ ([\cdot,\cdot]_\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi). \quad \blacksquare
\end{aligned}$$

LEMMA 2.3. *The commutator $[\cdot,\cdot]_\chi$ is H-covariant, in the sense that*

$$\rho_A \circ [\cdot,\cdot]_\chi = ([\cdot,\cdot]_\chi \otimes \mathrm{id}_H) \circ \rho_{A\otimes A}. \tag{2.7}$$

Proof. By [18], we have

$$\sigma(x_1, y_1)x_2y_2 = y_1x_1\sigma(x_2, y_2) \tag{2.8}$$

for all $x, y \in H$. Hence, for any $M, N \in \mathcal{M}^H$,

$$(\chi \otimes \mathrm{id}_H) \circ \rho_{M\otimes N} = \rho_{N\otimes M} \circ \chi. \tag{2.9}$$

Indeed, for any $m \in M$ and $n \in N$,

$$\begin{aligned}
(\chi \otimes \mathrm{id}_H) \circ \rho_{M\otimes N}(m \otimes n) &= (\chi \otimes \mathrm{id}_H)(m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}) \\
&= n_{(0)} \otimes m_{(0)} \otimes \sigma(m_{(1)}, n_{(1)})m_{(2)}n_{(2)} \\
&\overset{(2.8)}{=} n_{(0)} \otimes m_{(0)} \otimes n_{(1)}m_{(1)}\sigma(m_{(2)}, n_{(2)}) \\
&= \rho_{N\otimes M}(\sigma(m_{(1)}, n_{(1)})n_{(0)} \otimes m_{(0)}) \\
&= \rho_{N\otimes M} \circ \chi(m \otimes n),
\end{aligned}$$

as needed. Then,

$$\begin{aligned}\rho_A \circ [\cdot,\cdot]_\chi &= \rho_A \circ m \circ (\mathrm{id}_{A\otimes A} - \chi) = (m \otimes \mathrm{id}_H) \circ \rho_{A\otimes A} \circ (\mathrm{id}_{A\otimes A} - \chi)\\ &\overset{(2.9)}{=} (m \otimes \mathrm{id}_H) \circ [(\mathrm{id}_{A\otimes A} - \chi) \otimes \mathrm{id}_H] \circ \rho_{A\otimes A}\\ &= [m \circ (\mathrm{id}_{A\otimes A} - \chi) \otimes \mathrm{id}_H] \circ \rho_{A\otimes A},\end{aligned}$$

which coincides with (2.7). ■

LEMMA 2.4. *The commutator $[\cdot,\cdot]_\chi$ is q-antisymmetric, in the sense that*

$$[\cdot,\cdot]_\chi = -[\cdot,\cdot]_{\chi^{-1}} \circ \chi. \tag{2.10}$$

Proof. A direct calculation yields

$$\begin{aligned}[\cdot,\cdot]_\chi &= m \circ (\mathrm{id}_A - \chi) = -m \circ \chi + m\\ &= -m \circ (\mathrm{id}_A - \chi^{-1}) \circ \chi = -[\cdot,\cdot]_{\chi^{-1}} \circ \chi.\end{aligned}$$ ■

Assume that $\star_H : H \to H$ is a Hopf-star on H, and $\star_A : A \to A$ is compatible in the sense that

$$(\star_A \otimes 1) \circ \rho = (1 \otimes S) \circ (1 \otimes \star_H) \circ \rho \circ \star_A. \tag{2.11}$$

Then we can analyze an analogous conjugacy property of the commutator $[\cdot,\cdot]_\chi$ with respect to the star operation on an H-comodule algebra A.

LEMMA 2.5. *Let $\Pi : \mathbb{C} \to \mathbb{C}$ be the conjugacy map.*

(1) *If $\Pi \circ \sigma \circ (\star_H \otimes \star_H) = \sigma^{-1}$, then*

$$\star_A \circ [\cdot,\cdot]_\chi = [\cdot,\cdot]_{\chi^{-1}} \circ (\star_A \otimes \star_A) \circ \tau.$$

(2) *If $\Pi \circ \sigma \circ (\star_H \otimes \star_H) = \sigma \circ \tau$, then*

$$\star_A \circ [\cdot,\cdot]_\chi = [\cdot,\cdot]_\chi \circ (\star_A \otimes \star_A) \circ \tau.$$

Proof. Since $\sigma \circ (S \otimes S) = \sigma$, for any $a, b \in A$ we have

$$\begin{aligned}\star_A \circ [\cdot,\cdot]_\chi(a \otimes b) &= \star_A(ab - b_{(0)}a_{(0)}\sigma(a_{(1)}, b_{(1)}))\\ &= b^{\star_A}a^{\star_A} - {a_{(0)}}^{\star_A}{b_{(0)}}^{\star_A}\Pi \circ \sigma(a_{(1)}, b_{(1)})\\ &\overset{(2.11)}{=} b^{\star_A}a^{\star_A} - {a^{\star_A}}_{(0)}{b^{\star_A}}_{(0)}\Pi \circ \sigma\big(S({a^{\star_A}}_{(1)}{}^{\star_H}), S({b^{\star_A}}_{(1)}{}^{\star_H})\big)\\ &= b^{\star_A}a^{\star_A} - {a^{\star_A}}_{(0)}{b^{\star_A}}_{(0)}\Pi \circ \sigma({a^{\star_A}}_{(1)}{}^{\star_H}, {b^{\star_A}}_{(1)}{}^{\star_H}).\end{aligned}$$

(1) Now, if $\Pi \circ \sigma \circ (\star_H \otimes \star_H) = \sigma^{-1}$, then

$$\begin{aligned}\star_A \circ [\cdot,\cdot]_\chi(a \otimes b) &= b^{\star_A}a^{\star_A} - {a^{\star_A}}_{(0)}{b^{\star_A}}_{(0)}\sigma^{-1}({a^{\star_A}}_{(1)}, {b^{\star_A}}_{(1)})\\ &= [\cdot,\cdot]_{\chi^{-1}}(b^{\star_A} \otimes a^{\star_A}) = [\cdot,\cdot]_{\chi^{-1}} \circ (\star_A \otimes \star_A) \circ \tau(a \otimes b).\end{aligned}$$

It follows that $\star_A \circ [\cdot,\cdot]_\chi = [\cdot,\cdot]_{\chi^{-1}} \circ (\star_A \otimes \star_A) \circ \tau$.

(2) If $\Pi \circ \sigma \circ (\star_H \otimes \star_H) = \sigma \circ \tau$, then

$$\begin{aligned}\star_A \circ [\cdot,\cdot]_\chi(a \otimes b) &= b^{\star_A} a^{\star_A} - {a^{\star_A}}_{(0)} {b^{\star_A}}_{(0)} \sigma({b^{\star_A}}_{(1)}, {a^{\star_A}}_{(1)}) \\ &= [\cdot,\cdot]_\chi(b^{\star_A} \otimes a^{\star_A}) = [\cdot,\cdot]_\chi \circ (\star_A \otimes \star_A) \circ \tau(a \otimes b).\end{aligned}$$

It follows that $\star_A \circ [\cdot,\cdot]_\chi = [\cdot,\cdot]_\chi \circ (\star_A \otimes \star_A) \circ \tau$. ∎

Now we are able to prove one of the main results of this paper.

THEOREM 2.6. *Assume that (H, σ) is a coquasitriangular Hopf algebra over a fixed field k, A is an H-comodule algebra, and $[\cdot,\cdot]_\chi : A \otimes A \to A \otimes A$ is the quantum commutator on A.*

(a) *The quantum commutator $[\cdot,\cdot]_\chi$ is H-covariant and has the generalized q-antisymmetry and q-Leibniz properties.*
(b) *If H is cotriangular, then $[\cdot,\cdot]_\chi$ also satisfies the generalized Jacobi identity, turning A into a quantum Lie algebra.*

Proof. Suppose that H is coquasitriangular. Then the quantum commutator $[\cdot,\cdot]_\chi$ is H-covariant and has the generalized antisymmetry and Leibniz properties by Lemmas 2.2–2.4.

Comparing (1.5) with (2.2), we take

$$\sigma_{ij}^{mk} e_m \otimes e_k = \chi(e_i \otimes e_j) \quad \text{and} \quad C_{ij}^k e_k = [e_i, e_j]_\chi,$$

where $\{e_i\}$ are the vector space generators of A.

According to [18], χ obeys the relation

$$(\mathrm{id}_A \otimes \chi) \circ (\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi) = (\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi) \circ (\chi \otimes \mathrm{id}_A), \tag{2.12}$$

so the matrix σ_{ij}^{mk} satisfies the Yang–Baxter equation. Hence

$$\chi \circ ([\cdot,\cdot]_\chi \otimes \mathrm{id}_A) = (\mathrm{id}_A \otimes [\cdot,\cdot]_\chi) \circ (\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi), \tag{2.13}$$

$$\chi \circ (\mathrm{id}_A \otimes [\cdot,\cdot]_\chi) = ([\cdot,\cdot]_\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi) \circ (\chi \otimes \mathrm{id}_A). \tag{2.14}$$

Indeed,

$$\begin{aligned}\chi \circ ([\cdot,\cdot]_\chi \otimes \mathrm{id}_A) &= \chi \circ (m \otimes \mathrm{id}_A) \circ [\mathrm{id}_{A\otimes A\otimes A} - (\chi \otimes \mathrm{id}_A)] \\ &\overset{(2.4)}{=} (\mathrm{id}_A \otimes m) \circ (\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi) \circ [\mathrm{id}_{A\otimes A\otimes A} - (\chi \otimes \mathrm{id}_A)] \\ &\overset{(2.12)}{=} (\mathrm{id}_A \otimes m) \circ [\mathrm{id}_{A\otimes A\otimes A} - (\mathrm{id}_A \otimes \chi)] \circ (\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi) \\ &= (\mathrm{id}_A \otimes [\cdot,\cdot]_\chi) \circ (\chi \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \chi),\end{aligned}$$

and (2.14) can be proved similarly by using (2.6) and (2.12)

From (1.7), on one hand, we have

$$\begin{aligned}C_{ni}^k \sigma_{kq}^{pm} e_p \otimes e_m &= C_{ni}^k \chi(e_k \otimes e_q) = \chi \circ ([\cdot,\cdot]_\chi \otimes \mathrm{id}_A)(e_n \otimes e_i \otimes e_q) \\ &= \chi \circ (m \otimes \mathrm{id}_A) \circ [(\mathrm{id}_{A\otimes A} - \chi) \otimes \mathrm{id}_A](e_n \otimes e_i \otimes e_q).\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\sigma_{iq}^{sj}\sigma_{ns}^{pk}C_{kj}^{m}e_p\otimes e_m &= \sigma_{iq}^{sj}\sigma_{ns}^{pk}(\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)(e_p\otimes e_k\otimes e_j)\\
&= \sigma_{iq}^{sj}(\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)\circ(\chi\otimes\mathrm{id}_A)(e_n\otimes e_s\otimes e_j)\\
&= (\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)(e_n\otimes e_i\otimes e_q)\\
&= (\mathrm{id}_A\otimes m)\circ[\mathrm{id}_A\otimes(\mathrm{id}_{A\otimes A}-\chi)]\\
&\quad\ \circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)(e_n\otimes e_i\otimes e_q)\\
&= (\mathrm{id}_A\otimes m)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)(e_n\otimes e_i\otimes e_q)\\
&\quad\ -(\mathrm{id}_A\otimes m)\circ(\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)(e_n\otimes e_i\otimes e_q)\\
&\overset{(2.4),(2.12)}{=} \chi\circ(m\otimes\mathrm{id}_A)(e_n\otimes e_i\otimes e_q)-(\mathrm{id}_A\otimes m)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)\\
&\quad\ \circ(\chi\otimes\mathrm{id}_A)(e_n\otimes e_i\otimes e_q)\\
&= \chi\circ(m\otimes\mathrm{id}_A)(e_n\otimes e_i\otimes e_q)-\chi\circ(m\otimes\mathrm{id}_A)\\
&\quad\ \circ(\chi\otimes\mathrm{id}_A)(e_n\otimes e_i\otimes e_q)\\
&= \chi\circ(m\otimes\mathrm{id}_A)\circ[(\mathrm{id}_{A\otimes A}-\chi)\otimes\mathrm{id}_A](e_n\otimes e_i\otimes e_q).
\end{aligned}
$$

Let H be cotriangular. By (1.6), $[\cdot,\cdot]_\chi$ satisfies the generalized Jacobi identity

$$
[[\cdot,\cdot]_\chi,\cdot]_\chi=[\cdot,[\cdot,\cdot]_\chi]_\chi+[[\cdot,\cdot]_\chi,\cdot]_\chi\circ(1\otimes\chi). \tag{2.15}
$$

Hence, we get

$$
\begin{aligned}
[[\cdot,\cdot]_\chi,\cdot]_\chi &= m\circ(\mathrm{id}_{A\otimes A}-\chi)\circ([\cdot,\cdot]_\chi\otimes\mathrm{id}_A)\\
&= m\circ(\mathrm{id}_{A\otimes A}-\chi)\circ(m\otimes\mathrm{id}_A)\circ[(\mathrm{id}_{A\otimes A}-\chi)\otimes\mathrm{id}_A]\\
&\overset{(2.4)}{=} [m\circ(m\otimes\mathrm{id}_A)-m\circ(\mathrm{id}_A\otimes m)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)]\\
&\qquad\circ[(\mathrm{id}_{A\otimes A}-\chi)\otimes\mathrm{id}_A]\\
&= m\circ(m\otimes\mathrm{id}_A)\circ[\mathrm{id}_{A\otimes A\otimes A}-(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)]\\
&\qquad\circ[(\mathrm{id}_{A\otimes A}-\chi)\otimes\mathrm{id}_A],
\end{aligned}
$$

and by (2.6) we can similarly get

$$
\begin{aligned}
[\cdot,[\cdot,\cdot]_\chi]_\chi &= m\circ(\mathrm{id}_A\otimes m)\circ[\mathrm{id}_{A\otimes A\otimes A}-(\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)]\\
&\qquad\circ[\mathrm{id}_A\otimes(\mathrm{id}_{A\otimes A}-\chi)].
\end{aligned}
$$

Hence (2.15) can be translated into

$$
\begin{aligned}
&[\mathrm{id}_{A\otimes A\otimes A}-(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)]\circ[\mathrm{id}_{A\otimes A\otimes A}-(\chi\otimes\mathrm{id}_A)]\\
&\qquad= [\mathrm{id}_{A\otimes A\otimes A}-(\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)]\circ[\mathrm{id}_{A\otimes A\otimes A}-(\mathrm{id}_A\otimes\chi)]\\
&\qquad\quad+[\mathrm{id}_{A\otimes A\otimes A}-(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)]\\
&\qquad\quad\circ[\mathrm{id}_{A\otimes A\otimes A}-(\chi\otimes\mathrm{id}_A)]\circ(\mathrm{id}_A\otimes\chi).
\end{aligned}
$$

Then making use of the Yang–Baxter equation (2.12) for χ, we obtain the left side of the above equality:

$$\begin{aligned}\mathrm{L} &= \mathrm{id}_{A\otimes A\otimes A} - (\chi\otimes \mathrm{id}_A) - (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)\\ &\quad + (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)\\ &= \mathrm{id}_{A\otimes A\otimes A} - (\chi\otimes \mathrm{id}_A) - (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)\\ &\quad + (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi),\end{aligned}$$

and the right side:

$$\begin{aligned}\mathrm{R} &= [\mathrm{id}_{A\otimes A\otimes A} - (\mathrm{id}_A\otimes\chi) - (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A) + (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)]\\ &\quad + [\mathrm{id}_{A\otimes A\otimes A} - (\chi\otimes\mathrm{id}_A) - (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi) + (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)]\\ &\quad \circ(\mathrm{id}_A\otimes\chi)\\ &= \mathrm{id}_{A\otimes A\otimes A} - (\mathrm{id}_A\otimes\chi) - (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A) + (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)\circ(1\otimes\chi)\\ &\quad + (\mathrm{id}_A\otimes\chi) - (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi) - (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi^2)\\ &\quad + (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)\\ &\quad \circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)\\ &= \mathrm{id}_{A\otimes A\otimes A} - (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A) - (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi) - (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi^2)\\ &\quad + (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)\circ[\mathrm{id}_A\otimes(\chi-\chi^2)].\end{aligned}$$

Hence

$$\begin{aligned}0 &= \mathrm{R}-\mathrm{L}\\ &= (\chi\otimes\mathrm{id}_A) - (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A) - (\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi^2)\\ &\quad + (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi^2)]\\ &= [(\chi\otimes\mathrm{id}_A) - (\mathrm{id}_A\otimes\chi)\circ(\chi\otimes\mathrm{id}_A)]\circ[\mathrm{id}_{A\otimes A\otimes A} - (\mathrm{id}_A\otimes\chi^2)],\end{aligned}$$

which holds only if $\chi^2 = \mathrm{id}_{A\otimes A}$, i.e., if H is a cotriangular Hopf algebra.

Moreover, we have

$$\begin{aligned}(\sigma^{pj}_{im}C^n_{qp} + \delta^n_q C^j_{im})\sigma^{ks}_{nj}e_k\otimes e_s &= (\sigma^{pj}_{im}C^n_{qp} + \delta^n_q C^j_{im})\chi(e_n\otimes e_j)\\ &= \sigma^{pj}_{im}\chi\circ([\cdot,\cdot]_\chi\otimes\mathrm{id}_A)(e_q\otimes e_p\otimes e_j) + \delta^n_q\chi\circ(\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)(e_n\otimes e_i\otimes e_m)\\ &= \chi\circ([\cdot,\cdot]_\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)(e_q\otimes e_i\otimes e_m) + \chi\circ(\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)(e_q\otimes e_i\otimes e_m)\\ &\overset{(2.13),(2.14)}{=} [(\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)\circ(\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi^2) + ([\cdot,\cdot]_\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi)\\ &\quad \circ(\chi\otimes\mathrm{id}_A)](e_q\otimes e_i\otimes e_m)\\ &= [([\cdot,\cdot]_\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi) + (\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)]\circ(\chi\otimes\mathrm{id}_A)(e_q\otimes e_i\otimes e_m) \quad (\chi^2=\mathrm{id}_{A\otimes A})\\ &= \sigma^{jn}_{qi}[([\cdot,\cdot]_\chi\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\chi) + (\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)](e_j\otimes e_n\otimes e_m)\\ &= \sigma^{jn}_{qi}[\sigma^{ps}_{nm}([\cdot,\cdot]_\chi\otimes\mathrm{id}_A)(e_j\otimes e_p\otimes e_s) + \delta^k_j(\mathrm{id}_A\otimes[\cdot,\cdot]_\chi)(e_k\otimes e_n\otimes e_m)]\\ &= \sigma^{jn}_{qi}(\sigma^{ps}_{nm}C^k_{jp} + \delta^k_j C^s_{nm})e_k\otimes e_s.\end{aligned}$$

Hence (1.7) holds, so A is a quantum Lie algebra when H is cotriangular. ∎

EXAMPLE 2.7. Suppose that $\operatorname{char} k \neq 2$. Let

$$H_4 = k\langle 1, g, x, gx \mid g^2 = 1,\ x^2 = 1,\ xg = -gx\rangle$$

be Sweedler's Hopf algebra. The coalgebra structure and the antipode are

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$
$$S(g) = g \otimes g, \quad S(x) = -gx.$$

For any $\gamma \in k$, the map $\sigma_\gamma : H_4 \otimes H_4 \to k$ defined by

σ_γ	1	g	x	gx
1	1	1	0	0
g	1	-1	0	0
x	0	0	γ	γ
gx	0	0	γ	γ

is a coquasitriangular structure on H_4. Since H_4 is a right H_4-comodule algebra via its comultiplication, we can define $\chi : H_4 \otimes H_4 \to H_4 \otimes H_4$ by

χ	1	g	x	gx
1	$1\otimes 1$	$g \otimes 1$	$x \otimes 1$	$gx \otimes 1$
g	$1 \otimes g$	$-g \otimes g$	$x \otimes g$	$-gx \otimes g$
x	$1 \otimes x$	$g \otimes x$	$x \otimes x + \gamma g \otimes g$	$gx \otimes x + \gamma 1 \otimes g$
gx	$1 \otimes gx$	$-g \otimes gx$	$x \otimes gx + \gamma g \otimes 1$	$-gx \otimes gx + \gamma 1 \otimes 1$

Hence, the quantum commutator $[\cdot,\cdot]_\chi : H_4 \otimes H_4 \to H_4$ can be given by

$[\cdot,\cdot]_\chi$	1	g	x	gx
1	0	0	0	0
g	0	2	$2gx$	0
x	0	$2xg$	$-\gamma 1$	$-(2+\gamma)g$
gx	0	0	$(2-\gamma)g$	$-(2+\gamma)1$

Thus H_4 is a quantum Lie algebra by Theorem 2.6 since H_4 is also cotriangular for any $\gamma \in k$ by [16].

Example 2.8. Let $k = \mathbb{C}$ (the field of complex numbers), and denote by $\mathbb{C}\mathbb{Z}_n$ the group algebra. Let ω denote the nth primitive root of unity, $e^{2\pi i/n}$. Then by [16, Example 10.2.7], we have a coquasitriangular Hopf algebra $(\mathbb{C}\mathbb{Z}_n, \sigma)$ which is triangular when $n = 2$ (it is also an anyongenerating quantum group of [15]), with

$$\sigma : \mathbb{Z}_n \otimes \mathbb{Z}_n \to \mathbb{C}, \quad \sigma(i,j) = \omega^{ij}, \quad 0 \le i, j \le n-1.$$

Let $A = \bigoplus_{i \in \mathbb{Z}_n} A_i$ be a $\mathbb{Z}_n$-graded algebra. Then A is a $\mathbb{C}\mathbb{Z}_n$-comodule algebra with comodule structure $\rho(a_i) = a_i \otimes i$ for $a_i \in A_i$.

Define

$$\chi : A \otimes A \to A \otimes A, \quad \chi(a_i \otimes a_j) = \omega^{ij} a_j \otimes a_i,$$

and define the quantum commutator by

$$[\cdot,\cdot]_\chi : A \otimes A \to A, \quad [a_i, a_j]_\chi = a_i a_j - \omega^{ij} a_j a_i.$$

Then by Theorem 2.6, A is a quantum Lie algebra if $n = 2$.

EXAMPLE 2.9. Let us recall the definition of the coquasitriangular bialgebra $M_q(2)$ from [2] and [16], where $q \in k = \mathbb{C}$ (the field of complex numbers) is invertible with $q^2 \neq -1$.

We have $M_q(2) = k\langle a, b, c, d\rangle$ subject to the relations

$$ba = qab, \quad dc = qcd, \quad ca = qac, \quad db = qbd,$$
$$bc = cb, \quad da - ad = (q - q^{-1})bc.$$

The comultiplication on $M_q(2)$ is defined by

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

i.e.,

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$
$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,$$

The counit on $M_q(2)$ is defined by

$$\varepsilon(a) = \varepsilon(d) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0.$$

Thus $M_q(2)$ becomes a bialgebra. Moreover, $M_q(2)$ has a coquasitriangular structure when equipped with the bilinear form $\sigma : M_q(2) \otimes M_q(2) \to k$ determined by

σ	a	b	c	d
a	q	0	0	1
b	0	0	$q - q^{-1}$	0
c	0	0	0	0
d	1	0	0	q

The bialgebra $M_q(2)$ does not admit a Hopf algebra structure, but it possesses a remarkable group-like central element, called the quantum determinant:

$$\det_q = ad - q^{-1}bc,$$

which allows us to construct $\mathrm{SL}_q(2) = M_q(2)/(\det_q -1)$. It is known from [2] that $(\mathrm{SL}_q(2), \sigma)$ is a coquasitriangular Hopf algebra with the coquasitriangular bialgebra structure inherited from $(M_q(2), \sigma)$, and the antipode S is given by

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - q^{-1}bc)^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

Moreover, $(\mathrm{SL}_q(2), \sigma)$ is cotriangular when $q = 1$.

Let x and y be non-commuting variables, and let $k\{x, y\}$ be the free unital associative k-algebra generated by x and y. The standard quantum plane is the k-algebra A generated by 1, x and y with multiplication defined by $yx = qxy$. Then there is a right $\mathrm{SL}_q(2)$-comodule structure on A^{op}, whose structure map $\rho : A^{\mathrm{op}} \to A^{\mathrm{op}} \otimes \mathrm{SL}_q(2)$ is determined by

$$\rho\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \otimes S\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

so that A^{op} is a right $\mathrm{SL}_q(2)$-comodule algebra. In other words, we have

$$\begin{aligned} \rho(x) &= x \otimes (ad - q^{-1}bc)^{-1}d - y \otimes (ad - q^{-1}bc)^{-1}qb \\ &= x \otimes (\det_q)^{-1}d - y \otimes (\det_q)^{-1}qb, \end{aligned}$$

and

$$\begin{aligned} \rho(y) &= -x \otimes (ad - q^{-1}bc)^{-1}q^{-1}c + y \otimes (ad - q^{-1}bc)^{-1}a \\ &= -x \otimes (\det_q)^{-1}q^{-1}c + y \otimes (\det_q)^{-1}a. \end{aligned}$$

Since $\det_q$ is a group-like element of $M_q(2)$, we have $(\det_q)^{-1} = S(\det_q) = S^{-1}(\det_q)$ and $\varepsilon(\det_q) = \varepsilon((\det_q)^{-1}) = 1$. By [2, Lemma 6.1],

$$\sigma(\det_q, u) = \sigma(u, \det_q) = \varepsilon(u),$$

and from the fact that $\sigma = \sigma \circ (S \otimes S)$ we have

$$\sigma((\det_q)^{-1}, u) = \sigma(u, (\det_q)^{-1}) = \varepsilon(u)$$

for all $u \in M_q(2)$. Indeed,

$$\begin{aligned} \sigma((\det_q)^{-1}, u) = \sigma(S(\det_q), u) &= \sigma(S(\det_q), S(S^{-1}(u))) \\ &= \sigma(\det_q, S^{-1}(u)) = \varepsilon(u). \end{aligned}$$

Then by (1.2), (1.3) we can easily obtain

$$\sigma(\det_q u, v) = \sigma(u, \det_q v) = \sigma(u, v) = \sigma((\det_q)^{-1}u, v) = \sigma(u, (\det_q)^{-1}v)$$

for all $u, v \in M_q(2)$.

Define $\chi : A^{\mathrm{op}} \otimes A^{\mathrm{op}} \to A^{\mathrm{op}} \otimes A^{\mathrm{op}}$ by

χ	1	x	y
1	$1 \otimes 1$	$x \otimes 1$	$y \otimes 1$
x	$1 \otimes x$	$qx \otimes x$	$(q - q^{-1})x \otimes y + y \otimes x$
y	$1 \otimes y$	$x \otimes y$	$qy \otimes y$

Then the quantum commutator $[\cdot, \cdot]_\chi : A^{\mathrm{op}} \otimes A^{\mathrm{op}} \to A^{\mathrm{op}}$ is

$[\cdot, \cdot]_\chi$	1	x	y
1	0	0	0
x	0	$(1-q)x^2$	$(q-q^2)xy$
y	0	$(1-q)xy$	$(1-q)y^2$

Hence A^{op} is a quantum Lie algebra if $q = 1$. Now $A = A^{\mathrm{op}}$ is exactly the affine plane $k[x, y]$.

3. The case of $[\cdot,\cdot]_\chi = 0$. In this section, we mainly investigate projective and injective dimensions of an (A,H)-Hopf module over a quantum commutative comodule algebra. We denote the projective and injective (A,H)-Hopf modules by $\mathrm{Proj}\,{}_A\mathcal{M}^H$ and $\mathrm{Inj}\,{}_A\mathcal{M}^H$ respectively.

LEMMA 3.1. *Let (H,σ) be a coquasitriangular Hopf algebra, and A a right H-comodule algebra. Then A is a left H^{cop}-module algebra defined by*

$$h\cdot a = a_{(0)}\sigma(h, a_{(1)})$$

for all $a\in A, h\in H$.

Proof. It is easy to see that $1_H\cdot a = a$, and for any $a,b\in A$ and $h,g\in H^{\mathrm{cop}}$, we have

$$\begin{aligned} h\cdot(g\cdot a) &= h\cdot a_{(0)}\sigma(g,a_{(1)})\\ &= a_{(0)}\sigma(h,a_{(1)})\sigma(g,a_{(2)})\\ &\overset{(1.3)}{=} a_{(0)}\sigma(hg,a_{(1)}) = (hg)\cdot a, \end{aligned}$$

and

$$\begin{aligned} h\cdot(ab) &= a_{(0)}b_{(0)}\sigma(h,a_{(1)}b_{(1)})\\ &\overset{(1.2)}{=} a_{(0)}\sigma(h_2,a_{(1)})b_{(0)}\sigma(h_1,b_{(1)})\\ &= (h_2\cdot a)(h_1\cdot b),\\ h\cdot 1_A &= 1_A\sigma(h,1_H) = \varepsilon(h)1_A, \end{aligned}$$

as needed. ∎

Suppose that (H,σ) is coquasitriangular. If for any $a,b\in A$, we always have $[a,b]_\chi = 0$, that is,

$$ab = \sigma(a_{(1)},b_{(1)})b_{(0)}a_{(0)}, \tag{3.1}$$

then we call A *quantum commutative* with respect to (H,σ).

By (3.1), it is easy to see that A is quantum commutative with respect to (H,σ) if and only if

$$ab = \sigma^{-1}(b_{(1)},a_{(1)})b_{(0)}a_{(0)} \tag{3.2}$$

for all $a,b\in A$.

As a matter of fact, for any $a,b\in A$, if A is quantum commutative with respect to (H,σ), then

$$\begin{aligned} \sigma^{-1}(b_{(1)},a_{(1)})b_{(0)}a_{(0)} &\overset{(3.1)}{=} \sigma^{-1}(b_{(2)},a_{(2)})\sigma(b_{(1)},a_{(1)})a_{(0)}b_{(0)}\\ &= \varepsilon(b_{(1)}a_{(1)})a_{(0)}b_{(0)} = ab. \end{aligned}$$

Conversely, if (3.2) holds, then we can similarly obtain (3.1).

LEMMA 3.2. *Let H be a semisimple Hopf algebra, and A a left H-module algebra. Assume that A is also a quantum commutative right H-comodule* *algebra. Then* gl.dim $A \# H$ = gl.dim A, *where* $A \# H$ *denotes the usual smash product of* A *and* H.

Proof. Apply [12, Proposition 2]. ∎

PROPOSITION 3.3. *Let* (H, σ) *be a semisimple coquasitriangular Hopf algebra, and* A *a quantum commutative right* H*-comodule algebra. Then* gl.dim $A \# H^{\mathrm{cop}}$ = gl.dim A.

Proof. By Lemma 3.1, the smash product $A \# H^{\mathrm{cop}}$ can be formed. Then the conclusion is straightforward by Lemma 3.2. ∎

LEMMA 3.4. *Let* (H, σ) *be a coquasitriangular Hopf algebra, and* A *a quantum commutative right* H*-comodule algebra with respect to* (H, σ). *For* $M \in {}_A\mathcal{M}^H$, *define an* A*-action on* M *by*

$$m \cdot a = \sigma^{-1}(a_{(1)}, m_{(1)}) a_{(0)} \cdot m_{(0)}. \tag{3.3}$$

Then $M \in {}_A\mathcal{M}_A^H$.

Proof. The following proof will use the fact that $(H^{\mathrm{cop}}, \sigma^{-1})$ is a coquasitriangular Hopf algebra.

First, M is a right A-module, because

$$\begin{aligned}
(m \cdot a) \cdot b &\overset{(3.3)}{=} \sigma^{-1}(a_{(1)}, m_{(1)})(a_{(0)} \cdot m_{(0)}) \cdot b \\
&= \sigma^{-1}(a_{(2)}, m_{(2)})\sigma^{-1}(b_{(1)}, a_{(1)}m_{(1)}) b_{(0)} a_{(0)} \cdot m_{(0)} \\
&\overset{(1.4)}{=} \sigma^{-1}(b_{(2)}, a_{(2)})\sigma^{-1}(b_{(1)}a_{(1)}, m_{(1)}) b_{(0)} a_{(0)} \cdot m_{(0)} \\
&= \sigma^{-1}(b_{(1)}, a_{(1)}) m \cdot (b_{(0)} a_{(0)}) \overset{(3.2)}{=} m \cdot (ab).
\end{aligned}$$

Second, M is an A-bimodule, because

$$\begin{aligned}
(a \cdot m) \cdot b &= \sigma^{-1}(b_{(1)}, a_{(1)}m_{(1)}) b_{(0)} a_{(0)} \cdot m_{(0)} \\
&\overset{(3.1)}{=} \sigma^{-1}(b_{(2)}, a_{(2)}m_{(1)})\sigma(b_{(1)}, a_{(1)}) a_{(0)} b_{(0)} \cdot m_{(0)} \\
&\overset{(1.2)}{=} \sigma^{-1}(b_{(3)}, m_{(1)})\sigma^{-1}(b_{(2)}, a_{(2)})\sigma(b_{(1)}, a_{(1)}) a_{(0)} b_{(0)} \cdot m_{(0)} \\
&= \sigma^{-1}(b_{(1)}, m_{(1)}) a b_{(0)} \cdot m_{(0)} = a \cdot (m \cdot b).
\end{aligned}$$

Since $(H^{\mathrm{cop}}, \sigma^{-1})$ is coquasitriangular, by (2.8), we have

$$\sigma^{-1}(x_2, y_2) x_1 y_1 = y_2 x_2 \sigma^{-1}(x_1, y_1). \tag{3.4}$$

Lastly, M is a right (A, H)-Hopf module. This is because

$$\begin{aligned}
\rho(m \cdot a) &= a_{(0)} \cdot m_{(0)} \otimes a_{(1)} m_{(1)} \sigma^{-1}(a_{(2)}, m_{(2)}) \\
&\overset{(3.4)}{=} a_{(0)} \cdot m_{(0)} \otimes m_{(2)} a_{(2)} \sigma^{-1}(a_{(1)}, m_{(1)}) \\
&= m_{(0)} \cdot a_{(0)} \otimes m_{(1)} a_{(1)},
\end{aligned}$$

as required. ∎

Let us recall from [21] the definition of ${}_A\mathrm{HOM}(M,N)$, the rational part of ${}_A\mathrm{Hom}(M,N)$. Let H be a Hopf algebra with bijective antipode S, and A an H-comodule algebra. Let $M,N \in {}_A\mathcal{M}^H$. For every $f \in {}_A\mathrm{Hom}(M,N)$, we define $\rho(f) \in {}_A\mathrm{Hom}(M, N\otimes H)$ by

$$\rho(f)(m) = f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}. \tag{3.5}$$

Then f is rational if $\rho(f)(m) \in {}_A\mathrm{Hom}(M,N)\otimes H \subseteq {}_A\mathrm{Hom}(M,N\otimes H)$. In the following, we denote $\rho(f) = f_0 \otimes f_1$ for $f \in {}_A\mathrm{HOM}(M,N)$.

LEMMA 3.5. *Let (H,σ) be a coquasitriangular Hopf algebra, and A a quantum commutative right H-comodule algebra with respect to (H,σ). If $M,N \in {}_A\mathcal{M}^H$, then ${}_A\mathrm{HOM}(M,N) \in {}_A\mathcal{M}^H$.*

Proof. For every $f \in {}_A\mathrm{Hom}(M,N), a\in A$, we let $(a \triangleright f)(m) = f(m\cdot a)$. This action obviously makes ${}_A\mathrm{HOM}(M,N)$ into a left A-module, once the A-action is shown to be well defined.

On the one hand, $a \triangleright f \in {}_A\mathrm{Hom}(M,N)$: for any $b\in A$ and $m \in M$,

$$\begin{aligned}(a\triangleright f)(b\cdot m) &= f((b\cdot m)\cdot a) = f(b\cdot(m\cdot a))\\ &= b\cdot f(m\cdot a) = b\cdot (a\triangleright f)(m).\end{aligned}$$

On the other hand, $a\triangleright f \in {}_A\mathrm{HOM}(M,N)$:

$$\begin{aligned}\rho(a\triangleright f)(m) &= (a\triangleright f)(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)})(a\triangleright f)(m_{(0)})_{(1)}\\
&= f(m_{(0)}\cdot a)_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)}\cdot a)_{(1)}\\
&= f(a_{(0)}\cdot m_{(0)})_{(0)} \otimes \sigma^{-1}(a_{(1)}, m_{(1)}) S^{-1}(m_{(2)}) f(a_{(0)}\cdot m_{(0)})_{(1)}\\
&= a_{(0)}\cdot f(m_{(0)})_{(0)} \otimes \sigma^{-1}(a_{(2)}, m_{(1)}) S^{-1}(m_{(2)}) a_{(1)} f(m_{(0)})_{(1)}\\
&= a_{(0)}\cdot f(m_{(0)})_{(0)} \otimes \sigma(a_{(2)}, S^{-1}(m_{(1)})) S^{-1}(m_{(2)}) a_{(1)} f(m_{(0)})_{(1)}\\
&\overset{(3.4)}{=} a_{(0)}\cdot f(m_{(0)})_{(0)} \otimes \sigma(a_{(1)}, S^{-1}(m_{(2)})) a_{(2)} S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}\\
&= a_{(0)}\cdot f_0(m_{(0)}) \otimes \sigma^{-1}(a_{(1)}, m_{(1)}) a_{(2)} f_1\\
&= f_0(\sigma^{-1}(a_{(1)}, m_{(1)}) a_{(0)}\cdot m_{(0)}) \otimes a_{(2)} f_1\\
&= f_0(m\cdot a_{(0)}) \otimes a_{(1)} f_1 = (a_{(0)} \triangleright f_0)(m) \otimes a_{(1)} f_1.\end{aligned}$$

It follows that $\rho(a\triangleright f) \in {}_A\mathrm{Hom}(M,N)\otimes H$, i.e., $a\triangleright f$ is rational, and $\rho(a\triangleright f) = a_{(0)} \triangleright f_0 \otimes a_{(1)} f_1$. Therefore, ${}_A\mathrm{HOM}(M,N) \in {}_A\mathcal{M}^H$. ∎

COROLLARY 3.6. *If the quantum commutative right H-comodule algebra A is a faithfully flat right H-Galois extension of its coinvariant subalgebra A^{coH}, then ${}_A\mathrm{HOM}(M,N) \cong A\otimes_{A^{\mathrm{coH}}} {}_A\mathrm{Hom}^H(M,N)$ as left-right (A,H)-Hopf modules.*

Proof. By [5], we have ${}_A\mathrm{HOM}(M,N)^{\mathrm{coH}} = {}_A\mathrm{Hom}^H(M,N)$. Now the conclusion follows from Lemma 3.5 and the fundamental structure theorem for left-right (A,H)-Hopf modules. ∎

The following extends Proposition 2.14 of [5].

Lemma 3.7. *Let (H,σ) be a cotriangular Hopf algebra, and A a quantum commutative right H-comodule algebra with respect to (H,σ). Take $M,N,P\in {}_A\mathcal{M}^H$.*

(1) *There is a k-space isomorphism*

$$\widehat{\phi}: {}_A\mathrm{Hom}^H(M\otimes_A N,P)\to {}_A\mathrm{Hom}^H(M,{}_A\mathrm{HOM}(N,P)),$$
$$\widehat{\phi}(f)(m)(n)=f\circ\widehat{\chi}_{N,M}(n\otimes m),$$

where $\widehat{\chi}_{N,M}: M\otimes_A N\to N\otimes_A M$ is induced by the braiding $\chi_{N,M}$, and the A-action and H-coaction on $M\otimes_A N$ are given by $a\cdot(m\otimes n)=a\cdot m\otimes n$ and (2.1), respectively.

(2) *If $N\in \mathrm{Flat}\,{}_A\mathcal{M}^H$, then the functor ${}_A\mathrm{HOM}(N,-): {}_A\mathcal{M}^H\to {}_A\mathcal{M}^H$ preserves injective objects.*

Proof. (1) The right H-coaction on $M\otimes_A N$ is well defined since

$$\begin{aligned}\rho(m\cdot a\otimes n)&=m_{(0)}\cdot a_{(0)}\otimes n_{(0)}\otimes m_{(1)}a_{(1)}n_{(1)}\\&=m_{(0)}\otimes a_{(0)}\cdot n_{(0)}\otimes m_{(1)}a_{(1)}n_{(1)}=\rho(m\otimes a\cdot n).\end{aligned}$$

Consequently, $M\otimes_A N\in {}_A\mathcal{M}^H$:

$$\begin{aligned}\rho(a\cdot(m\otimes n))=\rho(a\cdot m\otimes n)&=a_{(0)}\cdot m_{(0)}\otimes n_{(0)}\otimes a_{(1)}m_{(1)}n_{(1)}\\&=a_{(0)}\cdot(m_{(0)}\otimes n_{(0)})\otimes a_{(1)}m_{(1)}n_{(1)}\\&=a_{(0)}\cdot(m\otimes n)_{(0)}\otimes a_{(1)}(m\otimes n)_{(1)}.\end{aligned}$$

The map $\widehat{\chi}_{M,N}: M\otimes_A N\to N\otimes_A M$ is a well-defined (A,H)-Hopf module isomorphism:

$$\begin{aligned}&\widehat{\chi}_{M,N}(m\cdot a\otimes n)=\sigma(m_{(1)}a_{(1)},n_{(1)})n_{(0)}\otimes m_{(0)}\cdot a_{(0)}\\&\quad=\sigma(m_{(2)}a_{(2)},n_{(1)})\sigma^{-1}(a_{(1)},m_{(1)})n_{(0)}\otimes a_{(0)}\cdot m_{(0)}\\&\quad=\sigma(m_{(2)}a_{(2)},n_{(1)})\sigma^{-1}(a_{(1)},m_{(1)})n_{(0)}\cdot a_{(0)}\otimes m_{(0)}\\&\quad=\sigma(m_{(2)}a_{(3)},n_{(2)})\sigma^{-1}(a_{(2)},m_{(1)})\sigma^{-1}(a_{(1)},n_{(1)})a_{(0)}\cdot n_{(0)}\otimes m_{(0)}\\&\quad\overset{(3.4)}{=}\sigma(a_{(2)}m_{(1)},n_{(2)})\sigma^{-1}(a_{(3)},m_{(2)})\sigma^{-1}(a_{(1)},n_{(1)})a_{(0)}\cdot n_{(0)}\otimes m_{(0)}\\&\quad\overset{(1.3)}{=}\sigma(a_{(2)},n_{(2)})\sigma(m_{(1)},n_{(3)})\sigma^{-1}(a_{(3)},m_{(2)})\sigma^{-1}(a_{(1)},n_{(1)})a_{(0)}\cdot n_{(0)}\otimes m_{(0)}\\&\quad=\sigma(m_{(1)},n_{(1)})\sigma^{-1}(a_{(1)},m_{(2)})a_{(0)}\cdot n_{(0)}\otimes m_{(0)}\\&\quad=\sigma(m_{(1)},n_{(1)})\sigma(m_{(2)},a_{(1)})a_{(0)}\cdot n_{(0)}\otimes m_{(0)}\quad(\sigma^{-1}=\sigma\circ\tau)\\&\quad\overset{(1.2)}{=}\sigma(m_{(1)},a_{(1)}n_{(1)})a_{(0)}\cdot n_{(0)}\otimes m_{(0)}\\&\quad=\widehat{\chi}_{M,N}(m\otimes a\cdot n),\end{aligned}$$

and

$$
\begin{aligned}
\rho \circ \widehat{\chi}_{M,N}(m \otimes n) &= \rho(\sigma(m_{(1)}, n_{(1)})n_{(0)} \otimes m_{(0)}) \\
&= n_{(0)} \otimes m_{(0)} \otimes \sigma(m_{(2)}, n_{(2)})n_{(1)}m_{(1)} \\
&\overset{(2.8)}{=} \sigma(m_{(1)}, n_{(1)})n_{(0)} \otimes m_{(0)} \otimes m_{(2)}n_{(2)} \\
&= \widehat{\chi}_{M,N}(m_{(0)} \otimes n_{(0)}) \otimes m_{(1)}n_{(1)} \\
&= (\widehat{\chi}_{M,N} \otimes 1) \circ \rho(m \otimes n), \\
\widehat{\chi}_{M,N}(a \cdot (m \otimes n)) &= \sigma(a_{(1)}m_{(1)}, n_{(1)})n_{(0)} \otimes a_{(0)} \cdot m_{(0)} \\
&\overset{(1.3)}{=} \sigma(a_{(1)}, n_{(1)})\sigma(m_{(1)}, n_{(2)})n_{(0)} \cdot a_{(0)} \otimes m_{(0)} \\
&= \sigma(a_{(2)}, n_{(2)})\sigma^{-1}(a_{(1)}, n_{(1)})\sigma(m_{(1)}, n_{(3)})a_{(0)} \cdot n_{(0)} \otimes m_{(0)} \\
&= \sigma(m_{(1)}, n_{(1)})a \cdot n_{(0)} \otimes m_{(0)} \\
&= a \cdot (n_{(0)} \otimes m_{(0)}\sigma(m_{(1)}, n_{(1)})) \\
&= a \cdot \widehat{\chi}_{M,N}(m \otimes n).
\end{aligned}
$$

Hence, $\widehat{\chi}_{M,N}$ is an (A, H)-Hopf module isomorphism.

Hence, by [1, Lemma 19.11] and [6, Proposition 1.2], there exist k-space isomorphisms

$$
{}_A\mathrm{Hom}^H(M, {}_A\mathrm{HOM}(N, P)) \cong {}_A\mathrm{Hom}^H(N \otimes_A M, P) \cong {}_A\mathrm{Hom}^H(M \otimes_A N, P).
$$

(2) Let $I \in \mathrm{Inj}\, {}_A\mathcal{M}^H$. Then the functor

$$
{}_A\mathrm{Hom}^H(-, I) : {}_A\mathcal{M}^H \to {}_k\mathcal{M}
$$

is exact. Since $N \in \mathrm{Flat}\, {}_A\mathcal{M}^H$, we know that $- \otimes_A N$ is exact, and it follows from (1) that

$$
{}_A\mathrm{Hom}^H(-, {}_A\mathrm{HOM}(N, I)) \cong {}_A\mathrm{Hom}^H(- \otimes_A N, I) : {}_A\mathcal{M}^H \to {}_k\mathcal{M}
$$

is exact. Thus ${}_A\mathrm{HOM}(N, I) \in \mathrm{Inj}\, {}_A\mathcal{M}^H$. ∎

PROPOSITION 3.8. *Let (H, σ) be a cotriangular Hopf algebra, and A a quantum commutative cleft right H-comodule algebra with respect to (H, σ). Take $M, N, P \in {}_A\mathcal{M}^H$ with $N \in \mathrm{Proj}\, {}_A\mathcal{M}^H$. Then*

$$
{}_A\mathrm{Ext}^{Hn}(M, {}_A\mathrm{HOM}(N, P)) \cong {}_A\mathrm{Ext}^{Hn}(M \otimes_A N, P),
$$

where ${}_A\mathrm{Ext}^{Hn}(-, -)$ is the right derived functor of ${}_A\mathrm{Hom}^H(-, -) : {}_A\mathcal{M}^H \times {}_A\mathcal{M}^H \to {}_k\mathcal{M}$.

Proof. By Lemma 3.7, the functor ${}_A\mathrm{Hom}^H(M, {}_A\mathrm{HOM}(N, -))$ is an isomorphism to ${}_A\mathrm{Hom}^H(M \otimes_A N, -)$. By [24], the functor ${}_A\mathrm{HOM}(N, -)$ is exact, and it preserves injective resolutions by Lemma 3.7.

Let $\mathbb{E}_{\widehat{P}}$ be a deleted injective resolution of P in ${}_A\mathcal{M}^H$. Consequently ${}_A\mathrm{HOM}(N, \mathbb{E}_{\widehat{P}})$ is an injective resolution of ${}_A\mathrm{HOM}(N, P)$ in ${}_A\mathcal{M}^H$, and

$$
{}_A\mathrm{Hom}^H(M, {}_A\mathrm{HOM}(N, \mathbb{E}_{\widehat{P}})) \cong {}_A\mathrm{Hom}^H(M \otimes_A N, \mathbb{E}_{\widehat{P}}).
$$

Taking homology on both sides of the above formula, we obtain

$$ {}_A\mathrm{Ext}^{Hn}(M, {}_A\mathrm{HOM}(N,P)) \cong {}_A\mathrm{Ext}^{Hn}(M \otimes_A N, P). \blacksquare $$

Let $M \in {}_A\mathcal{M}^H$ and $pd(M)$ and $id(M)$ denote the projective and injective dimensions of $M \in {}_A\mathcal{M}^H$, respectively. Then, by the above proposition, we have

Corollary 3.9. *Under the assumptions of Proposition* 3.8*, we have:*

(1) $pd(M \otimes_A N) \leq pd(M)$. *In particular,* $pd(N) \leq pd(A)$.
(2) $id({}_A\mathrm{HOM}(N,P)) \leq id(P)$.

Corollary 3.10. *Under the assumptions of Proposition* 3.8*, we also have:*

(1) *If* $M \in \mathrm{Proj}\,{}_A\mathcal{M}^H$*, then* $M \otimes_A N \in \mathrm{Proj}\,{}_A\mathcal{M}^H$.
(2) *If* $P \in \mathrm{Inj}\,{}_A\mathcal{M}^H$*, then* ${}_A\mathrm{HOM}(N,P) \in \mathrm{Inj}\,{}_A\mathcal{M}^H$.

Acknowledgements. The authors would like to thank the referee for the suggestions improving the exposition. The first author would like to thank Professor Gabriella Böhm for her help when she visited Nanjing Agricultural University.

This work was supported by the Natural Science Foundation of Jiangsu Province (BK20141358), the Fundamental Research Funds for the Central Universities (KYZ201125), the PhD Scientific Research Foundation of Jinling Institute of Technology (jit-6-201402), the Fundamental Research Funds of Jinling Institute of Technology (2014-jit-n-08) and the National Natural Science Foundation of China (11171149).

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Received 13 August 2014;
revised 28 September 2014 (6341)