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REGULARITY AND DECAY OF 3D INCOMPRESSIBLE MHD EQUATIONS WITH NONLINEAR DAMPING TERMS

BY

ZHUAN YE (Beijing)

Abstract. We prove the existence and uniqueness of global strong solutions to the Cauchy problem for 3D incompressible MHD equations with nonlinear damping terms. Moreover, the preliminary L^2 decay for weak solutions is also established.

1. Introduction. This paper focuses on the following Cauchy problem of 3D incompressible MHD equations with nonlinear damping terms:

$$(1.1)\quad \begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nu|u|^{\alpha-1}u + \nabla\pi = (b\cdot\nabla)b, & t\ge 0, x\in\mathbb{R}^3,\\ \partial_t b + (u\cdot\nabla)b - \Delta b + \eta|b|^{\beta-1}b = (b\cdot\nabla)u, & t\ge 0, x\in\mathbb{R}^3,\\ \nabla\cdot u = 0,\quad \nabla\cdot b = 0, & t\ge 0, x\in\mathbb{R}^3,\\ u(x,0) = u_0(x),\quad b(x,0) = b_0(x), & x\in\mathbb{R}^3, \end{cases}$$

where $\nu,\eta\ge 0$; $\alpha,\beta\ge 1$ are real exponents; and $u=u(x,t)\in\mathbb{R}^3$, $\pi=\pi(x,t)\in\mathbb{R}$ and $b=b(x,t)\in\mathbb{R}^3$ denote the velocity vector, scalar pressure and the magnetic field of the fluid, respectively. This model comes from porous media flow, friction effects, or some dissipative mechanisms, mainly as a limiting system from compressible flows. In the case when $\nu=\eta=0$ the system (1.1) reduces to the standard 3D MHD system. For the standard MHD, one has global existence and uniqueness of weak solutions in 2D and local existence of weak solutions in 3D. These results go back to Duvaut & Lions [DL] and Sermange & Temam [ST]. When $b=0$, the system (1.1) reduces to the Navier–Stokes equation with damping

$$(1.2)\quad \begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nu|u|^{\alpha-1}u + \nabla\pi = 0, & t\ge 0,\ x\in\mathbb{R}^3,\\ \nabla\cdot u = 0, & t\ge 0,\ x\in\mathbb{R}^3,\\ u(x,0) = u_0(x), & x\in\mathbb{R}^3. \end{cases}$$

The system (1.2) was studied first by Cai and Jiu [CJ] where the physical background is given. They proved that (1.2) has global weak solutions for any $\alpha\ge 1$, and global strong solutions for any $\alpha\ge 7/2$. Furthermore, the

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strong solution is unique for any $7/2 \le \alpha \le 5$. This result was improved significantly by Zhou [Z2] and Zhang–Wu–Lu [ZWL]. We emphasize that Zhou [Z2] proved that the strong solution exists globally for $\alpha \ge 3$. Fundamental mathematical issues such as the asymptotic behavior of solutions to three-dimensional Navier–Stokes equations with nonlinear damping (1.2) and related models have generated extensive research and many interesting results have been obtained (see, e.g., [AT, JZD, J, G1, G2, JZ]). As far as the authors know, the question of global existence or finite time blow-up of smooth solutions for the 3D MHD equations is still one of the most outstanding open problems in applied analysis, even though it has attracted significant attention. By taking advantage of some nonlinear damping terms, we are able to show that the system (1.1) admits a unique global strong solution for some α and β.

2. Preliminaries and the main theorem. Firstly, we state the definition of a weak solution to (1.1).

Definition 2.1. The triplet (u, b, π) is called a *weak solution* of the MHD equations (1.1) if:

- $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \cap L^{\alpha+1}(0,T;L^{\alpha+1}(\mathbb{R}^3))$,
  $b \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \cap L^{\beta+1}(0,T;L^{\beta+1}(\mathbb{R}^3))$;
- for any $\psi \in C_c^\infty([0,T) \times \mathbb{R}^3)$ with $\nabla \cdot \psi = 0$,
$$-\int_0^T\int_{\mathbb{R}^3} u\partial_t\psi\,dx\,dt + \int_0^T\int_{\mathbb{R}^3} \partial_i u_j \partial_i \psi_j\,dx\,dt + \int_0^T\int_{\mathbb{R}^3} (u\cdot\nabla u + |u|^{\alpha-1}u)\psi\,dx\,dt = \int_{\mathbb{R}^3} u_0\psi(x,0)\,dx,$$
  and
$$-\int_0^T\int_{\mathbb{R}^3} b\partial_t\psi\,dx\,dt + \int_0^T\int_{\mathbb{R}^3} \partial_i b_j \partial_i \psi_j\,dx\,dt + \int_0^T\int_{\mathbb{R}^3} (b\cdot\nabla b + |b|^{\beta-1}b)\psi\,dx\,dt = \int_{\mathbb{R}^3} b_0\psi(x,0)\,dx;$$
- $\nabla \cdot u(x,t) = \nabla \cdot b(x,t) = 0$ for a.e. $(x,t) \in \mathbb{R}^3 \times (0,T)$.

As is well-known, it is usually difficult to obtain the uniqueness of weak solutions. A weak solution becomes a strong solution provided that
$$\begin{aligned}(u,b) \in L^\infty(0,T;H^1(\mathbb{R}^3)) &\cap L^2(0,T;H^2(\mathbb{R}^3))\\ &\times L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)),\\ u \in L^\infty(0,T;L^{\alpha+1}(\mathbb{R}^3)),\ & b \in L^\infty(0,T;L^{\beta+1}(\mathbb{R}^3)),\end{aligned}$$
for any $T > 0$.

For the clarity of presentation, we denote

$$
\begin{aligned}
&\mathfrak{X}_1 \triangleq \{(\alpha,\beta) \mid \alpha \geq 4,\ \beta \geq 4\},\\
&\mathfrak{X}_2 \triangleq \left\{(\alpha,\beta) \;\middle|\; 3 < \alpha \leq \frac{7}{2},\ \frac{7}{2\alpha-5} \leq \beta \leq \frac{3\alpha+5}{\alpha-1}\right\},\\
&\mathfrak{X}_3 \triangleq \left\{(\alpha,\beta) \;\middle|\; \frac{7}{2} < \alpha < 4,\ \frac{5\alpha+7}{2\alpha} \leq \beta \leq \frac{3\alpha+5}{\alpha-1}\right\},\\
&\mathfrak{X}_4 \triangleq \left\{(\alpha,\beta) \;\middle|\; 4 \leq \alpha \leq \frac{17}{3},\ \frac{5\alpha+7}{2\alpha} \leq \beta < 4\right\},\\
&\mathfrak{X}_5 \triangleq \left\{(\alpha,\beta) \;\middle|\; \frac{17}{3} < \alpha \leq 7,\ \frac{5\alpha+7}{2\alpha} \leq \beta \leq \frac{\alpha+5}{\alpha-3}\right\}.
\end{aligned}
\tag{2.1}
$$

REMARK 2.2. It is not difficult to check that all the sets $\mathfrak{X}_i$, $i = 1, 2, \ldots, 5$ are nonempty and satisfy $\mathfrak{X}_j \cap \mathfrak{X}_k = \emptyset$ for any $j \neq k$, $1 \leq j,\ k \leq 5$.

Without loss of generality, ν and η are normalized to 1 in the rest of the paper. Our main result on the global existence and uniqueness of strong solutions to the system (1.1) is as follows.

THEOREM 2.3. *Suppose that* $(\alpha,\beta) \in \bigcup_{i=1}^{5} \mathfrak{X}_i$ *defined by* (2.1) *and* $(u_0, b_0) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$. *Then there exists a unique global strong solution pair* $(u(x,t),\ b(x,t),\ \pi(x,t))$ *for the system* (1.1). *Moreover,*

$$
\begin{aligned}
&(u,b) \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3))\\
&\qquad \times L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)),\\
&u \in L^\infty(0,T;L^{\alpha+1}(\mathbb{R}^3)),\ b \in L^\infty(0,T;L^{\beta+1}(\mathbb{R}^3)),\\
&(\partial_t u,\ \partial_t b),\ (|u|^{(\alpha-1)/2}\nabla u,\ |b|^{(\beta-1)/2}\nabla b),\ (\nabla |u|^{(\alpha+1)/2},\ \nabla |b|^{(\beta+1)/2})\\
&\qquad \in L^2(0,T;L^2(\mathbb{R}^3)) \times L^2(0,T;L^2(\mathbb{R}^3)),
\end{aligned}
$$

for any $T > 0$.

The following theorems concern the preliminary L^2 decay for weak solutions.

THEOREM 2.4. *Assume that* $\alpha, \beta > \frac{7}{3}$. *Let* $(u_0, b_0) \in (L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))^2$ *with* $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then there exists a weak solution* $(u(x,t), b(x,t))$ *to the system* (1.1) *such that*

$$
\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \leq C(1+t)^{-\min\{\frac{1}{2},\, \frac{3\alpha-7}{2(\alpha-1)},\, \frac{3\beta-7}{2(\beta-1)}\}}.
$$

THEOREM 2.5. *Suppose that* $\alpha, \beta \geq 1$. *Let* $(u_0, b_0) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then there exists a weak solution* $(u(x,t), b(x,t))$ *for the system* (1.1) *which satisfies*

$$
\|\nabla u(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \leq C(1+t)^{-1/2}.
$$

THEOREM 2.6. *Assume that* $\alpha,\beta\ge 10/3$. *Let* $(u_0,b_0)\in L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ *with* $\nabla\cdot u_0=0$ *and* $\nabla\cdot b_0=0$. *Suppose that*

$$\|e^{t\Delta}u_0\|_{L^2}+\|e^{t\Delta}b_0\|_{L^2}\le C(1+t)^{-\mu}\quad \textit{for some } \mu>0.$$

Then there exists a weak solution $(u(x,t),\,b(x,t))$ *to the system* (1.1) *which satisfies*

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\le C(1+t)^{-\min\{\mu,\,3/4\}}.$$

3. The proof of Theorem 2.3. To prove the main result, we first introduce the following conventions and notation. Throughout, C stands for real positive constants which may be different at each occurrence. We shall sometimes write $A\lesssim B$ for $A\le CB$.

We first establish the global existence of strong solutions to the system (1.1). The existence of global weak solutions can be obtained as in [CJ] by utilizing Galerkin approximation.

Before embarking on the proof of Theorem 2.3, we can easily derive the following energy estimate: take the inner product of $(1.1)_1$ with u and the inner product of $(1.1)_2$ with b, add and integrate to obtain

LEMMA 3.1. *For any solution* (u,b) *of* (1.1), *there exists a constant* C *such that for any* $T>0$,

$$\|(u,b)(t)\|_{L^2}^2+\int_0^T\{\|(\nabla u,\,\nabla b)\|_{L^2}^2+\|u\|_{L^{\alpha+1}}^{\alpha+1}+\|b\|_{L^{\beta+1}}^{\beta+1}\}(t)\,dt\le C(u_0,b_0,T)$$

for any $t\in[0,T]$.

Proof of Theorem 2.3. The proof of the global existence of strong solutions to (1.1) is split into two parts.

PART 1: $(\alpha,\beta)\in\mathfrak{X}_1$. Multiplying $(1.1)_1$ and $(1.1)_2$ with Δu and Δb, respectively, noting the incompressibility condition, adding them up and integrating over the space variable, we obtain

$$\begin{aligned}(3.1)\quad &\frac12\frac{d}{dt}\|(\nabla u,\nabla b)(t)\|_{L^2}^2+\|(\Delta u,\Delta b)(t)\|_{L^2}^2\\ &\quad+\|(|u|^{(\alpha-1)/2}\nabla u,|b|^{(\beta-1)/2}\nabla b)(t)\|_{L^2}^2+\frac{4(\alpha-1)}{(\alpha+1)^2}\|\nabla|u|^{(\alpha+1)/2}(t)\|_{L^2}^2\\ &\quad+\frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|b|^{(\beta+1)/2}(t)\|_{L^2}^2\\ &=\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\Delta u\,dx+\int_{\mathbb{R}^3}(u\cdot\nabla b)\cdot\Delta b\,dx-\int_{\mathbb{R}^3}(b\cdot\nabla b)\cdot\Delta u\,dx\\ &\quad-\int_{\mathbb{R}^3}(b\cdot\nabla u)\cdot\Delta b\,dx\\ &\le\tfrac14\|(\Delta u,\,\Delta b)(t)\|_{L^2}^2+C(J_1+J_2+J_3+J_4),\end{aligned}$$

where

$$J_1 = \int_{\mathbb{R}^3} |u\cdot\nabla u|^2\,dx,\quad J_2 = \int_{\mathbb{R}^3} |b\cdot\nabla u|^2\,dx,\quad J_3 = \int_{\mathbb{R}^3} |b\cdot\nabla b|^2\,dx,$$
$$J_4 = \int_{\mathbb{R}^3} |u\cdot\nabla b|^2\,dx.$$

Taking the inner product of $(1.1)_1$ and $(1.1)_2$ with $\partial_t u$ and $\partial_t b$, respectively, making use of the fact that $\nabla\cdot u = \nabla\cdot b = 0$, after a suitable integration by parts, one gets

$$\begin{aligned}(3.2)\qquad &\frac{1}{2}\frac{d}{dt}\|(\nabla u,\nabla b)(t)\|_{L^2}^2 + \|(\partial_t u,\partial_t b)(t)\|_{L^2}^2\\ &\quad + \frac{d}{dt}\left\{\frac{1}{\alpha+1}\|u\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1}\|b\|_{L^{\beta+1}}^{\beta+1}\right\}\\ &= \int_{\mathbb{R}^3} (b\cdot\nabla b)\cdot\partial_t u\,dx + \int_{\mathbb{R}^3} (b\cdot\nabla u)\cdot\partial_t b\,dx\\ &\quad - \int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot\partial_t u\,dx - \int_{\mathbb{R}^3} (u\cdot\nabla b)\cdot\Delta\partial_t b\,dx\\ &\le \tfrac{1}{4}\|(\partial_t u,\partial_t b)(t)\|_{L^2}^2 + C(J_1+J_2+J_3+J_4).\end{aligned}$$

In what follows, we will deal with the terms J_1, J_2, J_3, J_4 separately. First of all,

$$\begin{aligned}(3.3)\qquad J_1 &\le \int_{\mathbb{R}^3} |u|^2|\nabla u|^2\,dx \le \|u\|_{L^{\alpha+1}}^2\|\nabla u\|_{L^{\frac{2(\alpha+1)}{\alpha-1}}}^2\\ &\lesssim \|u\|_{L^{\alpha+1}}^2\|\nabla u\|_{L^2}^{\frac{2(\alpha-2)}{\alpha+1}}\|\Delta u\|_{L^2}^{\frac{6}{\alpha+1}}\\ &\le \frac{1}{8C}\|\Delta u\|_{L^2}^2 + C\|u\|_{L^{\alpha+1}}^{\frac{2(\alpha+1)}{\alpha-2}}\|\nabla u\|_{L^2}^2,\end{aligned}$$

where the Young inequality and Gagliardo–Nirenberg inequality are used.

Using the same method we obtain

$$\begin{aligned}(3.4)\qquad J_3 &\le \int_{\mathbb{R}^3} |b|^2|\nabla b|^2\,dx \le \|b\|_{L^{\beta+1}}^2\|\nabla b\|_{L^{\frac{2(\beta+1)}{\beta-1}}}^2\\ &\lesssim \|b\|_{L^{\beta+1}}^2\|\nabla b\|_{L^2}^{\frac{2(\beta-2)}{\beta+1}}\|\Delta b\|_{L^2}^{\frac{6}{\beta+1}}\\ &\le \frac{1}{8C}\|\Delta b\|_{L^2}^2 + C\|b\|_{L^{\beta+1}}^{\frac{2(\beta+1)}{\beta-2}}\|\nabla b\|_{L^2}^2.\end{aligned}$$

To bound J_2, again by the Young inequality and Gagliardo–Nirenberg inequality, we have

$$J_2 \le \int_{\mathbb{R}^3} |b|^2|\nabla u|^2\,dx \le \|b\|_{L^{\beta+1}}^2 \|\nabla u\|^2_{L^{\frac{2(\beta+1)}{\beta-1}}} \tag{3.5}$$
$$\lesssim \|b\|_{L^{\beta+1}}^2 \|\nabla u\|_{L^2}^{\frac{2(\beta-2)}{\beta+1}} \|\Delta u\|_{L^2}^{\frac{6}{\beta+1}}$$
$$\le \frac{1}{8C}\|\Delta u\|_{L^2}^2 + C\|b\|_{L^{\beta+1}}^{\frac{2(\beta+1)}{\beta-2}} \|\nabla u\|_{L^2}^2.$$

In the same manner as the bound for J_2, we obtain

$$J_4 \le \frac{1}{8C}\|\Delta b\|_{L^2}^2 + C\|u\|_{L^{\alpha+1}}^{\frac{2(\alpha+1)}{\alpha-2}} \|\nabla b\|_{L^2}^2. \tag{3.6}$$

Without loss of generality, we assume that $\|u\|_{L^{\alpha+1}}, \|b\|_{L^{\beta+1}} \ge 1$. As $\alpha, \beta \ge 4$, it is easy to show that $\frac{2(\alpha+1)}{\alpha-2} \le \alpha+1$ and $\frac{2(\beta+1)}{\beta-2} \le \beta+1$. Therefore,

$$\|u\|_{L^{\alpha+1}}^{\frac{2(\alpha+1)}{\alpha-2}} \lesssim \|u\|_{L^{\alpha+1}}^{\alpha+1}, \qquad \|b\|_{L^{\beta+1}}^{\frac{2(\beta+1)}{\beta-2}} \lesssim \|b\|_{L^{\beta+1}}^{\beta+1}.$$

Adding up (3.1) and (3.2), combining the estimates for J_1, J_2, J_3, J_4 and absorbing the dissipative term, we finally obtain

$$\frac{d}{dt}\|(\nabla u, \nabla b)(t)\|_{L^2}^2 + \frac{d}{dt}\left\{\frac{1}{\alpha+1}\|u\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1}\|b\|_{L^{\beta+1}}^{\beta+1}\right\} \tag{3.7}$$
$$+ \|(\partial_t u, \partial_t b)(t)\|_{L^2}^2 + \|(\Delta u, \Delta b)(t)\|_{L^2}^2$$
$$+ \|(|u|^{(\alpha-1)/2}\nabla u, |b|^{(\beta-1)/2}\nabla b)(t)\|_{L^2}^2$$
$$+ \frac{4(\alpha-1)}{(\alpha+1)^2}\|\nabla |u|^{(\alpha+1)/2}(t)\|_{L^2}^2$$
$$+ \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla |b|^{(\beta+1)/2}(t)\|_{L^2}^2$$
$$\le C\big(\|u\|_{L^{\alpha+1}}^{\frac{2(\alpha+1)}{\alpha-2}} + \|b\|_{L^{\beta+1}}^{\frac{2(\beta+1)}{\beta-2}}\big)\|(\nabla u,\, \nabla b)(t)\|_{L^2}^2$$
$$\le C\big(\|u\|_{L^{\alpha+1}}^{\alpha+1} + \|b\|_{L^{\beta+1}}^{\beta+1}\big)\|(\nabla u,\, \nabla b)(t)\|_{L^2}^2.$$

Applying the Gronwall-type inequality to (3.7) and using Lemma 3.1, we can immediately show that the system (1.1) admits global strong solutions. Thus, this completes the proof for this case.

PART 2: $(\alpha, \beta) \in \mathfrak{X}_2 \cup \mathfrak{X}_3 \cup \mathfrak{X}_4 \cup \mathfrak{X}_5$. We have to estimate each term J_1–J_4 on the right-hand side of (3.1) and (3.2). By taking advantage of the Young inequality and Gagliardo–Nirenberg inequality, it follows that

$$
\begin{aligned}
(3.8)\quad J_1 &\le \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2\,dx \\
&= \int_{\mathbb{R}^3} (|u|^{(\alpha-1)/2}|\nabla u|)^{\frac{2(5-\alpha)}{\alpha-1}} (|u|^{\alpha-3}|\nabla u|^{\frac{4(\alpha-3)}{\alpha-1}})\,dx \quad (3<\alpha<5) \\
&\le \frac{1}{8C}\big\||u|^{(\alpha-1)/2}\nabla u\big\|_{L^2}^2 + C\int_{\mathbb{R}^3} |u|^{(\alpha-1)/2}|\nabla u|^2\,dx \\
&\le \frac{1}{8C}\big\||u|^{(\alpha-1)/2}\nabla u\big\|_{L^2}^2 + C\|\nabla u\|^2_{L^{\frac{4(\alpha+1)}{\alpha+3}}}\|u\|^{(\alpha-1)/2}_{L^{\alpha+1}} \\
&\le \frac{1}{8C}\big\||u|^{(\alpha-1)/2}\nabla u\big\|_{L^2}^2 + C\|u\|_{L^{\alpha+1}}\|\Delta u\|_{L^2}\|u\|^{(\alpha-1)/2}_{L^{\alpha+1}} \\
&\le \frac{1}{8C}\big\||u|^{(\alpha-1)/2}\nabla u\big\|_{L^2}^2 + \frac{1}{8C}\|\Delta u\|^2_{L^2} + C\|u\|^{\alpha+1}_{L^{\alpha+1}}.
\end{aligned}
$$

Note that when $\alpha \ge 5$, the term J_1 can be bounded as J_2 in (3.5) above. Thus, repeating the arguments in (3.8), we have

$$
\begin{aligned}
(3.9)\quad J_3 &\le \int_{\mathbb{R}^3} |b|^2 |\nabla b|^2\,dx \\
&= \int_{\mathbb{R}^3} (|b|^{(\beta-1)/2}|\nabla b|)^{\frac{2(5-\beta)}{\beta-1}} (|b|^{\beta-3}|\nabla b|^{\frac{4(\beta-3)}{\beta-1}})\,dx \quad (3<\beta<5) \\
&\le \frac{1}{8C}\big\||b|^{(\beta-1)/2}\nabla b\big\|_{L^2}^2 + \frac{1}{8C}\|\Delta b\|^2_{L^2} + C\|b\|^{\beta+1}_{L^{\beta+1}}.
\end{aligned}
$$

Now, let us bound J_2 and J_4 which should be treated differently. For J_2, the Young inequality yields

$$
\begin{aligned}
(3.10)\quad J_2 &\le \int_{\mathbb{R}^3} |b|^2|\nabla u|^2\,dx \le \|b\|^2_{L^{\beta+1}}\|\nabla u\|^2_{L^{\frac{2(\beta+1)}{\beta-1}}} \\
&\lesssim \|b\|^2_{L^{\beta+1}}\|u\|^{2(1-\theta)}_{L^{\alpha+1}}\|\Delta u\|^{2\theta}_{L^2} \quad (\theta<1) \\
&\le \frac{1}{8C}\|\Delta u\|^2_{L^2} + C\|u\|^2_{L^{\alpha+1}}\|b\|^{\frac{2}{1-\theta}}_{L^{\beta+1}} \\
&\le \frac{1}{8C}\|\Delta u\|^2_{L^2} + C\|u\|^{2(\alpha+1)}_{L^{\alpha+1}} + C\|b\|^{\frac{2(\alpha+1)}{(1-\theta)\alpha}}_{L^{\beta+1}} \quad \left(\frac{2(\alpha+1)}{(1-\theta)\alpha} \le 2(\beta+1)\right) \\
&\le \frac{1}{8C}\|\Delta u\|^2_{L^2} + C\|u\|^{2(\alpha+1)}_{L^{\alpha+1}} + C\|b\|^{2(\beta+1)}_{L^{\beta+1}}.
\end{aligned}
$$

Note that we have used the following Gagliardo–Nirenberg inequality:

$$
\|\nabla u\|_{L^{\frac{2(\beta+1)}{\beta-1}}} \le C\|u\|^{1-\theta}_{L^{\alpha+1}}\|\Delta u\|^{\theta}_{L^2}, \quad \theta = \frac{2(\alpha+4)(\beta+1)-3(\alpha+1)(\beta-1)}{(\alpha+7)(\beta+1)},
$$

which holds true provided that $\theta \ge 1/2$. Collecting all the restrictions on θ, we can deduce the following inequalities:

$$\frac{1}{2} \leq \frac{2(\alpha+4)(\beta+1)-3(\alpha+1)(\beta-1)}{(\alpha+7)(\beta+1)} \leq \frac{\alpha\beta-1}{\alpha(\beta+1)}.$$

Direct calculations show that

$$\frac{5\alpha+7}{2\alpha} \leq \beta \leq \frac{3\alpha+5}{\alpha-1}. \tag{3.11}$$

In the same way, we estimate

$$\begin{aligned} J_4 &\leq \int_{\mathbb{R}^3} |u|^2|\nabla b|^2\,dx \leq \|u\|_{L^{\alpha+1}}^2 \|\nabla b\|_{L^{\frac{2(\alpha+1)}{\alpha-1}}}^2 \\ &\lesssim \|u\|_{L^{\alpha+1}}^2 \|b\|_{L^{\beta+1}}^{2(1-\lambda)} \|\Delta b\|_{L^2}^{2\lambda} \quad (\lambda<1) \\ &\leq \frac{1}{8C}\|\Delta b\|_{L^2}^2 + C\|b\|_{L^{\beta+1}}^2 \|u\|_{L^{\alpha+1}}^{\frac{2}{1-\lambda}} \\ &\leq \frac{1}{8C}\|\Delta b\|_{L^2}^2 + C\|b\|_{L^{\beta+1}}^{2(\beta+1)} + C\|u\|_{L^{\alpha+1}}^{\frac{2(\beta+1)}{(1-\lambda)\beta}} \quad \left(\frac{2(\beta+1)}{(1-\lambda)\beta} \leq 2(\alpha+1)\right) \\ &\leq \frac{1}{8C}\|\Delta b\|_{L^2}^2 + C\|b\|_{L^{\beta+1}}^{2(\beta+1)} + C\|u\|_{L^{\alpha+1}}^{2(\alpha+1)}. \end{aligned} \tag{3.12}$$

Similarly, we have used the following Gagliardo–Nirenberg inequality:

$$\|\nabla b\|_{L^{\frac{2(\alpha+1)}{\alpha-1}}} \leq C\|b\|_{L^{\beta+1}}^{1-\lambda} \|\Delta b\|_{L^2}^{\lambda},$$

$$\lambda = \frac{2(\alpha+1)(\beta+4)-3(\alpha-1)(\beta+1)}{(\alpha+1)(\beta+7)} \geq \frac{1}{2}.$$

Thus, we have

$$\frac{1}{2} \leq \frac{2(\alpha+1)(\beta+4)-3(\alpha-1)(\beta+1)}{(\alpha+1)(\beta+7)} \leq \frac{\alpha\beta-1}{(\alpha+1)\beta}.$$

Simple computations give rise to

$$\frac{7}{2\alpha-5} \leq \beta \leq \frac{\alpha+5}{\alpha-3}. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\max\left\{\frac{5\alpha+7}{2\alpha}, \frac{7}{2\alpha-5}\right\} \leq \beta \leq \min\left\{\frac{\alpha+5}{\alpha-3}, \frac{3\alpha+5}{\alpha-1}\right\}.$$

Observing the simple facts

$$\begin{aligned} 3<\alpha\leq \frac{7}{2} &\Rightarrow \frac{7}{2\alpha-5} \geq \frac{5\alpha+7}{2\alpha}, \quad \frac{7}{2\alpha-5} > 3, \\ 3<\alpha\leq 5 &\Rightarrow \frac{\alpha+5}{\alpha-3} \geq \frac{3\alpha+5}{\alpha-1}, \\ 3<\alpha\leq 7 &\Rightarrow \frac{5\alpha+7}{2\alpha} \geq 3, \quad \frac{5\alpha+7}{2\alpha} \leq \frac{\alpha+5}{\alpha-3}, \end{aligned}$$

we can divide the range of pairs (α,β) into $\mathfrak{X}_2$, $\mathfrak{X}_3$, $\mathfrak{X}_4$ and $\mathfrak{X}_5$ as defined in (2.1).

Substituting all the above estimates into (3.1) and (3.2), and adding them up, we conclude that

$$\begin{aligned}(3.14)\quad & \frac{d}{dt}\|(\nabla u,\nabla b)(t)\|_{L^2}^2+\frac{d}{dt}\left\{\frac{1}{\alpha+1}\|u\|_{L^{\alpha+1}}^{\alpha+1}+\frac{1}{\beta+1}\|b\|_{L^{\beta+1}}^{\beta+1}\right\}\\ &+\|(\partial_t u,\partial_t b)(t)\|_{L^2}^2+\|(\Delta u,\Delta b)(t)\|_{L^2}^2+\|(|u|^{(\alpha-1)/2}\nabla u,|b|^{(\beta-1)/2}\nabla b)(t)\|_{L^2}^2\\ &+\frac{4(\alpha-1)}{(\alpha+1)^2}\|\nabla|u|^{(\alpha+1)/2}(t)\|_{L^2}^2+\frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|b|^{(\beta+1)/2}(t)\|_{L^2}^2\\ &\le C(\|u\|_{L^{\alpha+1}}^{\alpha+1}+\|b\|_{L^{\beta+1}}^{\beta+1}+1)(\|u\|_{L^{\alpha+1}}^{\alpha+1}+\|b\|_{L^{\beta+1}}^{\beta+1}).\end{aligned}$$

A Gronwall argument and using Lemma 3.1 again show the desired conclusion. Thus, we have completed the proof of Part 2.

The next goal in this part is to prove the uniqueness of the strong solution. Let (u,b,π) and $(\overline{u},\overline{b},\overline{\pi})$ be two solutions to (1.1) with the initial datum $u(x,0)=\overline{u}(x,0)$, $b(x,0)=\overline{b}(x,0)$. Taking the difference, and then taking the inner product, we can easily obtain

$$\begin{aligned}(3.15)\quad & \frac{1}{2}\frac{d}{dt}(\|u-\overline{u}\|_{L^2}^2+\|b-\overline{b}\|_{L^2}^2)+\|\nabla(u-\overline{u})\|_{L^2}^2+\|\nabla(b-\overline{b})\|_{L^2}^2\\ &+\int_{\mathbb{R}^3}(|u|^{\alpha-1}u-|\overline{u}|^{\alpha-1}\overline{u})(u-\overline{u})\,dx+\int_{\mathbb{R}^3}(|b|^{\beta-1}b-|\overline{b}|^{\beta-1}\overline{b})(b-\overline{b})\,dx\\ &\le\int_{\mathbb{R}^3}|b-\overline{b}|\,|u-\overline{u}|\,|\nabla\overline{b}|\,dx+\int_{\mathbb{R}^3}|b-\overline{b}|^2\,|\nabla\overline{b}|\,dx+\int_{\mathbb{R}^3}|b-\overline{b}|^2\,|\nabla\overline{u}|\,dx\\ &+\int_{\mathbb{R}^3}|u-\overline{u}|^2\,|\nabla\overline{u}|\,dx\triangleq K_1+K_2+K_3+K_4.\end{aligned}$$

Then we estimate each K_i, $i=1,2,3,4$. For K_1, from the Young inequality and the Gagliardo–Nirenberg inequality it follows that

$$\begin{aligned}(3.16)\quad K_1&\le\|u-\overline{u}\|_{L^4}\|b-\overline{b}\|_{L^4}\|\nabla\overline{b}\|_{L^2}\\ &\lesssim\|u-\overline{u}\|_{L^2}^{1/4}\|\nabla(u-\overline{u})\|_{L^2}^{3/4}\|b-\overline{b}\|_{L^2}^{1/4}\|\nabla(b-\overline{b})\|_{L^2}^{\frac{3}{4}}\|\nabla\overline{b}\|_{L^2}\\ &\le\tfrac{1}{16}(\|\nabla(u-\overline{u})\|_{L^2}^2+\|\nabla(b-\overline{b})\|_{L^2}^2)+C\|u-\overline{u}\|_{L^2}\|b-\overline{b}\|_{L^2}\|\nabla\overline{b}\|_{L^2}^4.\end{aligned}$$

In the same way, we get

$$\begin{aligned}(3.17)\quad K_2&\le\|b-\overline{b}\|_{L^4}^2\|\nabla\overline{b}\|_{L^2}\lesssim\|b-\overline{b}\|_{L^2}^{1/2}\|\nabla(b-\overline{b})\|_{L^2}^{3/2}\|\nabla\overline{b}\|_{L^2}\\ &\le\tfrac{1}{16}\|\nabla(b-\overline{b})\|_{L^2}^2+C\|\nabla\overline{b}\|_{L^2}^4\|b-\overline{b}\|_{L^2}^2;\end{aligned}$$

$$\begin{aligned}(3.18)\quad K_3&\le\|b-\overline{b}\|_{L^4}^2\|\nabla\overline{u}\|_{L^2}\lesssim\|b-\overline{b}\|_{L^2}^{1/2}\|\nabla(b-\overline{b})\|_{L^2}^{3/2}\|\nabla\overline{u}\|_{L^2}\\ &\le\tfrac{1}{16}\|\nabla(b-\overline{b})\|_{L^2}^2+C\|\nabla\overline{u}\|_{L^2}^4\|b-\overline{b}\|_{L^2}^2;\end{aligned}$$

$$K_4 \le \|u-\overline{u}\|_{L^4}^2\|\nabla\overline{u}\|_{L^2} \lesssim \|u-\overline{u}\|_{L^2}^{1/2}\|\nabla(u-\overline{u})\|_{L^2}^{3/2}\|\nabla\overline{u}\|_{L^2} \le \tfrac{1}{16}\|\nabla(u-\overline{u})\|_{L^2}^2 + C\|\nabla\overline{u}\|_{L^2}^4\|u-\overline{u}\|_{L^2}^2. \tag{3.19}$$

Finally, we will show that the integrals $\int_{\mathbb{R}^3}(|u|^{\alpha-1}u-|\overline{u}|^{\alpha-1}\overline{u})(u-\overline{u})\,dx$ and $\int_{\mathbb{R}^3}(|b|^{\beta-1}b-|\overline{b}|^{\beta-1}\overline{b})(b-\overline{b})\,dx$ on the left hand side of (3.15) are both nonnegative.

Indeed, several applications of the Hölder inequality imply

$$\begin{aligned}\int_{\mathbb{R}^3}&(|u|^{\alpha-1}u-|\overline{u}|^{\alpha-1}\overline{u})(u-\overline{u})\,dx \\ &= \int_{\mathbb{R}^3}|u|^{\alpha+1}\,dx - \int_{\mathbb{R}^3}|\overline{u}|^{\alpha-1}\overline{u}u\,dx - \int_{\mathbb{R}^3}|u|^{\alpha-1}u\overline{u}\,dx + \int_{\mathbb{R}^3}|\overline{u}|^{\alpha+1}\,dx \\ &\ge \|u\|_{L^{\alpha+1}}^{\alpha+1} - \|\overline{u}\|_{L^{\alpha+1}}^{\alpha}\|u\|_{L^{\alpha+1}} - \|u\|_{L^{\alpha+1}}^{\alpha}\|\overline{u}\|_{L^{\alpha+1}} + \|\overline{u}\|_{L^{\alpha+1}}^{\alpha+1} \\ &= (\|u\|_{L^{\alpha+1}}^{\alpha} - \|\overline{u}\|_{L^{\alpha+1}}^{\alpha})(\|u\|_{L^{\alpha+1}} - \|\overline{u}\|_{L^{\alpha+1}}) \ge 0.\end{aligned} \tag{3.20}$$

By the same argument, we conclude that

$$\int_{\mathbb{R}^3}(|b|^{\beta-1}b-|\overline{b}|^{\beta-1}\overline{b})(b-\overline{b})\,dx \ge 0. \tag{3.21}$$

Inserting (3.16)–(3.21) into (3.15), neglecting the two nonnegative terms and absorbing the dissipative terms, we have

$$\frac{1}{2}\frac{d}{dt}(\|u-\overline{u}\|_{L^2}^2+\|b-\overline{b}\|_{L^2}^2) \le C(\|\nabla\overline{u}\|_{L^2}^4+\|\nabla\overline{b}\|_{L^2}^4)(\|u-\overline{u}\|_{L^2}^2+\|b-\overline{b}\|_{L^2}^2).$$

A standard Gronwall-type argument shows that

$$\|u-\overline{u}\|_{L^2}^2+\|b-\overline{b}\|_{L^2}^2 \le \big(\|u(x,0)-\overline{u}(x,0)\|_{L^2}^2+\|b(x,0)-\overline{b}(x,0)\|_{L^2}^2\big) \times \exp\Big\{C\int_0^T H(t)\,dt\Big\},$$

where $H(t)=\|\nabla\overline{u}(\cdot,t)\|_{L^2}^4+\|\nabla\overline{b}(\cdot,t)\|_{L^2}^4$. Thus, we get $u=\overline{u}$, $b=\overline{b}$ in the L^2 sense due to $u(x,0)=\overline{u}(x,0)$, $b(x,0)=\overline{b}(x,0)$. This completes the proof of the uniqueness part, and so the proof of the whole Theorem 2.3. ∎

4. The proof of Theorems 2.4–2.6.

4. The proof of Theorems 2.4–2.6. Here we mainly take advantage of the Fourier-splitting method which was introduced to study the decay of solutions to parabolic conservation laws in [S1], and refined in [S2, SW].

Proof of Theorem 2.4. First, from Lemma 3.1 we get the global bound

$$\|(u,b)(t)\|_{L^2}^2 + \int_0^T \{\|(\nabla u,\nabla b)(s)\|_{L^2}^2+\|u(s)\|_{L^{\alpha+1}}^{\alpha+1}+\|b(s)\|_{L^{\beta+1}}^{\beta+1}\}\,ds \le C. \tag{4.1}$$

It is easy to check that one can rewrite $(1.1)_1$ and $(1.1)_2$ as

$$\partial_t u - \Delta u = -\mathbb{P}(|u|^{\alpha-1}u) + \mathbb{P}\nabla\cdot\{(b\otimes b)-(u\otimes u)\}, \tag{4.2}$$

$$\partial_t b - \Delta b = -|b|^{\beta-1}b + \nabla\cdot\{(b\otimes u)-(u\otimes b)\}, \tag{4.3}$$

where $\mathbb{P}$ is the Leray projection operator defined by $\mathbb{P}f = f+\nabla(-\Delta)^{-1}\nabla\cdot f$. By taking the Fourier transform, we can easily deduce that

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-|\xi|^2(t-s)}G(\xi,s)\,ds, \tag{4.4}$$

$$\hat{b}(\xi,t) = e^{-|\xi|^2 t}\hat{b}_0(\xi) + \int_0^t e^{-|\xi|^2(t-s)}H(\xi,s)\,ds, \tag{4.5}$$

where

$$G(\xi,s) = \left(1-\frac{\xi\otimes\xi}{|\xi|^2}\right)\left\{\xi\cdot(\widehat{b\otimes b})(\xi,s) - \xi\cdot(\widehat{u\otimes u})(\xi,s) - (\widehat{|u|^{\alpha-1}u})(\xi,s)\right\},$$

$$H(\xi,s) = \xi\cdot(\widehat{b\otimes u})(\xi,s) - \xi\cdot(\widehat{u\otimes b})(\xi,s) - (\widehat{|b|^{\beta-1}b})(\xi,s).$$

It follows from the Hölder inequality and the definition of the Fourier transform that

$$\begin{aligned} |G(\xi,s)| &\le |\xi|\,|(\widehat{b\otimes b})(\xi,s)| + |\xi|\,|(\widehat{u\otimes u})(\xi,s)| + |(\widehat{|u|^{\alpha-1}u})(\xi,s)| \\ &\lesssim |\xi|\,\|b\otimes b\|_{L^1} + |\xi|\,\|u\otimes u\|_{L^1} + \big\||u|^{\alpha-1}u\big\|_{L^1} \\ &\lesssim |\xi|\,\|b\|_{L^2}^2 + |\xi|\,\|u\|_{L^2}^2 + \|u\|_{L^\alpha}^\alpha \\ &\lesssim |\xi|\,\|b_0\|_{L^2}^2 + |\xi|\,\|u_0\|_{L^2}^2 + \|u\|_{L^2}^{\frac{2}{\alpha-1}}\|u\|_{L^{\alpha+1}}^{\frac{(\alpha-2)(\alpha+1)}{\alpha-1}} \\ &\le C|\xi| + C\|u\|_{L^{\alpha+1}}^{\frac{(\alpha-2)(\alpha+1)}{\alpha-1}}, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} |H(\xi,s)| &\le |\xi|\,|(\widehat{b\otimes u})(\xi,s)| + |\xi|\,|(\widehat{u\otimes b})(\xi,s)| + |(\widehat{|b|^{\beta-1}b})(\xi,s)| \\ &\lesssim |\xi|\,\|b\otimes u\|_{L^1} + \big\||b|^{\beta-1}b\big\|_{L^1} \\ &\lesssim |\xi|\,\|b\|_{L^2}\|u\|_{L^2} + \|b\|_{L^\beta}^\beta \\ &\lesssim |\xi|\,\|b_0\|_{L^2}\|u_0\|_{L^2} + \|b\|_{L^2}^{\frac{2}{\beta-1}}\|b\|_{L^{\beta+1}}^{\frac{(\beta-2)(\beta+1)}{\beta-1}} \\ &\le C|\xi| + C\|b\|_{L^{\beta+1}}^{\frac{(\beta-2)(\beta+1)}{\beta-1}}. \end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.4) and (4.5), respectively, and applying the Hölder inequality we obtain

$$
\begin{aligned}
(4.8)\qquad |\hat{u}(\xi,t)| &\lesssim e^{-|\xi|^2 t}|\hat{u}_0(\xi)| + \int_0^t e^{-|\xi|^2(t-s)}\big(|\xi| + \|u(s)\|_{L^{\alpha+1}}^{\frac{(\alpha-2)(\alpha+1)}{\alpha-1}}\big)\,ds \\
&\lesssim e^{-|\xi|^2 t}\|u_0\|_{L^1} + |\xi|^{-1} + \Big(\int_0^t e^{-(\alpha-1)|\xi|^2(t-s)}\,ds\Big)^{\frac{1}{\alpha-1}} \\
&\quad\times\Big(\int_0^t \|u(s)\|_{L^{\alpha+1}}^{\alpha+1}\,ds\Big)^{\frac{\alpha-2}{\alpha-1}} \\
&\le C + C|\xi|^{-1} + C|\xi|^{-\frac{2}{\alpha-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.9)\qquad |\hat{b}(\xi,t)| &\lesssim e^{-|\xi|^2 t}|\hat{b}_0(\xi)| + \int_0^t e^{-|\xi|^2(t-s)}\big(|\xi| + \|b(s)\|_{L^{\beta+1}}^{\frac{(\beta-2)(\beta+1)}{\beta-1}}\big)\,ds \\
&\lesssim e^{-|\xi|^2 t}\|b_0\|_{L^1} + |\xi|^{-1} + \Big(\int_0^t e^{-(\beta-1)|\xi|^2(t-s)}\,ds\Big)^{\frac{1}{\beta-1}} \\
&\quad\times\Big(\int_0^t \|b(s)\|_{L^{\beta+1}}^{\beta+1}\,ds\Big)^{\frac{\beta-2}{\beta-1}} \\
&\le C + C|\xi|^{-1} + C|\xi|^{-\frac{2}{\beta-1}}.
\end{aligned}
$$

Next with the estimates (4.8) and (4.9) at our disposal, we use the well-known Fourier-splitting argument to obtain the desired result. Let

$$
S(t) \triangleq \Big\{\xi\in\mathbb{R}^3 : |\xi| \le \sqrt{\frac{3}{2(1+t)}}\Big\}.
$$

Taking the inner product of $(1.1)_1$ with u and the inner product of $(1.1)_2$ with b, adding them up and ignoring the positive terms on the left-hand side, we get

$$
\begin{aligned}
(4.10)\qquad \frac{d}{dt}\int_{\mathbb{R}^3}(|u|^2+|b|^2)\,dx &\le -2\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla b|^2)\,dx \\
&= -2\int_{\mathbb{R}^3}(|\xi|^2|\hat{u}|^2+|\xi|^2|\hat{b}|^2)\,d\xi \\
&= -2\int_{S(t)}(|\xi|^2|\hat{u}|^2+|\xi|^2|\hat{b}|^2)\,d\xi - 2\int_{S^c(t)}(|\xi|^2|\hat{u}|^2+|\xi|^2|\hat{b}|^2)\,d\xi
\end{aligned}
$$

$$= -2 \int\limits_{S(t)} (|\xi|^2|\hat{u}|^2 + |\xi|^2|\hat{b}|^2)\, d\xi - \frac{3}{t+1} \int\limits_{S^c(t)} (|\hat{u}|^2 + |\hat{b}|^2)\, d\xi$$
$$\le -\frac{3}{t+1} \int\limits_{\mathbb{R}^3} (|\hat{u}|^2 + |\hat{b}|^2)\, d\xi + \frac{3}{t+1} \int\limits_{S(t)} (|\hat{u}|^2 + |\hat{b}|^2)\, d\xi.$$

Applying the Plancherel identity and multiplying by $(t+1)^3$, we obtain

$$\frac{d}{dt}\Big[(t+1)^3 \int\limits_{\mathbb{R}^3} (|\hat{u}|^2 + |\hat{b}|^2)\, d\xi\Big] \le 3(t+1)^2 \int\limits_{S(t)} (|\hat{u}|^2 + |\hat{b}|^2)\, d\xi. \tag{4.11}$$

Making use of (4.8) and (4.9), one has

$$\int\limits_{S(t)} (|\hat{u}|^2 + |\hat{b}|^2)\, d\xi$$
$$\le C \int\limits_{S(t)} \big(1 + |\xi|^{-2} + |\xi|^{-\frac{4}{\alpha-1}} + |\xi|^{-\frac{4}{\beta-1}}\big)\, d\xi$$
$$\le C\big((t+1)^{-3/2} + (t+1)^{-1/2} + (t+1)^{-\frac{3\alpha-7}{2(\alpha-1)}} + (t+1)^{-\frac{3\beta-7}{2(\beta-1)}}\big),$$

where we have used the fact that $\alpha, \beta > 7/3$. Thus, we get

$$\frac{d}{dt}\Big[(t+1)^3 \int\limits_{\mathbb{R}^3} (|\hat{u}|^2 + |\hat{b}|^2)\, d\xi\Big]$$
$$\le C\big((t+1)^{3/2} + (t+1)^{\alpha+3/2(\alpha-1)} + (t+1)^{\frac{\beta+3}{2(\beta-1)}}\big).$$

Integrating the above differential type inequality in time yields

$$\int\limits_{\mathbb{R}^3} (|\hat{u}|^2 + |\hat{b}|^2)\, d\xi \tag{4.12}$$
$$\le C(t+1)^{-3} \int\limits_{\mathbb{R}^3} (|\hat{u}_0|^2 + |\hat{b}_0|^2)\, d\xi$$
$$+ \big((t+1)^{-1/2} + (t+1)^{-\frac{3\alpha-7}{2(\alpha-1)}} + (t+1)^{\frac{3\beta-7}{2(\beta-1)}}\big)$$
$$\le C(1+t)^{-\min\{\frac{1}{2}, \frac{3\alpha-7}{2(\alpha-1)}, \frac{3\beta-7}{2(\beta-1)}\}},$$

which together with the Plancherel identity yields the desired result. ∎

Proof of Theorem 2.5. From (3.1), by using the Gagliardo–Nirenberg inequality and the Hölder inequality we get

$$
\begin{aligned}
(4.13)\quad & \frac{1}{2}\frac{d}{dt}\|(\nabla u,\nabla b)(t)\|_{L^2}^2+\|(\Delta u,\Delta b)(t)\|_{L^2}^2 \\
&+\|(|u|^{(\alpha-1)/2}\nabla u,|b|^{(\beta-1)/2}\nabla b)(t)\|_{L^2}^2+\frac{4(\alpha-1)}{(\alpha+1)^2}\|\nabla|u|^{(\alpha+1)/2}(t)\|_{L^2}^2 \\
&+\frac{4(\beta-1)}{(\beta+1)^2}\|\nabla|b|^{(\beta+1)/2}(t)\|_{L^2}^2 \\
&=\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\Delta u\,dx+\int_{\mathbb{R}^3}(u\cdot\nabla b)\cdot\Delta b\,dx-\int_{\mathbb{R}^3}(b\cdot\nabla b)\cdot\Delta u\,dx \\
&\quad-\int_{\mathbb{R}^3}(b\cdot\nabla u)\cdot\Delta b\,dx \\
&\le\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}+\|u\|_{L^\infty}\|\nabla b\|_{L^2}\|\Delta b\|_{L^2}+\|b\|_{L^\infty}\|\nabla b\|_{L^2}\|\Delta u\|_{L^2} \\
&\quad+\|b\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta b\|_{L^2} \\
&\lesssim\|\Delta u\|_{L^2}^2\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}+\|\Delta u\|_{L^2}^{3/4}\|\Delta b\|_{L^2}^{5/4}\|u\|_{L^2}^{1/4}\|b\|_{L^2}^{1/4}\|\nabla b\|_{L^2}^{1/2} \\
&\quad+\|\Delta u\|_{L^2}\|\Delta b\|_{L^2}\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}+\|\Delta u\|_{L^2}^{1/4}\|\Delta b\|_{L^2}^{7/4}\|u\|_{L^2}^{1/4}\|b\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{1/2} \\
&\le C(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)(\|u_0\|_{L^2}^{1/2}+\|b_0\|_{L^2}^{1/2})(\|\nabla u\|_{L^2}^{1/2}+\|\nabla b\|_{L^2}^{1/2}).
\end{aligned}
$$

Above, the following fact has been used several times (see. e.g. [Z1]):

$$
\begin{aligned}
\int_{\mathbb{R}^3}(f\cdot\nabla g)\cdot\Delta h\,dx&\le\|f\|_{L^\infty}\|\nabla g\|_{L^2}\|\Delta h\|_{L^2} \\
&\le C\|f\|_{L^2}^{1/4}\|\Delta f\|_{L^2}^{3/4}\|\nabla g\|_{L^2}^{1/2}\|g\|_{L^2}^{1/4}\|\Delta g\|_{L^2}^{1/4}\|\Delta h\|_{L^2}.
\end{aligned}
$$

Thus, it is easy to deduce from (4.13) that

$$
\begin{aligned}
(4.14)\quad & \frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2) \\
&\le(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)\{C(\|u_0\|_{L^2}^{1/2}+\|b_0\|_{L^2}^{1/2})(\|\nabla u\|_{L^2}^{1/2}+\|\nabla b\|_{L^2}^{1/2})-1\}.
\end{aligned}
$$

Due to the energy estimate $\int_0^\infty\big(\|\nabla u(s)\|_{L^2}^2+\|\nabla b(s)\|_{L^2}^2\big)\,ds\le 2(\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2)$, we can choose a T_* such that

$$
\|\nabla u\|_{L^2}^{1/2}+\|\nabla b\|_{L^2}^{1/2}\le\frac{1}{C(\|u_0\|_{L^2}^{1/2}+\|b_0\|_{L^2}^{1/2})}.
$$

By the continuity argument for ordinary differential equations,

$$
\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2)\le 0\quad\text{ for any } t\ge T_*.
$$

As a consequence,

$$(4.15)\qquad (t-T_*)(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2)\le \int_{T_*}^{t}(\|\nabla u(s)\|_{L^2}^2+\|\nabla b(s)\|_{L^2}^2)\,ds$$
$$\le \int_0^t(\|\nabla u(s)\|_{L^2}^2+\|\nabla b(s)\|_{L^2}^2)\,ds\le 2(\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2).$$

Therefore, it is easy to check that

$$\|\nabla u(t)\|_{L^2}+\|\nabla b(t)\|_{L^2}\le C\big(1+t\big)^{-1/2}\quad \text{for any } t\ge 0.$$

Hence, this completes the proof of Theorem 2.5. ∎

Proof of Theorem 2.6. The strategy is quite similar as that in proving Theorem 2.4. Moreover, the proof is more or less motivated by [JZD].

From (4.6) and (4.7), we have

$$|G(\xi,s)|\lesssim|\xi|(\|u\|_{L^2}^2+\|b\|_{L^2}^2)+\|u\|_{L^\alpha}^\alpha,$$
$$|H(\xi,s)|\lesssim|\xi|(\|u\|_{L^2}^2+\|b\|_{L^2}^2)+\|b\|_{L^\beta}^\beta.$$

It follows from Lemma 3.1, the Gagliardo–Nirenberg inequality and the Hölder inequality that

$$(4.16)\qquad \int_0^\infty\|u\|_{L^\alpha}^\alpha\,ds\le\int_0^\infty\|u\|_{L^{10/3}}^{\theta\alpha}\|u\|_{L^{\alpha+1}}^{(1-\theta)\alpha}\,ds$$
$$\le\Big(\int_0^\infty\|u\|_{L^{10/3}}^{10/3}\,ds\Big)^{3\alpha\theta/10}\Big(\int_0^\infty\|u\|_{L^{\alpha+1}}^{\alpha+1}\,ds\Big)^{\frac{(1-\theta)\alpha}{\alpha+1}}\le C,$$

where

$$\frac{1}{\alpha}=\frac{3\theta}{10}+\frac{1-\theta}{\alpha+1},\qquad \alpha\ge\frac{10}{3}.$$

Similarly, applying the same argument one can immediately obtain

$$\int_0^\infty\|b\|_{L^\beta}^\beta\,ds\le C,\qquad \beta\ge 10/3.$$

Therefore, it is easy to check

$$(4.17)\qquad |\hat u(\xi,t)|\lesssim e^{-|\xi|^2t}|\hat u_0(\xi)|+\int_0^t e^{-|\xi|^2(t-s)}\big(|\xi|(\|u\|_{L^2}^2+\|b\|_{L^2}^2)+\|u\|_{L^\alpha}^\alpha\big)\,ds$$
$$\lesssim e^{-|\xi|^2t}|\hat u_0(\xi)|+|\xi|\int_0^t(\|u\|_{L^2}^2+\|b\|_{L^2}^2)\,ds+\int_0^t\|u\|_{L^\alpha}^\alpha\,ds$$
$$\lesssim 1+e^{-|\xi|^2t}|\hat u_0(\xi)|+|\xi|\int_0^t(\|u\|_{L^2}^2+\|b\|_{L^2}^2)\,ds$$

and

$$|\hat{b}(\xi,t)| \lesssim 1 + e^{-|\xi|^2 t}|\hat{b}_0(\xi)| + |\xi| \int_0^t (\|u\|_{L^2}^2 + \|b\|_{L^2}^2)\,ds. \tag{4.18}$$

Let us denote

$$B(t) = \Big\{\xi \in \mathbb{R}^3 : |\xi| \le \sqrt{\frac{3+2\mu}{2(1+t)}}\Big\}.$$

Similarly, we can show

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3} (|\hat{u}|^2 + |\hat{b}|^2)\,dx &\le -2\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla b|^2)\,dx \\
&= -2\int_{\mathbb{R}^3} (|\xi|^2|\hat{u}|^2 + |\xi|^2|\hat{b}|^2)\,d\xi \\
&= -2\int_{B(t)} (|\xi|^2|\hat{u}|^2 + |\xi|^2|\hat{b}|^2)\,d\xi - 2\int_{B^c(t)} (|\xi|^2|\hat{u}|^2 + |\xi|^2|\hat{b}|^2)\,d\xi \\
&= -2\int_{B(t)} (|\xi|^2|\hat{u}|^2 + |\xi|^2|\hat{b}|^2)\,d\xi - \frac{3+2\mu}{t+1}\int_{B^c(t)} (|\hat{u}|^2 + |\hat{b}|^2)\,d\xi \\
&\le -\frac{3+2\mu}{t+1}\int_{\mathbb{R}^3} (|\hat{u}|^2 + |\hat{b}|^2)\,d\xi + \frac{3+2\mu}{t+1}\int_{B(t)} (|\hat{u}|^2 + |\hat{b}|^2)\,d\xi.
\end{aligned} \tag{4.19}$$

The last term on the right-hand side of (4.19) can be estimated as follows:

$$\begin{aligned}
&\int_{B(t)} (|\hat{u}|^2 + |\hat{b}|^2)\,d\xi \\
&\quad\lesssim \int_{B(t)} \{1 + e^{-2|\xi|^2 t}|\hat{u}_0(\xi)|^2 + e^{-2|\xi|^2 t}|\hat{b}_0(\xi)|^2\}\,d\xi \\
&\qquad + \int_{B(t)} \Big\{|\xi|^2\Big(\int_0^t (\|u\|_{L^2}^2 + \|b\|_{L^2}^2)\,ds\Big)^2\Big\}\,d\xi \\
&\quad\lesssim (t+1)^{-3/2} + \|e^{t\Delta}u_0\|_{L^2}^2 + \|e^{t\Delta}b_0\|_{L^2}^2 \\
&\qquad + \int_{B(t)} |\xi|^2\,d\xi\,\Big(\int_0^t (\|u\|_{L^2}^2 + \|b\|_{L^2}^2)\,ds\Big)^2 \\
&\quad\lesssim (t+1)^{-3/2} + (t+1)^{-2\mu} + (t+1)^{-5/2}t\int_0^t (\|u\|_{L^2}^4 + \|b\|_{L^2}^4)\,ds \\
&\quad\lesssim (t+1)^{-3/2} + (t+1)^{-2\mu} + (t+1)^{-3/2}\int_0^t (\|u\|_{L^2}^2 + \|b\|_{L^2}^2)\,ds,
\end{aligned} \tag{4.20}$$

where in the last inequality, we have also used the fact that $\|u\|_{L^2}+\|b\|_{L^2} \le \|u_0\|_{L^2}+\|b_0\|_{L^2}$.

Combining (4.19) and (4.20) yields

$$\frac{d}{dt}\int_{\mathbb{R}^3}(|\hat{u}|^2+|\hat{b}|^2)\,dx+\frac{3+2\mu}{t+1}\int_{\mathbb{R}^3}(|\hat{u}|^2+|\hat{b}|^2)\,d\xi \lesssim (t+1)^{-5/2}+(t+1)^{-2\mu-1}+(t+1)^{-5/2}\int_0^t(\|u\|_{L^2}^2+\|b\|_{L^2}^2)\,ds. \tag{4.21}$$

Multiplying (4.21) by $(t+1)^{3+2\mu}$, we have

$$\frac{d}{dt}\Big[(t+1)^{3+2\mu}\int_{\mathbb{R}^3}(|\hat{u}|^2+|\hat{b}|^2)\,d\xi\Big] \lesssim (t+1)^{1/2+2\mu}+(t+1)^2+(t+1)^{1/2+2\mu}\int_0^t(\|u\|_{L^2}^2+\|b\|_{L^2}^2)\,ds. \tag{4.22}$$

Integrating the above differential type inequality in time and dividing it by $(t+1)^{3+2\mu}$ leads to

$$\begin{aligned}\int_{\mathbb{R}^3}(|u|^2+|b|^2)\,dx &\lesssim (t+1)^{-3/2}+(t+1)^{-2\mu}+(t+1)^{-3-2\mu}\\ &\quad\times\int_0^t\Big\{(s+1)^{1/2+2\mu}\int_0^s(\|u\|_{L^2}^2+\|b\|_{L^2}^2)\,d\tau\Big\}\,ds\\ &\lesssim (t+1)^{-3/2}+(t+1)^{-2\mu}+(t+1)^{-3/2}\int_0^t(\|u(s)\|_{L^2}^2+\|b(s)\|_{L^2}^2)\,ds.\end{aligned} \tag{4.23}$$

We denote $\gamma=\min\{2\mu,3/2\}$. The above inequality reduces to

$$\begin{aligned}\|u\|_{L^2}^2+\|b\|_{L^2}^2 &\lesssim (t+1)^{-\gamma}+(t+1)^{-3/2}\int_0^t(\|u(s)\|_{L^2}^2+\|b(s)\|_{L^2}^2)\,ds\\ &\lesssim (t+1)^{-\gamma}+(t+1)^{-\gamma}(t+1)^{\gamma-3/2}\int_0^t(\|u(s)\|_{L^2}^2+\|b(s)\|_{L^2}^2)\,ds\\ &\lesssim (t+1)^{-\gamma}+(t+1)^{-\gamma}\int_0^t(s+1)^{-3/2}(s+1)^{\gamma}(\|u(s)\|_{L^2}^2+\|b(s)\|_{L^2}^2)\,ds.\end{aligned}$$

Consequently, the above inequality is equivalent to

$$(t+1)^{\gamma}(\|u\|_{L^2}^2+\|b\|_{L^2}^2)\le C+C\int_0^t (s+1)^{-3/2}(s+1)^{\gamma}(\|u(s)\|_{L^2}^2+\|b(s)\|_{L^2}^2)\,ds.$$

The standard Gronwall inequality gives rise to

$$(t+1)^{\gamma}(\|u\|_{L^2}^2+\|b\|_{L^2}^2)\le C.$$

Hence, we can obtain

$$\|u\|_{L^2}+\|b\|_{L^2}\le C(t+1)^{-\min\{\mu,3/4\}}. \blacksquare$$

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Zhuan Ye
School of Mathematical Sciences
Beijing Normal University
Laboratory of Mathematics and Complex Systems
Ministry of Education
Beijing 100875, People's Republic of China
E-mail: yezhuan815@126.com

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