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SELF-AFFINE MEASURES THAT ARE L^p-IMPROVING

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Abstract. A measure is called L^p-improving if it acts by convolution as a bounded operator from L^q to L^2 for some $q < 2$. Interesting examples include Riesz product measures, Cantor measures and certain measures on curves. We show that equicontractive, self-similar measures are L^p-improving if and only if they satisfy a suitable linear independence property. Certain self-affine measures are also seen to be L^p-improving.

1. Introduction. A measure μ on the d-dimensional torus, $\mathbb{T}^d = [0,1]^d$, is said to be L^p*-improving* if μ acts by convolution as a bounded linear operator from L^q to L^2 for some $q < 2$.

If $\mu = f\,dx$ for some $f \in L^r$ with $r > 1$, then an application of Young's inequality shows that μ is L^p-improving. The Hausdorff–Young inequality implies that any measure μ on $[0,1]$ with the property that $\widehat{\mu} \in l^p(\mathbb{Z})$ for some $p < \infty$ is also L^p-improving. More interestingly, there are L^p-improving measures whose Fourier transform does not tend to zero. Examples include Riesz product measures ([1], [17]) and uniform Cantor measures supported on Cantor sets with ratios of dissection bounded away from zero, such as the classical middle-third Cantor set. This was first established for the classical Cantor measure by Oberlin [15] using an iterative argument and was subsequently extended to Cantor measures on Cantor sets with ratios of dissection bounded away from zero by Christ [3]. The L^p-improving behaviour of measures on curves has also been extensively studied; we refer the reader to [21], for example, and the references cited therein.

The iterative construction of the Cantor measure is key to both the Oberlin and Christ proofs that the Cantor measures are L^p-improving. As an invariant probability measure associated with an iterated function system (IFS) of contractions also has an iterative construction, it is natural to ask if it too is L^p-improving.

The main result of this paper is to prove that an invariant measure associated with an equicontractive IFS of similarities is L^p-improving if and only if the similarities satisfy a suitable linear independence property. Our

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method is a generalization of that of [3]. A modification of the argument shows that invariant measures associated with a self-affine, equicontractive IFS, whose linear maps are diagonalizable over $\mathbb{R}$ and satisfy the same linear independence condition, are also L^p-improving. In addition to Cantor measures, examples of such measures include Bernoulli convolutions and invariant measures supported on Sierpiński carpets.

As one application, we show that the energy dimension of the k-fold convolution of any such measure on $\mathbb{T}^d$ tends to d as k tends to ∞.

2. Set up. A measure μ on $\mathbb{T}^d$ is said to be L^p-improving if there is some $q < 2$ and constant C such that

$$\|\mu * f\|_2 \leq C\|f\|_q \quad \text{for all } f \in L^q(\mathbb{T}^d). \tag{2.1}$$

Since (2.1) holds if and only if the same inequality holds with the measure μ^* defined by $\mu^*(E) = \overline{\mu(-E)}$, a duality argument shows that if (2.1) holds then we also have

$$\|\mu * f\|_{q'} \leq C\|f\|_2 \quad \text{for all } f \in L^{q'}(\mathbb{T}^d)$$

when q' is the conjugate index to q, meaning $1/q + 1/q' = 1$. As all measures act under convolution as bounded operators from L^p to L^p for all $1 \leq p \leq \infty$, an interpolation argument shows that if μ is L^p-improving, then for every $1 < p < \infty$ there is some $q > p$ such that μ maps boundedly from L^p to L^q.

Consider the iterated function system (IFS) of affine contractions on $\mathbb{R}^d$,

$$\{\mathcal{S}_i(x) = S_i x + b_i,\ i = 0, \ldots, m\} \tag{2.2}$$

where $b_i \in \mathbb{R}^d$ and S_i are linear maps. It is a classical result of Hutchinson [13] that there is a unique set K, called the *attractor*, satisfying $K = \bigcup_{i=0}^m \mathcal{S}_i(K)$. Furthermore, given any probabilities $\{p_i\}_{i=0}^m$, i.e., real numbers satisfying $p_i > 0$ and $\sum_{i=0}^m p_i = 1$, there is a unique, compactly supported probability measure μ satisfying

$$\mu(E) = \sum_{i=0}^m p_i \mu(\mathcal{S}_i^{-1}(E)) \quad \text{for all Borel sets } E \subseteq \mathbb{R}^d. \tag{2.3}$$

We will refer to the measure μ as the *self-affine* (or *invariant*) *measure* associated with the IFS (2.2) and probabilities $\{p_i\}_{i=0}^m$.

Without loss of generality we can assume $b_0 = 0$ and that the attractor is a subset of $[0,1]^d = \mathbb{T}^d$. We will suppose that all $S_i = S$; such IFS are sometimes called *equicontractive*. We are interested in two special cases:

1. The linear map S is a similarity. In this case $S = rR$ where R is an orthogonal transformation and $0 < r < 1$ is the contraction factor. We call the IFS an *equicontractive similarity*. The IFS that generates the classical Cantor set (see below) is an example.

2. The linear map S is diagonalizable over $\mathbb{R}$. We call this an *equicontractive diagonalizable IFS*. A Sierpiński carpet (see [6]) is an example of the attractor of such an IFS.

Notice that S is both a similarity and diagonalizable over $\mathbb{R}$ if and only if the rotation R is the identity map or its negative.

By an *equicontractive, self-similar measure* we mean an invariant measure associated with an IFS (as in (2.3)) that is an equicontractive similarity. An example of an equicontractive, self-similar measure on $[0,1]$ is the p-Cantor measure supported on a Cantor set with fixed ratio of dissection $r < 1/2$. This measure is generated by the IFS of similarities $\{S_0(x) = rx,\ S_1(x) \leq rx + 1 - r\}$ and probabilities $p_0 = p$, $p_1 = 1 - p$, and is purely singular with respect to Lebesgue measure. The classical, uniform Cantor measure is the special case $r = 1/3$ and $p = 1/2$. When $r \geq 1/2$ and $p_0 = p_1 = 1/2$, the equicontractive IFS $\{rx, rx+1-r\}$ generates a Bernoulli convolution measure. The Bernoulli convolutions are well known to have an L^2 density function for a.e. $r \geq 1/2$ [18], but are purely singular when r is a Pisot number [4], [5].

Given $\{b_j\} \subseteq \mathbb{R}^d$ and probabilities $\{p_j\}$, put

$$p(z) = \sum_{j=0}^{m} p_j \exp i(b_j \cdot z).$$

It is known (see [20, p. 342]) that the Fourier transform of the self-affine measure μ defined by (2.3) is given by

$$\widehat{\mu}(z) = \prod_{k=0}^{\infty} p(T^k(z)) \quad \text{where } T = S^*.$$

This infinite product structure is key to proving that such measures are typically L^p-improving. To be precise, we will prove the following.

THEOREM 2.1. *Suppose μ is a measure on $\mathbb{T}^d$ associated with the IFS $\{S_i(x) = S(x) + b_i\}_{i=0}^m$, where $b_0 = 0$ and S is either a similarity or diagonalizable over $\mathbb{R}$. Assume the vectors $b_1, \ldots, b_m$ span $\mathbb{R}^d$. There is a constant C and $q < 2$ such that*

$$\|\mu * f\|_2 \leq C \|f\|_q \quad \text{for all } f \in L^q(\mathbb{T}^d).$$

The proof will be given for the similarity case in Section 3 and for the diagonalizable case in Section 4. Before turning to this, we show how to deduce the characterization of L^p-improving, equicontractive, self-similar measures mentioned in the Introduction.

COROLLARY 2.2. *Suppose μ is an equicontractive, self-similar measure on $\mathbb{T}^d$ associated with the IFS $\{S_i(x) = Sx + b_i\}_{i=0}^m$, where $b_0 = 0$ and S is* *a similarity. Then μ is L^p-improving if and only if*

$$W_n := \{S^k(b_j) : k = 0, \dots, n-1;\ j = 1, \dots, m\}$$

spans $\mathbb{R}^d$ for some n.

Proof. The measure μ is also the invariant measure arising from the equicontractive IFS consisting of the collection of functions $\{S_{i_1} \circ \dots \circ S_{i_n} : 0 \le i_j \le m\}$ (for any fixed n) and probabilities $\{p_{i_1}, \dots, p_{i_n}\}$. We have $S_{i_1} \circ \dots \circ S_{i_n}(x) = S^n(x) + \sum_{j=0}^{n-1} S^j(b_{i_{j+1}})$ and thus, according to the theorem, μ is L^p-improving if

$$\Big\{ \sum_{j=0}^{n-1} S^j(b_{i_{j+1}}) : i_j \in \{0, \dots, m\} \Big\}$$

spans $\mathbb{R}^d$ for some n. As $b_0 = 0$, this is the same as saying W_n spans $\mathbb{R}^d$.

Conversely, suppose W_n does not span $\mathbb{R}^d$ for any n. Let V_n be the vector subspace spanned by W_n. By assumption, each V_n is a proper subspace of $\mathbb{R}^d$, and as they are nested there must be an integer n_0 such that $V_n = V_{n_0}$ for all $n \ge n_0$. The attractor of the IFS is the closure of $\bigcup W_n$ and hence is contained in the closure of V_{n_0}. But V_{n_0} is a finite-dimensional subspace and so is already closed. Furthermore, being a proper subspace it has Lebesgue measure zero.

As measure zero is preserved when passing to the quotient space $\mathbb{T}^d$, it follows that μ is supported on a closed subgroup of infinite index and measure zero in $\mathbb{T}^d$. But this is not possible for an L^p-improving measure (see [9]). ∎

REMARK 2.3. The property that W_n spans $\mathbb{R}^d$ is equivalent to the statement that some subset of W_n of cardinality d is linearly independent. This linear independence property can hold without the open set condition being satisfied by the IFS. Indeed, when $d = 1$, it is equivalent to the requirement that the IFS includes two equations, Sx and $Sx + b$ where $b \neq 0$. Thus all Bernoulli convolutions are L^p-improving measures. In $\mathbb{R}^2$, the linear independence property (W_n spans $\mathbb{R}^d$) is satisfied by the two-function IFS $\{Sx, Sx + b\}$ if and only if $b \neq 0$ and b is not an eigenvector of S.

3. Self-similar measures that are L^p-improving

3.1. Preliminary results. As in [3], we begin with two technical results; these are essentially known. The first is a version of the Littlewood–Paley theorem.

NOTATION. Given a bounded function $\phi : \mathbb{Z}^d \to \mathbb{C}$, we define a multiplier, M_ϕ, by $\widehat{M_\phi(f)}(n) = \phi(n)\widehat{f}(n)$ for $n \in \mathbb{Z}^d$. We write $\|M_\phi\|_{p,q}$ for the operator norm of M_ϕ as a mapping $L^p(\mathbb{T}^d) \to L^q$.

LEMMA 3.1. *Suppose $F_j \subseteq F_{j+1}$ are subsets of $\mathbb{Z}^d$ with $j = 1, 2, \ldots$ and F_0 is empty. Assume* $\operatorname{dist}(F_j, F_{j+1}^c) \geq 2 \operatorname{diam} F_j$ *and* $\bigcup F_j = \mathbb{Z}^d$.

(a) *Given a trigonometric polynomial f on $\mathbb{T}^d$, define*

$$f_j(x) = \sum_{n \in F_j \setminus F_{j-1}} \widehat{f}(n) e^{in\cdot x}, \qquad j = 1, 2, \ldots.$$

There is a constant C, independent of the choice of sets $\{F_j\}_j$, such that for all trigonometric polynomials f,

$$\|f\|_4 \leq C \Big\| \Big(\sum |f_j|^2 \Big)^{1/2} \Big\|_4.$$

(b) *Given any $A > 1$, there is some $p > 2$ such that if $\phi : \mathbb{Z}^d \to \mathbb{C}$ is a bounded function and $\phi_j = \phi 1_{F_j \setminus F_{j-1}}$, then the multiplier M_ϕ has operator norm*

$$\|M_\phi\|_{2,p} \leq A \sup_j (\|M_{\phi_j}\|_{2,p}).$$

Proof. (a) For completeness, we give a proof in the spirit of [10].

First, we remark that there is no loss of generality in assuming that if $f_j \neq 0$, then $f_{j-1} = f_{j+1} = 0$. This is because if the result is established for all such polynomials f, then taking an arbitrary polynomial f and putting $g_1 = \sum_{j \text{ even}} f_j$ and $g_2 = \sum_{j \text{ odd}} f_j$ gives

$$\begin{aligned} C \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_4^4 &\geq C \int \Big(\sum_{j \text{ even}} |f_j|^2 \Big)^2 + C \int \Big(\sum_{j \text{ odd}} |f_j|^2 \Big)^2 \\ &\geq \|g_1\|_4^4 + \|g_2\|_4^4 \geq 2^{-4} \|f\|_4^4. \end{aligned}$$

So assume f is a trigonometric polynomial with $f_{j-1} = f_{j+1} = 0$ whenever $f_j \neq 0$. Put $G_j = \sum_{k=1}^{j-1} f_k$ and $B_j = \sum_{k=j+1}^{\infty} f_k$. We claim

(i) $\int |G_j|^2 (f_j \overline{G_j} + \overline{f_j} G_j) = 0$,
(ii) $\int |f_j|^2 (B_j \overline{G_j} + \overline{B_j} G_j + \overline{f_j} B_j + \overline{B_j} f_j) = 0$.

This is trivially true if $f_j = 0$, so assume otherwise. In this case, we have $G_j = \sum_{k=1}^{j-2} f_k$ and $B_j = \sum_{k=j+2}^{\infty} f_k$, so

$$\operatorname{supp} \widehat{G_j} \subseteq \bigcup_{k=1}^{j-2} \operatorname{supp} \widehat{f_k} \subseteq F_{j-2} \quad \text{and} \quad \operatorname{supp} \widehat{B_j} \subseteq \bigcup_{k=j+2}^{\infty} \operatorname{supp} \widehat{f_k} \subseteq F_{j+1}^c.$$

If $n \in \operatorname{supp} \widehat{|G_j|^2}$, then $n = n_1 - n_2$ where $n_1, n_2 \in \operatorname{supp} \widehat{G_j}$. Hence $\|n\| = \|n_1 - n_2\| \leq \operatorname{diam} F_{j-2}$.

If $n \in \operatorname{supp} \widehat{f_j \overline{G_j}}$ or $n \in \operatorname{supp} \widehat{G_j \overline{f_j}}$ then $n = \pm(n_1 - n_2)$ where $n_1 \in \operatorname{supp} \widehat{f_j} \subseteq F_j \setminus F_{j-1}$ and $n_2 \in \operatorname{supp} \widehat{G_j}$. Thus $\|n\| = \operatorname{dist}(n_1, n_2) \geq \operatorname{dist}(F_{j-1}^c, F_{j-2})$

$\geq 2\operatorname{diam} F_{j-2}$. This shows that $|G_j|^2$ is orthogonal to both $f_j\overline{G_j}$ and $\overline{f_j}G_j$, so the integral in (i) is zero.

Similarly, if $n \in \operatorname{supp} \widehat{|f_j|^2}$, then $\|n\| \leq \operatorname{diam} F_j$. If $n \in \operatorname{supp} \widehat{B_j\overline{G_j}}$ or $n \in \operatorname{supp} \widehat{G_j\overline{B_j}}$, then $\|n\| \geq \operatorname{dist}(F_{j-2}, F_{j+1}^c) \geq 2\operatorname{diam} F_j$, so $\int |f_j|^2(B_j\overline{G_j} + \overline{B_j}G_j) = 0$. A similar argument shows that the remaining integrals in (ii) are zero.

Put $G_1 = 0$ and let $P_j = |G_j + f_j|^4 - |G_j|^4 = \sum c(a,b)G_j^{2-a}\overline{G}_j^{2-b} f_j^a \overline{f_j}^b$ where the sum is over $a, b \in \{0,1,2\}$ with a, b not both zero and $c(a,b)$ are suitable binomial coefficients.

If $a + b = 1$, then (i) implies that $\int G_j^{2-a}\overline{G}_j^{2-b} f_j^a \overline{f_j}^b = 0$. Thus we may assume $a + b \geq 2$, and then we have

$$\left|\int G_j^{2-a}\overline{G}_j^{2-b} f_j^a \overline{f_j}^b\right| \leq \int (|f_j|^4 + |f_j|^2|G_j|^2).$$

The orthogonality relations of (ii) imply that

$$\begin{aligned}\int |f_j|^2|f|^2 &= \int |f_j|^2|G_j + f_j + B_j|^2 \\ &= \int |f_j|^2\big(|G_j|^2 + |f_j|^2 + |B_j|^2 + f_j\overline{G_j} + \overline{f_j}G_j\big) \\ &\geq \int |f_j|^2(|G_j|^2 - 2|f_jG_j|)\end{aligned}$$

so that

$$\int |f_j|^2|G_j|^2 \leq \int |f_j|^2|f|^2 + 2|f_j|^3|G_j|.$$

Applying the elementary inequality $s^x t^{n-x} \leq \varepsilon s^n + c(\varepsilon, x)t^n$ for $s, t \geq 0$, $0 \leq x \leq n$, with $\varepsilon = 1/4$ gives

$$\int |f_j|^3|G_j| \leq \frac{1}{4}\int |f_j|^2|G_j|^2 + c\int |f_j|^4,$$

hence

$$\int |f_j|^2|G_j|^2 \leq 2\int |f_j|^2|f|^2 + 4c\int |f_j|^4.$$

Thus, we deduce that

$$\left|\int G_j^{2-a}\overline{G}_j^{2-b} f_j^a \overline{f_j}^b\right| \leq c\int (|f_j|^4 + |f_j|^2|f|^2)$$

(for a new constant c). Summing over j gives

$$\|f\|_4^4 = \int \sum P_j \leq \sum_j c\int (|f_j|^4 + |f_j|^2|f|^2).$$

Applying the elementary inequality again to $|f_j|^2|f|^2$, with small enough ε, and simplifying gives the desired result,

$$\|f\|_4^4 \leq c\sum_j \int |f_j|^4 \leq \int \Big(\sum |f_j|^2\Big)^2.$$

(b) We use the same notation as in (a). Since the inequality

$$\|f\|_p \le C\Big\|\Big(\sum |f_j|^2\Big)^{1/2}\Big\|_p$$

holds for $p = 4$ by (a) and for $p = 2$ with $C = 1$ by Parseval's theorem, the vector-valued version of the Riesz–Thorin interpolation theorem implies that for any $2 < p < 4$ with $1/p = t/4 + (1-t)/2$ we have

$$\|f\|_p \le C^t\Big\|\Big(\sum |f_j|^2\Big)^{1/2}\Big\|_p.$$

Given $A > 1$, choose $t > 0$ small enough (equivalently, p sufficiently close to 2) so that $C^t \le A$. Since $(M_\phi f)_j = M_{\phi_j}(f_j)$, with this p and Minkowski's inequality we obtain

$$\begin{aligned}\|M_\phi f\|_p &\le A\Big\|\Big(\sum_j |(M_\phi f)_j|^2\Big)^{1/2}\Big\|_p \le A\Big(\sum \|M_{\phi_j}(f_j)\|_p^2\Big)^{1/2}\\ &\le A\max_k \|M_{\phi_k}\|_{2,p}\Big(\sum_j \|f_j\|_2^2\Big)^{1/2} = A\max_k \|M_{\phi_k}\|_{2,p}\|f\|_2\end{aligned}$$

with the (final) equality holding because the functions f_j are mutually orthogonal. ∎

A similar interpolation argument gives a related result for a finite decomposition.

LEMMA 3.2. *Suppose L is fixed and $F_1, \dots, F_L \subseteq \mathbb{Z}^d$ are disjoint sets. Given any $A > 1$, there is some $p > 2$ such that if $\phi : \mathbb{Z}^d \to \mathbb{C}$ is a bounded function and $\phi_j = \phi 1_{F_j}$, then*

$$\|M_\phi\|_{2,p} \le A\max(\|M_{\phi_j}\|_{2,p} : j = 1, \dots, L).$$

3.2. Proof of Theorem 2.1 in the self-similar case. Recall that $\widehat{\mu}(z) = \prod_{k=0}^\infty p(T^k(z))$ where $T = S^*$ and $p(z) = \sum_{j=0}^m p_j e^{ib_j\cdot z}$. For each $z \in \mathbb{Z}^d$ we have $|p(z)| \le 1$, thus for any positive integer J,

$$|\widehat{\mu}(z)| \le \Big|\prod_{k=0}^\infty p(T^{kJ}(z))\Big| =: \Phi_J(z).$$

Consequently, $\|\mu * f\|_2 \le \|M_{\Phi_J}(f)\|_2$ for all f, and thus it is enough to prove $M_{\Phi_J} : L^q \to L^2$ is a bounded multiplier for some J. Since the contraction factor of T^J is r^J, where r is the contraction factor of T, it follows that there is no loss of generality in assuming $r \le 1/9$.

As the vectors $\{b_1, \dots, b_m\}$ are assumed to span $\mathbb{R}^d$, there is also no loss of generality in assuming $b_1, \dots, b_d$ are linearly independent.

Elementary trigonometry arguments show that there is a constant $c > 0$ such that for any $\varepsilon > 0$, $|p(z)| \le 1 - \varepsilon$ if $|1 - e^{ib_j\cdot z}| > c\sqrt{\varepsilon}$ for any j. Let

$k = (k_1, \dots, k_d) \in \mathbb{Z}^d$. For each j, the set

$$\{z \in \mathbb{R}^d : b_j \cdot z = k_j\}$$

is a $(d-1)$-dimensional hyperplane. The linear independence of $\{b_1, \dots, b_d\}$ ensures that for each d-tuple $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, there is a unique $z \in \mathbb{R}^d$ such that $b_j \cdot z = k_j$ for all $j = 1, \dots, d$. Further, $\{z : |b_j \cdot z - k_j| \le c\sqrt{\varepsilon}\}$ is the region between two hyperplanes with distance $O(\sqrt{\varepsilon})$, and thus

$$\{z : |b_j \cdot z - k_j| \le c\sqrt{\varepsilon} \text{ for } j = 1, \dots, d\}$$

is contained in a d-dimensional cube $\Lambda_k = \Lambda_k(\varepsilon) \subseteq \mathbb{R}^d$ of side lengths $O(\sqrt{\varepsilon})$ and centred at the unique solution to $\{b_j \cdot z = k_j : j = 1, \dots, d\}$. For small enough $\varepsilon > 0$, say $\varepsilon \le \varepsilon_0$, the cubes $\Lambda_k(\varepsilon)$ are disjoint for different $k \in \mathbb{Z}^d$. Outside these cubes, $|p(z)| \le 1 - \varepsilon$.

Fix a sphere S_0 whose diameter is so small that even τS_0, the tripled sphere of S_0—meaning the sphere with the same centre and three times the radius—has the property that any translate of τS_0 can intersect at most one of the cubes $\Lambda_k(\varepsilon)$ for any fixed $\varepsilon \le \varepsilon_0$.

If r is the contraction factor of T, then $T^{-1}(\Lambda_k(\varepsilon))$ is also a cube of diameter $O(\sqrt{\varepsilon})r^{-1}$. By taking sufficiently small $\varepsilon \le \min(\varepsilon_0, 1/2)$, we can assume that for each k, $T^{-1}(\Lambda_k(\varepsilon))$ is contained in a translate of S_0. This choice of ε is now fixed.

Take q to be the conjugate index to the minimal of the $p > 2$ that are found in Lemmas 3.1 and 3.2 with $A = 1 + \varepsilon^4$, and L the number of translates of S_0 required to cover $\tau(T^{-1}(S_0))$, where $\tau(T^{-1}(S_0))$ denotes the tripled sphere of $T^{-1}(S_0)$. Note that linearity of T implies $\tau(T^{-1}(S_0)) = T^{-1}(\tau S_0)$.

Define the function $\phi_n : \mathbb{Z}^d \to \mathbb{C}$ by

$$\phi_n(z) = \Big| \prod_{k=0}^{n} p(T^k(z)) \Big| \quad \text{for } z \in \mathbb{Z}^d,$$

and let M_n denote the multiplier M_{ϕ_n}. Since $|\widehat{\mu}(z)| \le |\phi_n(z)|$ for all n, it is enough to prove there is a constant C such that $\|M_n\|_{q,2} \le C$ for all n. In fact, it is enough to show

$$\|M_n 1_{T^{-n}(S_0)}\|_{q,2} \le C \quad \text{for all } n, \tag{3.1}$$

where by 1_E we mean the multiplier M_ϕ with $\phi = 1_{E \cap \mathbb{Z}^d}$. This is because if f is any trigonometric polynomial, then $\operatorname{supp} \widehat{f} \subseteq T^{-n}(S_0)$ for some n, and thus, assuming (3.1) holds,

$$\|\mu * f\|_2 \le \|M_n(f)\|_2 = \|M_n 1_{T^{-n}(S_0)}(f)\|_2 \le C\|f\|_q.$$

We will actually prove that if $S_n = T^{-n}(S_0)$ and S'_n is any translate of S_n, then $\|M_n 1_{S'_n}\|_{q,2} \le C$ for all n. We will do this by an induction argument on n.

Let B_0 be an upper bound on the number of integer vectors in $\mathbb{Z}^d$ contained in any translate of τS_0, and put $C = 2B_0$. Since $|p(z)| \le 1$, we even have $\|M_0 1_{S_0'}\|_{q,2} \le \sqrt{B_0} \le C/2$ whenever S_0' is a translate of S_0.

Now we proceed by induction, assuming $\|M_j 1_{S_j'}\|_{q,2} \le C$ for all $j \le n-1$. Fix a translate, S_n', of $T^{-n}(S_0)$ and let $\tau S_n'$ be its tripled sphere. Then $\tau S_n' = T^{-n}(\tau S_0')$ for some translate S_0' of S_0.

Recall that $|p(z)| \le 1-\varepsilon$ except if $z \in \bigcup_{j\in\mathbb{Z}^d} \Lambda_j$. Thus $|p(T^n(z))| \le 1-\varepsilon$ except if $z \in T^{-n}(\bigcup \Lambda_j)$. As there is at most one choice of j such that $\Lambda_j \cap \tau S_0'$ is non-empty, there is also at most one choice of j such that

$$T^{-n}(\Lambda_j) \cap T^{-n}(\tau S_0') = T^{-n}(\Lambda_j) \cap \tau S_n'$$

is non-empty.

Case 1: $T^{-n}(\bigcup \Lambda_j) \cap S_n'$ is empty. In this case, $|p(T^n(z))| \le 1-\varepsilon$ for all $z \in S_n'$.

We know that $T^{-1}(S_0')$ can be covered by at most L translates of S_0. Thus $S_n' = T^{-(n-1)}(T^{-1}(S_0'))$ can be covered by at most L translates of $T^{-(n-1)}(S_0)$, say $S_{n-1}^{(i)} = T^{-(n-1)}(S_0^{(i)})$ for $i = 1, \ldots, L$. By the induction assumption,

$$\|M_{n-1} 1_{T^{-(n-1)}(S_0^{(i)})}\|_{q,2} = \|M_{n-1} 1_{S_{n-1}^{(i)}}\|_{q,2} \le C$$

for each translate $S_0^{(i)}$ of S_0. As $M_{n-1} 1_{S_n'} \le \sum_{i=1}^{L} M_{n-1} 1_{S_{n-1}^{(i)}}$, it follows from Lemma 3.2 (for the choice of q that has been made) that

$$\|M_{n-1} 1_{S_n'}\|_{q,2} \le (1+\varepsilon^4) \max(\|M_{n-1} 1_{S_{n-1}^{(i)}}\|_{q,2} : i = 1, \ldots, L) \le (1+\varepsilon^4)C.$$

Since $M_n(z) = p(T^n(z)) M_{n-1}(z)$ and $|p(T^n(z))| \le 1-\varepsilon$ for all $z \in S_n'$, we deduce that

$$\|M_n 1_{S_n'}\|_{q,2} \le (1-\varepsilon)\|M_{n-1} 1_{S_n'}\|_{q,2} \le (1-\varepsilon^2)C \le C$$

and we are done.

Case 2: There is one choice of $j = j(n)$ such that $T^{-n}(\Lambda_j) \cap S_n'$ is non-empty. In this case $T^{-n}(\Lambda_k) \cap \tau S_n'$ is empty for all $k \ne j$, and therefore $|p(T^n(z))| \le 1-\varepsilon$ for all $z \in \tau S_n' \setminus T^{-n}(\Lambda_j)$.

Since $T^{-1}(\Lambda_j)$ is contained in a translate of S_0, $T^{-n}(\Lambda_j)$ is contained in a translate of $T^{-(n-1)}(S_0)$, say S_{n-1}'. The set $\tau S_n'$ can be covered by at most L translates of $T^{-(n-1)}(S_0)$, hence the same reasoning as above shows that

$$\|M_n 1_{\tau S_n' \setminus S_{n-1}'}\|_{q,2} \le (1-\varepsilon^2)C.$$

Now we repeat the argument. We know that $\tau S_{n-1}'$ can intersect at most one set $T^{-(n-1)}(\Lambda_j)$. If S_{n-1}' misses all the sets $T^{-(n-1)}(\Lambda_j)$, then

$|p(T^{n-1}(z))| \le 1-\varepsilon$ for all $z \in S'_{n-1}$. Arguing as in Case 1 above, we find that

$$\|M_n 1_{S'_{n-1}}\|_{q,2} \le \|M_{n-1} 1_{S'_{n-1}}\|_{q,2} \le (1-\varepsilon^2)C.$$

Applying Lemma 3.2 completes the induction step since

$$\begin{aligned}\|M_n 1_{\tau S'_n}\|_{q,2} &\le (1+\varepsilon^4)\max(\|M_n 1_{\tau S'_n \setminus S'_{n-1}}\|_{q,2}, \|M_n 1_{S'_{n-1}}\|_{q,2})\\ &\le (1+\varepsilon^4)(1-\varepsilon^2)C \le (1-\varepsilon^4)C.\end{aligned}$$

Thus we can suppose there is one choice of j such that $T^{-(n-1)}(\Lambda_j) \cap S'_{n-1}$ is non-empty. Then we are back to the beginning of the Case 2 scenario, but with the index n replaced by $n-1$. Consequently,

$$\|M_n 1_{\tau S'_{n-1} \setminus \tau S'_{n-2}}\|_{q,2} \le \|M_n 1_{\tau S'_{n-1} \setminus S'_{n-2}}\|_{q,2} \le (1-\varepsilon^2)C$$

for S'_{n-2} a suitable translate of $T^{-(n-2)}(S_0)$.

We continue to repeat these arguments, producing sets S'_{n-j} satisfying

$$\|M_n 1_{\tau S'_{n-j+1} \setminus \tau S'_{n-j}}\|_{q,2} \le \|M_n 1_{\tau S'_{n-j+1} \setminus S'_{n-j}}\|_{q,2} \le (1-\varepsilon^2)C,$$

until either some set S'_{n-J} misses all the sets $T^{-(n-J)}(\Lambda_k)$, or $J = n$.

In the former case, $\left|p(T^{n-J}(z))\right| \le 1-\varepsilon$ for all $z \in S'_{n-J}$, and then as in the Case 1 argument,

$$\|M_n 1_{S'_{n-J}}\|_{q,2} \le \|M_{n-J} 1_{S'_{n-J}}\|_{q,2} \le (1-\varepsilon^2)C.$$

We therefore have

$$\begin{aligned}\|M_n 1_{\tau S'_{n-J}}\|_{q,2} &\le \|M_n 1_{\tau S'_{n-J+1}}\|_{q,2}\\ &\le (1+\varepsilon^4)\max\big(\|M_n 1_{\tau S'_{n-J+1} \setminus S'_{n-J}}\|_{q,2}, \|M_n 1_{S'_{n-J}}\|_{q,2}\big)\\ &\le (1-\varepsilon^4)C.\end{aligned}$$

We recall that C was chosen so that $\|M_n 1_{\tau S'_0}\|_{q,2} \le C/2$, so that also in the case when $n = J$, we have $\|M_n 1_{\tau S'_{n-J}}\|_{q,2} \le (1-\varepsilon^4)C$.

The construction process ensures that $S'_{k-1} \cap S'_k$ is non-empty provided $k \ge n-J+1$. Since $r^{-1} \ge 9$, one can check that $\tau S'_{k-1} \subseteq \tau S'_k$, and since $\operatorname{diam} \tau S'_{k-1} = 3r^{-(k-1)} \operatorname{diam} S_0$, we even have

$$\begin{aligned}\operatorname{dist}(\tau S'_{k-1}, (\tau S'_k)^c) &\ge \operatorname{diam} S'_k - 3 \operatorname{diam} S'_{k-1}\\ &\ge r^{-(k-1)}(r^{-1}-3) \operatorname{diam} S_0\\ &\ge 6r^{-(k-1)} \operatorname{diam} S_0 = 2 \operatorname{diam} \tau S'_{k-1}.\end{aligned}$$

Thus we are now in a position to apply Lemma 3.1(b), taking the nested sets $F_k = \tau S'_{n-J+k-1}$ for $k = 1, \ldots, J+1$ and $m = M_n 1_{\tau S'_n}$. Appealing to that lemma we see that

$$\begin{aligned}&\|M_n 1_{\tau S'_n}\|_{q,2}\\ &\qquad\le (1+\varepsilon^4)\max\big(\|M_n 1_{\tau S'_{n-J}}\|_{q,2}, \|M_n 1_{\tau S'_{n-j+1}\setminus \tau S'_{n-j}}\|_{q,2} : j=1,\dots,J\big)\\ &\qquad\le (1-\varepsilon^8)C,\end{aligned}$$

completing the proof. ∎

4. Self-affine measures that are L^p-improving

4.1. Preliminary results. In this section we argue similarly to establish the L^p-improving properties of the measures associated with the IFS $\{Sx+b_i : i=0,\dots,m\}$, when S is a linear map on $\mathbb{R}^d$ that is diagonalizable over $\mathbb{R}$.

To begin, we introduce additional notation. Let $e_1,\dots,e_d$ be a linearly independent set of vectors in $\mathbb{R}^d$. By *d-dimensional parallelepipeds oriented in the directions* $e_1,\dots,e_d$ we mean sets of the form

$$F_j = \Big\{v=\sum_{i=1}^d v_i e_i \in \mathbb{R}^d : v_i \in [a_{ij}, b_{ij}]\Big\}.$$

These sets are nested if $[a_{i,j-1}, b_{i,j-1}] \subseteq [a_{ij}, b_{ij}]$ for all i,j, and in that case we say the ith *coordinate distance* between F_{j-1} and F_j^c is

$$D_i(F_{j-1}, F_j^c) := \min(|a_{ij}-a_{i,j-1}|, |b_{ij}-b_{i,j-1}|).$$

We call v_i the ith *coordinate* of v and write

$$l_i(F_j) \equiv b_{ij} - a_{ij}.$$

Our first lemma is analogous to Lemma 3.1.

LEMMA 4.1. *Suppose the sets F_j are nested, d-dimensional parallelepipeds oriented in the directions $e_1,\dots,e_d$. Assume $\bigcup F_j = \mathbb{R}^d$ and that for all i,j,*

$$D_i(F_{j-1}, F_j^c) > l_i(F_{j-1}).$$

(a) *Given a trigonometric polynomial f on $\mathbb{T}^d$, define*

$$f_j(x) = \sum_{n\in F_j\setminus F_{j-1}} \widehat{f}(n) e^{in\cdot x}, \qquad j=1,2,\dots$$

(where F_0 is the empty set). There is a constant C (independent of the choice of sets $\{F_j\}$) such that for all trigonometric polynomials f,

$$\|f\|_4 \le C\Big\|\Big(\sum |f_j|^2\Big)^{1/2}\Big\|_4.$$

(b) *Given any* $A > 1$, *there is some* $p > 2$ *such that if* $\phi : \mathbb{Z}^d \to \mathbb{C}$ *is a bounded function and* $\phi_j = \phi 1_{F_j \setminus F_{j-1}}$, *then*

$$\|M_\phi\|_{2,p} \le A \sup_j (\|M_{\phi_j}\|_{2,p}).$$

Proof. The proof is quite similar to that of Lemma 3.1. We can assume f is a trigonometric polynomial with $f_{j-1} = f_{j+1} = 0$ whenever $f_j \neq 0$. Put $G_j = \sum_{k=1}^{j-1} f_k$ and $B_j = \sum_{k=j+1}^{\infty} f_k$. As in the proof of Lemma 3.1 it will suffice to show that

(i) $\int |G_j|^2 (f_j \overline{G_j} + \overline{f_j} G_j) = 0$,
(ii) $\int |f_j|^2 (B_j \overline{G_j} + \overline{B_j} G_j + \overline{f_j} B_j + \overline{B_j} f_j) = 0$.

Of course this is obvious if $f_j = 0$, so we can assume $G_j = \sum_{k=1}^{j-2} f_k$ and $B_j = \sum_{k=j+2}^{\infty} f_k$. Thus if $n \in \operatorname{supp} \widehat{|G_j|^2}$, then $n = n_1 - n_2$ where $n_1, n_2 \in F_{j-2}$. Hence $|(n_1 - n_2)_k| \le l_k(F_{j-2})$ for all k.

If $m \in \operatorname{supp} \widehat{f_j \overline{G_j}}$ or $\operatorname{supp} \widehat{\overline{f_j} G_j}$, then $m = m_1 - m_2$ where $m_1 \in F_{j-1}^c$ and $m_2 \in F_{j-2}$ (or vice versa). For some coordinate k, $|(m_1 - m_2)_k| \ge D_k(F_{j-2}, F_{j-1}^c)$. As $D_k(F_{j-2}, F_{j-1}^c) > l_k(F_{j-2})$, we cannot have $m = n$. Thus $|G_j|^2$ is orthogonal to $f_j \overline{G_j}$ and $\overline{f_j} G_j$, and that establishes (i). Identity (ii) is similar.

The remainder of the proof follows exactly as before. ■

4.2. Proof of Theorem 2.1 for the diagonalizable case. With the revised lemma, the proof of Theorem 2.1 in the equicontractive, diagonalizable case is quite similar to the self-similar case, with the main difference being that we replace spheres by parallelepipeds. We note that T, the adjoint of S, is also diagonalizable over $\mathbb{R}$, and we will assume $e_1, \ldots, e_d$ is a basis of eigenvectors of T corresponding to eigenvalues $r_1, \ldots, r_d$. As S is a contraction, each $|r_j| < 1$. When we say parallelepiped, we will mean a d-dimensional parallelepiped oriented in the directions of these vectors $e_1, \ldots, e_d$.

To begin, we let $\Lambda_k(\varepsilon)$ denote a parallelepiped with equal side lengths ε, centred at the unique solution to $\{b_j \cdot z = k_j : j = 1, \ldots, d\}$ (with the same notation as for the similarity case) and choose $\varepsilon_0 > 0$ so that if $\varepsilon \le \varepsilon_0$, these sets are disjoint for distinct k. Fix a parallelepiped, S_0, which is so small that even its *triple*, τS_0, the parallelepiped with the same centre and triple the side lengths, has the property that any translate of τS_0 can intersect at most one of the parallelepipeds $\Lambda_k(\varepsilon)$ for any $\varepsilon \le \varepsilon_0$.

The linear map T^{-1} is also diagonalizable, with the same eigenvectors as T, and $T^{-1}(\Lambda_k(\varepsilon))$ is a parallelepiped with side lengths $|r_j^{-1}| O(\varepsilon)$, $j = 1, \ldots, d$. By taking ε sufficiently small we can assume each $T^{-1}(\Lambda_k(\varepsilon))$ is contained in a translate of S_0. Fix this ε.

We remark that $T^{-n}(S_0)$ and its triple, $\tau T^{-n}(S_0) = T^{-n}(\tau S_0)$, are also parallelepipeds (with the obvious modifications to their meaning).

Now proceed as in the proof of the similarity case (and with the analogous notation) taking $F_k = \tau S'_{n-J+k-1}$ for $k = 1, \ldots, J+1$. The fact that $S'_k \cap S'_{k-1}$ is non-empty and both S'_k, S'_{k-1} are parallelepipeds oriented in the directions $e_1, \ldots, e_d$ allows one to establish that the sets F_k satisfy the hypothesis of Lemma 4.1. The proof is completed by appealing to the lemma. ∎

5. Consequences. Many self-similar measures are known to have an average decay in their Fourier transform. For instance, if μ is the self-similar measure associated with an IFS on $\mathbb{R}^d$, satisfying the open set condition (1), with contractions $\{r_j\}_{j=0}^m$ and probabilities $\{p_j\}_{j=0}^m$, then

$$\sup_R \frac{1}{R^{d-\beta}} \int\limits_{B(0,R)} |\widehat{\mu}(z)|^2 \, dz < \infty, \tag{5.1}$$

where β satisfies the equation $1 = \sum_{i=0}^m p_i^2 r_i^{-\beta}$. For more about this see [7], [14], [19] and [20].

There is a discrete version of (5.1) known for L^p-improving measures, and thus, in particular, for suitable self-affine measures.

Corollary 5.1 ([11]). *If there is some $q < 2$ and constant C such that $\|\mu * f\|_2 \le C\|f\|_q$ for all $f \in L^2(\mathbb{T}^d)$, then*

$$\sup_N \frac{1}{N^{2d/q'}} \sum_{\|n\| \le N} |\widehat{\mu}(n)|^2 < \infty. \tag{5.2}$$

The *Hausdorff dimension* of a measure μ is defined as $\dim_H \mu = \inf\{\dim_H F : \mu(F) \neq 0\}$. When μ is the self-similar measure as above, then $\dim_H \mu = s$ where s solves the equation $s = \sum p_j \log p_j / \sum p_j \log r_j$ (see [2] or [6]). A related dimension is the *energy dimension*, defined as

$$\dim_e \mu = \sup\Big\{ t : \int\limits_{\mathbb{T}^d} \int\limits_{\mathbb{T}^d} \frac{d\mu(x)\, d\mu(y)}{\|x-y\|^t} < \infty \Big\}.$$

It is well known that for all measures μ, $\dim_e \mu \le \dim_H \mu$ and often the dimensions are equal. It is also known (see [12]) that

$$\dim_e \mu = \sup\Big\{ t : \sup_N \sum_{\|n\| \le N} |\widehat{\mu}(n)|^2 \|n\|^{t-d} < \infty \Big\}. \tag{5.3}$$

Combining this with (5.2), one immediately sees that

(1) An IFS, $\{\mathcal{S}_i\}$, is said to satisfy the *open set condition* if there is a non-empty, bounded, open set U such that $\bigcup \mathcal{S}_i(U) \subseteq U$ and the sets $\mathcal{S}_i(U)$ are disjoint. For example, the IFS $\{rx, rx + 1 - r\}$ satisfies the open set condition if and only if $r \le 1/2$.

Corollary 5.2. *If there is some $q < 2$ and constant C so that $\|\mu * f\|_2 \leq C\|f\|_q$ for all $f \in L^q(\mathbb{T}^d)$, then* $\dim_e \mu \geq d(2/q - 1)$.

Example 5.3. If μ is the uniform Cantor measure on the classical middle-third Cantor set, then the Hausdorff and energy dimensions are both $\log 2/\log 3$. Thus if $\mu : L^q \to L^2$, then $2/q \leq 1 + \log 2/\log 3$. This lower bound on q was also obtained by Oberlin [16], using other methods.

If μ is an equicontractive, self-affine measure, then so is μ^k for any positive integer k, where this notation means the k-fold convolution product of μ with itself. But even if the IFS associated with μ satisfies the open set condition, the IFS generating μ^k does not, in general, have this property.

However, all convolution powers of an L^p-improving measure are L^p-improving. In fact, if μ is L^p-improving, say $\mu : L^2 \to L^r$ is a bounded operator for some $r > 2$, an interpolation argument can be used to show that μ^k is a bounded operator from L^2 to L^{r_k} where $r_k = r^k/2^{k-1}$. Using this observation we can deduce the following facts about L^p-improving measures. The notation $\mathrm{Grp}(E)$ means the subgroup of $\mathbb{T}^d$ generated by E.

Corollary 5.4. *Suppose there is some $r > 2$ and constant C such that $\|\mu * f\|_r \leq C\|f\|_2$ for all $f \in L^2(\mathbb{T}^d)$. For each $k = 1, 2, \ldots$ let $p_k = 1 - 2^k/r^k$. Then for some constants C_k we have*

$$\frac{1}{N^{d(2/r)^k}} \sum_{\|n\| \leq N} |\widehat{\mu}(n)|^{2k} \leq C_k \quad \text{for all } N. \tag{5.4}$$

Thus $\dim_e \mu^k \geq dp_k$ and $\dim_e \mu^k \to d$ as $k \to \infty$. Furthermore, if μ is concentrated on $E \subseteq \mathbb{R}^d$, then $\dim_H \mathrm{Grp}(E) = d$.

Proof. Since μ^k maps L^2 to L^{r_k} where $r_k = r^k/2^{k-1}$ and $2/r_k' - 1 = p_k$, (5.4) and the statements about the energy dimension follow from Corollary 5.1 and (5.3). The final claim holds because if μ is concentrated on E, then μ^k is concentrated on the k-fold sum of E, and hence on the group generated by E. Thus $\dim_H \mathrm{Grp}(E) \geq \dim_e \mu^k \to d$ as $k \to \infty$. ∎

The fact that $\dim_e \mu^k \to 1$ for (non-trivial) equicontractive, self-similar measures on $[0, 1]$ was previously established in [8]. For further dimensional properties of L^p-improving measures we refer the reader to [11].

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