C O L L O Q U I U M M A T H E M A T I C U M

VOL. 139 2015 NO. 2

ASYMPTOTIC PERIOD IN DYNAMICAL SYSTEMS IN METRIC SPACES

BY

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Abstract. We introduce the notions of asymptotic period and asymptotically periodic orbits in metric spaces. We study some properties of these notions and their connections with ω-limit sets. We also discuss the notion of growth rate of such orbits and describe its properties in an extreme case.

1. Introduction. Topological properties of dynamical systems, particularly shapes and dynamics of orbits and limit sets are among important topics in qualitative theory of differential equations. We usually start from a fixed point or a periodic orbit and ask about the behaviour of the system in its vicinity. Many tools can be chosen for that research. However, we usually assume the presence of a fixed point, a periodic orbit or different kind of set, especially limit sets. A natural question is: how much can we learn about the system if we have limited information? For instance, we may not know whether there is any fixed point but there are other indicators or values that we can derive and that may imply other useful information.

In dynamics we can find various generalizations of periodicity in systems with both continuous and discrete time. For the continuous case one can find such a generalization in [1] where the notion of asymptotic period for pseudo-systems is introduced. However, that notion requires strong conditions. To avoid this problem and to start from less amount of information we introduce a new notion of asymptotic period and asymptotically periodic orbits. Several basic properties, such as the characterization of orbits with zero asymptotic period on compact metric space, will be established. We will also give examples of abstract flows with asymptotically non-periodic orbits and discuss the growth rate of such orbits. Our main contribution is to extend Theorem 1.1 of [2] using the notion of asymptotic periodicity.

We now introduce notation and basic definitions. Let (X, d) be a metric space.

2010 *Mathematics Subject Classification*: Primary 37C25; Secondary 37C15, 37C35.
Key words and phrases: dynamical systems, asymptotic periodicity, ω-limit set.

DOI: 10.4064/cm139-2-6

DEFINITION 1.1. A *dynamical system* (or *flow*) is a continuous function $\phi\colon \mathbb{R}\times X \to X$ such that $\phi(0,x) = x$ and for any x, s, and t, $\phi(t,(\phi(s,x))) = \phi(t+s,x)$.

DEFINITION 1.2. The *ω-limit set* of a point $x \in X$ is defined to be the set

$$\omega(x) = \{y \in X \mid \exists\ (t_n)_{n\in\mathbb{N}} \subset \mathbb{R} : \phi(t_n,x) \to y \wedge t_n \nearrow +\infty\}.$$

The *α-limit set* of x is

$$\alpha(x) = \{y \in X \mid \exists\ (t_n)_{n\in\mathbb{N}} \subset \mathbb{R} : \phi(t_n,x) \to y \wedge t_n \searrow -\infty\}.$$

DEFINITION 1.3. The *orbit* $o(x)$ of x is

$$o(x) = \{\phi(t,x) \mid t \in \mathbb{R}\}.$$

The *positive orbit* $o^+(x)$ of x is

$$o^+(x) = \{\phi(t,x) \mid t \geq 0\}.$$

The point x is *T-periodic* for some $T > 0$ if $\phi(T,x) = x$.

The value

$$\mathrm{Per}(x) := \inf\{T > 0 \mid \phi(T,x) = x\}$$

is said to be the *period* of x.

DEFINITION 1.4. Two flows $\phi\colon \mathbb{R}\times X \to X$ and $\psi\colon \mathbb{R}\times Y \to Y$ are *equivalent* if there exists a homeomorphism $h\colon X \to Y$ that sends each orbit of ϕ onto an orbit of ψ while preserving time orientation.

Let (X,d) be a metric space and let ϕ be a flow on X. Fix $x \in X$ and $\varepsilon > 0$, and define

$$A(x,\varepsilon) := \{t \geq 0 \mid d(\phi(t,x),x) > \varepsilon\}.$$

This set is the union of at most countably many pairwise disjoint and open intervals (q_i,r_i). The equality $r_i = +\infty$ for some i is allowed. Define

$$w_t := \begin{cases} 0, & t \notin A(x,\varepsilon), \\ \mathrm{diam}(q_i,r_i), & t \in (q_i,r_i). \end{cases}$$

The set $W := \{w_t\}_{t\geq 0}$ contains at most countably many different non-negative real numbers or $+\infty$. Note that if ε decreases, then the elements of W do not decrease.

Set

$$W(x,\varepsilon) := \limsup_{t\to+\infty} w_t.$$

We have $W(x,\varepsilon) = 0$ if $o^+(x) \subset \overline{B}(x,\varepsilon)$, and $W(x,\varepsilon) = +\infty$ if there exists an index i_0 such that $r_{i_0} = +\infty$.

DEFINITION 1.5. The *asymptotic period* of a point x is defined as

$$\mathrm{AP}(x) := \lim_{\varepsilon\to 0}\limsup_{t\to+\infty} W(\phi(t,x),\varepsilon).$$

This limit always exists, and it is either finite or infinite. If $\mathrm{AP}(x) = 0$, then x is called *asymptotically fixed*. If x has a finite asymptotic period, then it is called *asymptotically periodic*. If $\mathrm{AP}(x) = +\infty$, then x is called *asymptotically non-periodic*. The respective definitions can be formulated for the orbit of x.

The main advantage of the notion introduced above is that it is independent of the limit set: the value $\mathrm{AP}(x)$ is determined by the behaviour of the orbit of x.

REMARK 1.6. If x is T-periodic, then it is asymptotically periodic and $\mathrm{AP}(x) = T$. If x is a fixed point, then it is asymptotically fixed.

REMARK 1.7. Depending on the position of the ball $B(\phi(t,x),\varepsilon)$, the sets $A(\phi(t,x),\varepsilon)$ and W may change completely. As an example consider a planar dynamical system composed of a single periodic orbit of circle shape and an orbit starting from a fixed point that rolls onto the circle (so the circle is a limit set). The velocity of the motion is constant for the entire flow, except in some small neighbourhood of the fixed point. Figure 1 shows a part of this system. Notice that for $i = 1$ the set $A(\phi(t_i,x),\varepsilon)$ is a single open interval $(t_0,+\infty)$ for some time t_0. For $i = 2$ we get infinitely many time intervals for which the point x is outside the ball and the lengths of those intervals are similar.

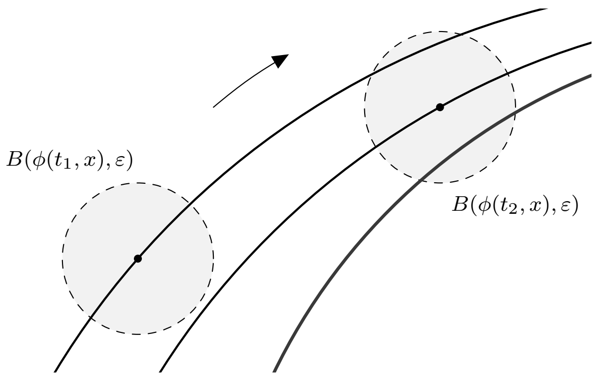

Fig. 1. A sketch of a close up to a system described in Remark 1.7. The arrow indicates the direction of motion.

2. Properties of asymptotic period. In this section we describe some properties of asymptotically periodic orbits. We investigate how the dynamics on an ω-limit set influences asymptotic periodicity.

LEMMA 2.1. *Assume that (X,d) is a compact metric space, ϕ is a flow on X and $x \in X$. Fix $\varepsilon > 0$. Then there exists $T > 0$ such that for each $t > T$ we have $d(\phi(t,x),\omega(x)) \leq \varepsilon$.*

Proof. Assume the contrary: for any $T > 0$ there is a time $t > T$ such that $d(\phi(t, x), \omega(x)) > \varepsilon$. Take $T_1 > 0$ and $t_1 > T_1$ satisfying this condition. Now choose $T_2 > t_1$ and $t_2 > T_2$ in the same way. By induction we can construct a sequence $(t_n)_{n\in\mathbb{N}}$ diverging to infinity such that for each $n \in \mathbb{N}$ we have

$$d(\phi(t_n, x), \omega(x)) > \varepsilon.$$

By compactness there is a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ and $x_s \in X$ such that $\phi(t_{n_k}, x) \to x_s$ and $d(x_s, \omega(x)) \geq \varepsilon$. By the definition x_s has to be an element of $\omega(x)$, which contradicts our assumption. ∎

THEOREM 2.2. *Assume that (X, d) is a compact metric space and ϕ is a flow on X. For a given point $x \in X$ the following equivalence holds:*

$$\mathrm{AP}(x) = 0 \;\Leftrightarrow\; \exists y \in X\ \omega(x) = \{y\}.$$

Proof. ($\Rightarrow$) Assume that $y, z \in \omega(x)$ and $y \neq z$. Let $\varepsilon < \frac{1}{5}d(y, z)$ and let t be such that $d(\phi(t, x), y) \leq \varepsilon$. Take all time shifts s such that $\phi(t+s, x) \in B(y, 2\varepsilon)$. Since $z \in \omega(x)$, there exist countably many times t_n and s_n such that $\phi(t + t_n, x) \in \partial B(y, 2\varepsilon)$ and $\phi(t + s_n, x) \in \partial B(z, \varepsilon)$. The sequence $(s_n)_{n\in\mathbb{N}}$ is such that $s_n > t_n$ and the times s_n are the smallest possible for which the above holds. Note that it follows from the continuity of ϕ that there is some α such that $s_n - t_n > \alpha > 0$.

Since $W(\phi(t, x), \varepsilon)$ is the upper limit of a sequence whose countably many terms are bounded from below by α (reaching a boundary of $B(y, 2\varepsilon)$ requires some positive time), the limit itself is bounded by α. One can take an arbitrarily large initial time t such that $\phi(t, x) \in B(y, 2\varepsilon)$. Using the same argument we obtain the same lower bound α (we took all possible time shifts between two balls around y and z). Thus we conclude that $\limsup_{t\to+\infty} W(\phi(t, x), \varepsilon) \geq \alpha$ and so $\mathrm{AP}(x) \geq \alpha$.

($\Leftarrow$) Assume that $\omega(x) = \{y\}$. Then from Lemma 2.1, for each $\varepsilon > 0$ we obtain $T > 0$ such that for all $t > T$ we have $\phi(x, t) \in B(\phi(T, x), \varepsilon)$. Hence $\mathrm{AP}(x) = 0$. ∎

THEOREM 2.3. *Assume that (X, d) is a compact metric space and ϕ is a flow on X. For a given point $x \in X$, if $\{y\} \subsetneq \omega(x)$ and y is a fixed point, then* $\mathrm{AP}(x) = +\infty$.

Proof. It is sufficient to show that the time needed for a point x to traverse an arbitrarily small neighbourhood of a fixed point y is unbounded.

Assume the contrary. Let K be an upper bound. Take any $z \in \omega(x) \setminus \{y\}$ and pick $\varepsilon > 0$ such that $d(y, z) > \varepsilon$. Take $(t_n)_{n\in\mathbb{N}}$ satisfying $\phi(t_n, x) \to y$ and $d(\phi(t_n, x), y) < \varepsilon$.

Let s_n be the smallest time such that $\phi(t_n + s_n, x) \notin B(y, \varepsilon)$. The sequence $(s_n)_{n\in\mathbb{N}}$ is bounded by K, hence we can pick a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ such

that $s_{n_k} \to s$ for some $s \le K$ and
$$\phi(t_{n_k} + s_{n_k}, x) \to w \notin B(y, \varepsilon).$$
On the other hand, by continuity we have
$$\phi(t_{n_k} + s_{n_k}, x) = \phi(s_{n_k}, \phi(t_{n_k}, x)) \to \phi(s, y) = y,$$
a contradiction. ∎

THEOREM 2.4. *Assume that ϕ is a flow on a compact metric space (X, d). Let $x \in X$ be such that $\omega(x)$ contains two periodic orbits. Then* $\mathrm{AP}(x) = +\infty$.

Proof. Assume the contrary. Let $\mathrm{AP}(x) = T > 0$ and let y and z be two periodic points of two different periodic orbits with positive distance between them. Choose $\varepsilon > 0$ such that $d(o(z), o(y)) > \frac{5}{2}\varepsilon$ and take δ such that $\limsup_{t\to+\infty} W(\phi(t, x), \varepsilon) < T + \delta$. We can choose a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\limsup_{n\to+\infty} W(\phi(t_n, x), \varepsilon) < T + \delta$. Then for sufficiently large $N > 0$ and $n > N$ we have $W(\phi(t_n, x), \varepsilon) < T + 2\delta$. Choose one such $\tilde{n} > N$ and denote $s_0 := t_{\tilde{n}}$. Without loss of generality (see Lemma 2.1) we can assume that $B(\phi(s_0, x), \varepsilon) \cap o(y) \neq \emptyset$. From now on, $\phi(s_0, x)$ is our point of reference.

Take $(s_n)_{n\ge1}$ and $(r_n)_{n\ge1}$ such that $\varphi(s_n, x) \in \partial B(\phi(s_0, x), \varepsilon)$ and $\phi(s_n + r_n, x) \to z$, and the orbit of $\phi(s_0, x)$ does not re-enter $B(\phi(s_0, x), \varepsilon)$ in the interval (s_n, r_n). Note that the times r_n are bounded by some elements from W, which implies that $\limsup_{n\to+\infty} r_n \le \limsup_{t\ge s_0} w_t$ and $\limsup_{t\ge s_0} w_t$ is finite. Choose a convergent subsequence $(r_{n_k})_{k\in\mathbb{N}}$ such that $r_{n_k} \to r$. Using compactness one can choose another subsequence $(s_{n_{k_m}})_{m\in\mathbb{N}}$ such that $\phi(s_{n_{k_m}}, x) \to x_s$ for some $x_s \in \partial B(\phi(s_0, x), \varepsilon)$. By continuity we have
$$\phi(s_{n_{k_m}} + r_{n_{k_m}}, x) \to z,$$
but on the other hand
$$\phi(s_{n_{k_m}} + r_{n_{k_m}}, x) = \phi(r_{n_{k_m}}, \phi(s_{n_{k_m}}, x)) \to \phi(r, x_s).$$
However, the distance between $o(z)$ and x_s is positive so $\phi(r, x_s)$ cannot be an element of $o(z)$. This contradiction completes the proof. ∎

The following example describes a situation where, although the dynamics of the ω-limit set can change dramatically, the asymptotic period remains unchanged. It also implies, along with previous results, that an infinite asymptotic period does not imply any specific dynamics of the limit set.

EXAMPLE 2.5. Let $\alpha \in [0, \pi/2)$ and denote $I = [0, 1]$. Let $T \subset \mathbb{R}^3$ be the torus $I^2/\!\sim$, where $\sim$ is the relation gluing opposite sides of the square with the orientation preserved.

To construct a flow ϕ on T, we first build the positive orbit of $(0, 0)$ by drawing a straight line from $(0, 0)$ to the top of the square at angle α to the bottom of the square. The past is defined in the same way. Depending on the rationality of α that orbit can be dense or periodic. In either case we fill T homogeneously with such orbits. We assume that any point on the torus moves along each orbit with a constant speed $\eta > 0$.

Choose a decreasing sequence $(\alpha_n)_{n\in\mathbb{N}\setminus\{0\}}$ of positive numbers converging to 0 such that $\tan(\alpha+\alpha_n) \notin \mathbb{Q}$ for each $n \in \mathbb{N}\setminus\{0\}$. Consider the family of tori T_k created from squares of side $1+1/k$ for $k = 1, 2, \ldots$. They are placed in $\mathbb{R}^3$ so that the following properties are satisfied:

- they have the same axis of rotation and there is one special plane orthogonal to this axis that intersects all tori in their internal and external equators,
- T_{k+1} is inside T_k for each $k \geq 1$,
- T is inside all T_k's (so it is the innermost torus).

We now construct the orbit of $x \in T_1$.

Step 1 (first covering). On T_1 we build a piece of $o(x)$ using the same method as for T. There are three additional restrictions: x need not be $(0,0)$, the angle we use is $\alpha+\alpha_1$ and the orbit runs until T_1 is covered with accuracy $\frac{1}{1}$, which means that for some t_1 we have

$$\forall x' \in T_1 \quad d(x', \phi([0,t_1],x)) \leq \frac{1}{1}.$$

The point x moves with constant velocity η. The past can be defined in any manner but so as not to disturb the dynamics on any tori T_i and T (for instance the α-limit set of x is a fixed point outside T_1). We do not fill T with more orbits.

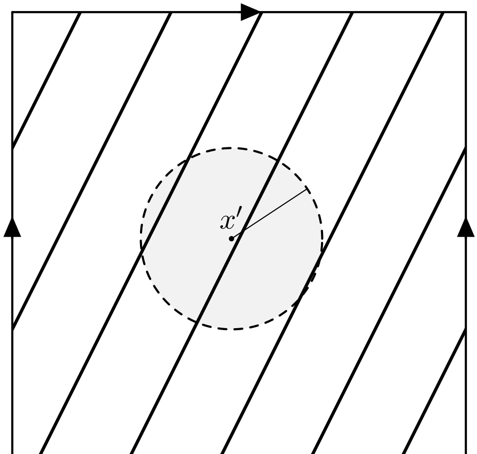

Fig. 2. Illustration of the covering with accuracy $\frac{1}{5}$

Step 2 (first jump). We pass from T_1 to T_2. If t_1 is as in Step 1, then we project $y_1 := \phi(x,t_1)$ orthogonally onto T_2. This gives us the point y_2' which now plays the role of x from Step 1. The points y_1 and y_2' are connected with a straight line (see Figure 3) where x continues its motion with the same velocity η. Starting from y_2' we cover T_2 with accuracy $\frac{1}{2}$ (and therefore producing t_2 and y_2) and with angle $\alpha + \alpha_2$. We maintain the velocity η.

Step 3 (induction). Having constructed one jump we can now proceed by induction and jump from T_k to T_{k+1}, change to an angle $\alpha + \alpha_{k+1}$ and cover T_{k+1} with accuracy $\frac{1}{k+1}$. All the time, x moves with the same velocity η.

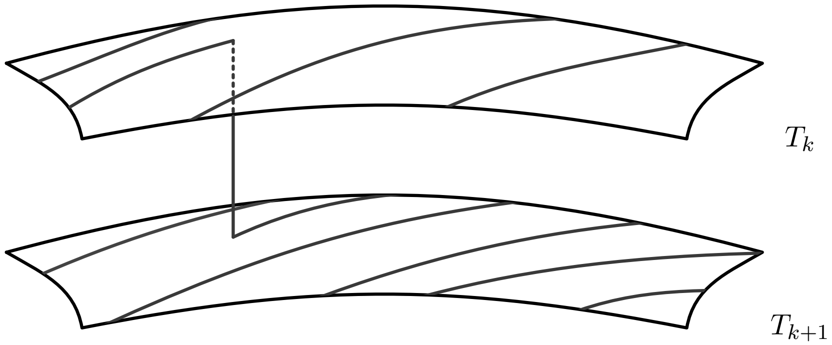

Fig. 3. Sketch of how an orbit switches between T_k and T_{k+1}. The bottom torus has a denser orbit and a different angle.

Our phase space is composed of T, the set $o(x)$ we have just constructed, and $\alpha(x)$. By construction, $\omega(x) = T$. Depending on the choice of α we obtain two different dynamics of the ω-limit set. Moreover, irrationality of angles on tori T_k surrounding T implies $\mathrm{AP}(x) = +\infty$. This can be derived from the following sketch of argument.

Consider the mapping $f : S^1 \to S^1$ given by $f(u) = u + \beta \mod 2\pi$, where $\beta \notin \mathbb{Q}$. We can find N such that the orbit $\{f^n(0)\}_{n\in\mathbb{N}}$ has the following property: the upper limit of the number of consecutive iterations for which the orbit of 0 is outside the ball $B(0, \varepsilon)$ is greater than or equal to N. Given N, consider the set $\{f^n(0) : n = 0, \dots, N\}$ and take $\varepsilon_N := \min\{d(0, f^n(0)) : n = 1, \dots, N\}$. Then we need at least N iterations of 0 to jump into $B(0, \varepsilon)$ provided $\varepsilon \le \varepsilon_N$. If we take the upper limit, the desired value will be obtained.

The above reasoning can now be translated to the case described in the example. Iterations become circuits (that is: one segment of the orbit from the bottom to the top of the corresponding square) along the orbit on each T_k and the first jump into the ball becomes the time when $o(x)$ enters the ball again. Now this gives the information that the smaller the ε we choose, the more time the orbit $o(x)$ needs to hit the ball $B(\phi(x, t), \varepsilon)$ for some initial $t > 0$, regardless of t.

Each periodic orbit is also asymptotically periodic, so it is natural that we should distinguish the two notions. We introduce the following notion.

Definition 2.6. A point $x \in X$ (and so its orbit) is called *essentially asymptotically periodic* if it is asymptotically periodic and not periodic.

This definition is well-motivated and explained in the next section, where we discuss the growth rate of asymptotically periodic orbits.

3. Growth rate of asymptotically periodic orbits. We introduce the definition of growth rate of asymptotically periodic orbits which extends the definition of growth rate of periodic orbits in [2]. Let us recall the latter. Let $\phi\colon \mathbb{R} \times X \to X$ be a flow on a metric space (X, d). Given $A \in \mathbb{R}^+$ we define the number of periodic orbits of period at most A by

$$\pi(\phi, A) := \max\bigl\{1, \#\{o(x) \subset X \mid \phi(x, a) = x \text{ for some } 0 < a \leq A\}\bigr\}.$$

If the set of orbits with period A is infinite, then we set $\pi(\phi, A) = +\infty$. Let

$$p(\phi, A) := \frac{1}{A} \log \pi(\phi, A), \qquad p(\phi) := \limsup_{A \to +\infty} p(\phi, A).$$

$p(\phi)$ is called the *growth rate of periodic orbits* for ϕ. Note that $p(\phi) \in [0, +\infty]$.

Now we introduce the growth rate for asymptotically periodic orbits.

Definition 3.1. Given $A \in \mathbb{R}^+$ we define the number of asymptotically periodic orbits of period at most A by

$$\pi_{\mathrm{AP}}(\phi, A) := \max\bigl\{1, \#\{o(x) \subset X \mid 0 < \mathrm{AP}(x) \leq A\}\bigr\}.$$

And similarly

$$p_{\mathrm{AP}}(\phi, A) := \frac{1}{A} \log \pi_{\mathrm{AP}}(\phi, A), \qquad p_{\mathrm{AP}}(\phi) := \limsup_{A \to +\infty} p_{\mathrm{AP}}(\phi, A).$$

$p_{\mathrm{AP}}(\phi)$ is called the *growth rate of asymptotically periodic orbits.*

In the same way we introduce the growth rate of essentially asymptotically periodic orbits.

Definition 3.2. Given $A \in \mathbb{R}^+$ we define the number of essentially asymptotically periodic orbits of period at most A by

$$\pi_{\mathrm{EAP}}(\phi, A) := \max\bigl\{1, \#(\{o(x) \subset X \mid 0 < \mathrm{AP}(x) \leq A\} \setminus P_{\phi,A})\bigr\},$$

where $P_{\phi,A} := \{o(x) \subset X \mid 0 < \mathrm{Per}(x) \leq A\}$. Similarly we introduce the *growth rate of essentially asymptotically periodic orbits*:

$$p_{\mathrm{EAP}}(\phi, A) := \frac{1}{A} \log \pi_{\mathrm{EAP}}(\phi, A), \qquad p_{\mathrm{EAP}}(\phi) := \limsup_{A \to +\infty} p_{\mathrm{EAP}}(\phi, A).$$

Motivation for both definitions seems very natural. We want to describe separately properties involving periodic and asymptotically periodic but not periodic orbits. We already noted in Remark 1.6 that in easy cases they coincide. We can also provide a much stronger result that uses both notions.

Theorem 1.1 from a recent work of Sun and Zhang [2] gives us an example of an extreme growth rate only for periodic orbits. In this section we will extend this result to both asymptotically periodic and essentially asymptotically periodic orbits. Because each periodic orbit is also asymptotically periodic (see Remark 1.6), we can rewrite [2, Theorem 1.1] for $p_{\mathrm{AP}}(\phi)$ in place of $p(\phi)$ and obtain the same result. Thus we will focus on the second kind of orbits.

Before we state and prove our main theorem (Theorem 3.8), let us describe separately two cases of extreme growth rate of specific orbits.

Remark 3.3. Consider the example in Remark 1.7. Assume that the velocity (which is constant) is equal to v (linear velocity) or ω (angular velocity). Denote by T_v and T_ω the respective periods of a circular orbit. Then for a given x belonging to an orbit rolling onto a circle we have $\mathrm{AP}(x) = T_v$ or $\mathrm{AP}(x) = T_\omega$.

Proposition 3.4 (see [2, Theorem 1.1]). *For any $a, b \in [0, +\infty]$, there exist compact metric spaces X and Y, and a pair of equivalent flows $\phi\colon \mathbb{R} \times X \to X$ and $\psi\colon \mathbb{R} \times Y \to Y$ with fixed points such that*

$$p(\phi) = a, \qquad p(\psi) = b.$$

Proof. This was already proven in [2] in the simple case $a = 0$ and $b = +\infty$. We give a different construction.

Assume that $\lambda \in (0, +\infty)$. We construct a flow (Z, ξ_λ) on a subset of $\mathbb{R}^2$ such that $p(\xi_\lambda) = \lambda$. For simplicity we write ξ instead of ξ_λ.

Note that

$$\frac{1}{A} \log 2^{\frac{\lambda}{\log 2} A} = \lambda.$$

If we define

$$\mu(A, \lambda) := \lfloor 2^{\frac{\lambda}{\log 2} A} \rfloor,$$

then

$$2^{\frac{\lambda}{\log 2} A} \leq \mu(A, \lambda) \leq 2^{\frac{\lambda}{\log 2} A} + 1.$$

Let $\mathcal{X} := \{X_n : x \in \mathbb{N}\}$, where

$$X_0 := \{(0,0)\}, \qquad X_n := \partial B((0,0), 1/n), \qquad n \geq 1,$$

and $B((a,b), r)$ denotes an open ball in $\mathbb{R}^2$ equipped with the Euclidean metric. Now we define the dynamics (and consequently the flow ξ) on the set $Z := \bigcup \mathcal{X}$. Suppose X_0 is a fixed point and X_n is a periodic orbit of the following period:

- a 1-periodic orbit for $n = 1, \dots, \mu(1, \lambda)$,
- a 2-periodic orbit for $n = \mu(1, \lambda) + 1, \dots, \mu(2, \lambda)$,
- a k-periodic orbit for $n = \mu(k-1, \lambda) + 1, \dots, \mu(k, \lambda)$.

In the case of $\mu(k, \lambda) = \mu(k+1, \lambda)$ we take no $k+1$-periodic orbits.

Here and later (in Proposition 3.6) we assume that the motion on each periodic orbit is with constant velocity. We also assume that the motion on all orbits (both periodic and later, with minor modifications, asymptotically periodic) is in the positive direction of the plane.

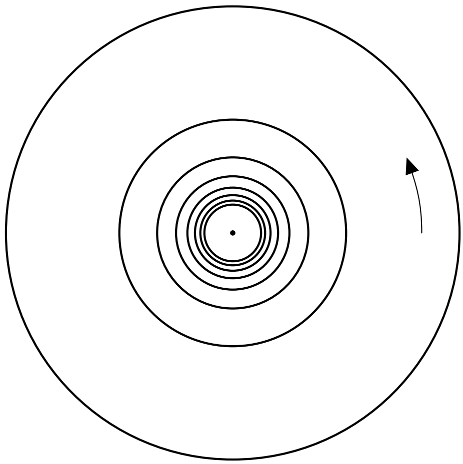

Fig. 4. Plot of a system in Proposition 3.4. A few orbits are drawn. The arrow indicates the direction of motion.

It follows directly from the construction that for the flow $\xi : \mathbb{R} \times Z \to Z$ we have

$$2^{\frac{\lambda}{\log 2} A} \leq \pi(\xi, A) \leq 2^{\frac{\lambda}{\log 2} A} + 1,$$

and so $p(\xi) = \lambda$.

Assume now that $\lambda = 0$. Then we define the dynamics on the circles as follows:

$$\text{For } n \in \mathbb{N} \setminus \{0\} \text{ the set } X_n \text{ is an } n\text{-periodic orbit,}$$

and X_0 remains a fixed point. In this case $\pi(\xi, A) = A$ and hence $p(\xi) = 0$.

Finally, assume that $\lambda = +\infty$. Similarly, X_0 is a fixed point and X_1 is a 1-periodic orbit. For $n \geq 2$ we introduce the following dynamics:

$$X_{2^{(k-1)^2}+1}, \dots, X_{2^{k^2}} \text{ are } k\text{-periodic orbits, for } k = 1, 2, \dots.$$

In this case $\pi(\xi, A) = 2^{A^2}$ for $A \geq 2$, so $p(\xi) = +\infty$.

Clearly, any two flows as above are equivalent: the identity is the desired homeomorphism. ∎

REMARK 3.5. For any $\lambda \in [0, +\infty]$ the flow (Z, ξ) constructed in Proposition 3.4 satisfies the equality $p_{\mathrm{EAP}}(\xi) = 0$.

PROPOSITION 3.6. *For any $c, d \in [0, +\infty]$, there exist compact metric spaces X and Y, and a pair of equivalent flows $\phi\colon \mathbb{R} \times X \to X$ and $\psi\colon \mathbb{R} \times Y \to Y$ with fixed points such that*

$$p_{\mathrm{EAP}}(\phi) = c, \qquad p_{\mathrm{EAP}}(\psi) = d.$$

Proof. Assume that $\lambda \in (0, +\infty)$. We will construct a flow (Z, ξ_λ) on a subset of $\mathbb{R}^2$ such that $p_{\mathrm{EAP}}(\xi_\lambda) = \lambda$. We use the idea from Proposition 3.4 and we also use the same notion of μ and the definition of X_n's. We additionaly set $X_1' := \partial B((0,0), 2)$ and $Z' := \bigcup \mathcal{X} \cup X_1'$. For simplicity we write ξ instead of ξ_λ.

We will define a flow ξ' on Z'. Suppose X_0 is a fixed point and X_1' is a 1-periodic orbit. Finally, assume X_n's are n-periodic orbits. Between X_{n-1} and X_n for $n \geq 2$, and between X_1 and X_1', we place essentially asymptotically periodic orbits Y_n^k using the following description:

- For $n = 1$ we place orbits Y_1^k, $k = 1, \ldots, \mu(1, \lambda)$, so that their α-limit set is X_1' and ω-limit set is X_1.

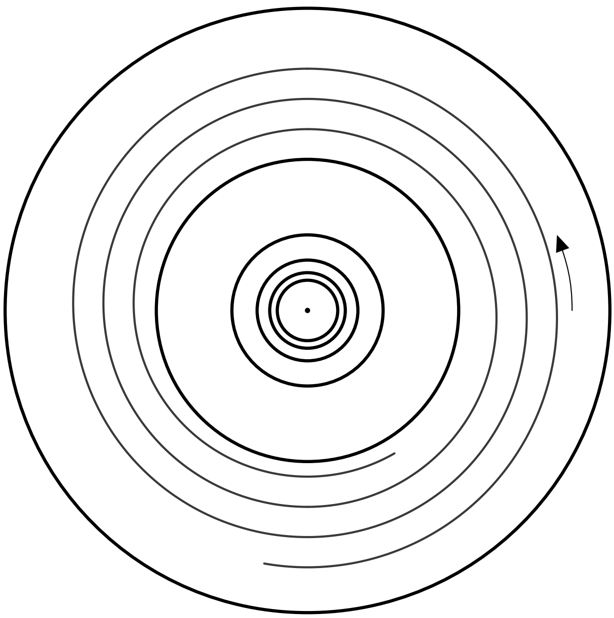

Fig. 5. Partial plot of a system in Proposition 3.6. A few periodic orbits and one asymptotically periodic orbit are drawn. The arrow indicates the direction of motion.

- For $n=2$ we place orbits Y_2^k, $k=\mu(1,\lambda)+1,\dots,\mu(2,\lambda)$, so that their α-limit set is X_1 and ω-limit set is X_2.
- For $n=\ell$ we place orbits Y_ℓ^k, $k=\mu(\ell-1,\lambda)+1,\dots,\mu(\ell,\lambda)$, so that their α-limit set is $X_{\ell-1}$ and ω-limit set is X_ℓ.

If $\mu(k,\lambda)=\mu(k+1,\lambda)$, then we take no such orbits.

Define

$$Z := Z' \cup \bigcup_{n=1}^{+\infty} \bigcup_{k=\mu(n-1,\lambda)+1}^{\mu(n,\lambda)} Y_n^k.$$

The motion of the entire flow is as follows. On periodic orbits we move with a constant velocity, in the positive direction of the plane. The motion of a point $x\in Y_i^j$ depends on how close it is from the closest circle. It decelerates close to $\omega(x)$ and then maintains the same velocity as the velocity on $\omega(x)$ (see Remark 1.7 where we described a similar case). Similarly we describe the motion close to $\alpha(x)$. This extends ξ' to ξ defined on the entire set Z.

It follows from our construction that for (Z,ξ) we obtain $p_{\mathrm{EAP}}(\xi)=\lambda$.

The cases $\lambda=0$ and $\lambda=+\infty$ are similar to the respective cases in Proposition 3.4, hence the details are omitted. ∎

REMARK 3.7. For any $\lambda\in[0,+\infty]$ the flow (Z,ξ) constructed in Proposition 3.6 satisfies $p(\xi)=0$.

MAIN THEOREM 3.8. *For any $a,b,c,d\in[0,+\infty]$, there exist compact metric spaces X and Y, and a pair of equivalent flows $\phi\colon\mathbb{R}\times X\to X$ and $\psi\colon\mathbb{R}\times Y\to Y$ with fixed points such that*

$$p(\phi)=a,\qquad p(\psi)=b,\qquad p_{\mathrm{EAP}}(\phi)=c,\qquad p_{\mathrm{EAP}}(\psi)=d.$$

Proof. Using Propositions 3.4 and 3.6 we construct two compact metric spaces C_1 and C_2, and two flows ϕ and ψ, such that

$$p(\phi|_{C_1})=a,\ p_{\mathrm{EAP}}(\phi|_{C_1})=0,\qquad p(\psi|_{C_1})=b,\ p_{\mathrm{EAP}}(\psi|_{C_1})=0,$$

and similarly

$$p(\phi|_{C_2})=0,\ p_{\mathrm{EAP}}(\phi|_{C_2})=c,\qquad p(\psi|_{C_2})=0,\ p_{\mathrm{EAP}}(\phi|_{C_2})=d.$$

Take $X=Y=C_1\sqcup C_2$. Then the flows (X,ϕ) and (Y,ψ) satisfy the desired conditions. ∎

Our construction also guarantees that we can provide a system of ordinary differential equations describing the flow. One can write them using polar coordinates on the plane.

4. Conclusion. The tools described in Section 2 can provide us some more information based on asymptotic behaviour of an orbit. As an example, Theorem 2.2 gives a sufficient condition for detecting a fixed point. The remaining part explains that the case $\mathrm{AP}(x) = +\infty$ cannot determine any specific behaviour of the limit set.

There are several more questions that one can ask. Is there a characterization for periodic orbits similar to the one for fixed points? Can any of these constructions be refined so as to obtain extreme properties of entropy?

Acknowledgements. The author would like to thank Marcin Kulczycki for his helpful remarks and ideas given while preparing this paper. The author would also like to thank Professor Włodzimierz Zwonek for pointing out language errors and the referee for his comments and suggestions to clarify and rephrase several things in the second section.

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Received 17 April 2014;
revised 28 September 2014 and 6 February 2015 (6240)