COLLOQUIUM MATHEMATICUM

VOL. 139 2015 NO. 2

TURÁN'S PROBLEM AND RAMSEY NUMBERS FOR TREES

BY

ZHI-HONG SUN (Huaian), LIN-LIN WANG (Xuzhou) and YI-LI WU (Huaian)

Abstract. Let $T_n^1 = (V, E_1)$ and $T_n^2 = (V, E_2)$ be the trees on n vertices with $V = \{v_0, v_1, \dots, v_{n-1}\}$, $E_1 = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-4}v_{n-2}, v_{n-3}v_{n-1}\}$ and $E_2 = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-3}v_{n-1}\}$. For $p \ge n \ge 5$ we obtain explicit formulas for $\mathrm{ex}(p; T_n^1)$ and $\mathrm{ex}(p; T_n^2)$, where $\mathrm{ex}(p; L)$ denotes the maximal number of edges in a graph of order p not containing L as a subgraph. Let $r(G_1, G_2)$ be the Ramsey number of the two graphs G_1 and G_2. We also obtain some explicit formulas for $r(T_m, T_n^i)$, where $i \in \{1, 2\}$ and T_m is a tree on m vertices with $\Delta(T_m) \le m - 3$.

1. Introduction. In this paper, all graphs are simple graphs. For a graph $G = (V(G), E(G))$ let $e(G) = |E(G)|$ be the number of edges in G and let $\Delta(G)$ be the maximal degree of G. For a forbidden graph L, let $\mathrm{ex}(p; L)$ denote the maximal number of edges in a graph of order p not containing any copies of L. The corresponding Turán problem is to evaluate $\mathrm{ex}(p; L)$. For a graph G of order p, if G does not contain any copies of L and $e(G) = \mathrm{ex}(p; L)$, we say that G is an *extremal graph*. In this paper we also use $\mathrm{Ex}(p; L)$ to denote the set of extremal graphs of order p not containing L as a subgraph.

Let $\mathbb{N}$ be the set of positive integers. Let $p, n \in \mathbb{N}$ with $p \ge n \ge 2$. For a given tree T_n on n vertices, it is difficult to determine the value of $\mathrm{ex}(p; T_n)$. The famous Erdős–Sós conjecture asserts that $\mathrm{ex}(p; T_n) \le \frac{(n-2)p}{2}$. For the progress on the Erdős–Sós conjecture, see for example [8, 11]. Write $p = k(n-1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Let P_n be the path on n vertices. In [4] Faudree and Schelp showed that

$$\mathrm{ex}(p; P_n) = k\binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}. \tag{1.1}$$

Let $K_{1,n-1}$ denote the unique tree on n vertices with $\Delta(K_{1,n-1}) = n-1$, and let T_n' denote the unique tree on n vertices with $\Delta(T_n') = n-2$. For $n \ge 4$ let $T_n^* = (V, E)$ be the tree on n vertices with $V = \{v_0, v_1, \dots, v_{n-1}\}$ and $E = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In [10] we determine $\mathrm{ex}(p; K_{1,n-1})$, $\mathrm{ex}(p; T_n')$ and $\mathrm{ex}(p; T_n^*)$. For $i = 1, 2$ let $T_n^i = (V, E_i)$ be the tree on n vertices

2010 *Mathematics Subject Classification*: 05C55, 05C35, 05C05.
Key words and phrases: Ramsey number, tree, Turán's problem.

DOI: 10.4064/cm139-2-8

with

$$\begin{aligned}V &= \{v_0, v_1, \dots, v_{n-1}\},\\ E_1 &= \{v_0v_1, \dots, v_0v_{n-3}, v_{n-4}v_{n-2}, v_{n-3}v_{n-1}\},\\ E_2 &= \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-3}v_{n-1}\}.\end{aligned}$$

In this paper, for $p \ge n \ge 5$ we obtain explicit formulas for $\mathrm{ex}(p; T_n^1)$ and $\mathrm{ex}(p; T_n^2)$ (see Theorems 2.1 and 3.1).

For a graph G, as usual $\overline{G}$ denotes the complement of G. Let G_1 and G_2 be two graphs. The *Ramsey number* $r(G_1, G_2)$ is the smallest positive integer p such that, for every graph G with p vertices, either G contains a copy of G_1 or else $\overline{G}$ contains a copy of G_2.

Let $n \in \mathbb{N}$, $n \ge 6$, and let T_n be a tree on n vertices. As mentioned in [7], recently Zhao proved the following conjecture of Burr and Erdős [2]: $r(T_n, T_n) \le 2n-2$. Let $m, n \in \mathbb{N}$. In 1973 Burr and Roberts [3] showed that for $m, n \ge 3$,

$$r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m+n-3 & \text{if } 2 \nmid mn,\\ m+n-2 & \text{if } 2 \mid mn.\end{cases} \tag{1.2}$$

In 1995, Guo and Volkmann [5] proved that for $n > m \ge 4$,

$$r(K_{1,m-1}, T_n') = \begin{cases} m+n-3 & \text{if } 2 \mid m(n-1),\\ m+n-4 & \text{if } 2 \nmid m(n-1).\end{cases} \tag{1.3}$$

Recently the first author evaluated the Ramsey number $r(T_m, T_n^*)$ for T_m in $\{P_m, K_{1,m-1}, T_m', T_m^*\}$. In particular, he proved that (see [9]) for $n > m \ge 7$,

$$r(K_{1,m-1}, T_n^*) = \begin{cases} m+n-3 & \text{if } m-1 \mid n-3,\\ m+n-4 & \text{if } m-1 \nmid n-3.\end{cases} \tag{1.4}$$

Suppose $m, n \in \mathbb{N}$ and $i, j \in \{1, 2\}$. In this paper, using the formula for $\mathrm{ex}(p; T_n^i)$ and the method in [9] we evaluate $r(T_m, T_n^i)$ for $T_m \in \{K_{1,m-1}, T_m', T_m^*, T_m^j\}$. In particular, we have the following typical results:

$$\begin{aligned}
r(T_n^i, T_n^j) &= 2n-6-(1-(-1)^n)/2, \quad r(P_n, T_n^j) = 2n-7 \quad \text{for } n \ge 17,\\
r(T_n^i, T_n') &= r(T_n^i, T_n^*) = 2n-5 \quad \text{for } n \ge 8,\\
r(K_{1,m-1}, T_n^i) &= m+n-4 \quad \text{for } n > m \ge 7,\ 2 \mid mn,\\
r(T_m^i, T_n^j) &= m+n-5 \quad \text{for } m \ge 7,\ n \ge (m-3)^2+3,\ m-1 \nmid n-4,\\
r(T_m', T_n^i) &= \begin{cases} m+n-4 & \text{if } m-1 \mid n-4,\\ m+n-6 & \text{if } n = m+1 \equiv 1 \pmod 2,\\ m+n-5 & \text{otherwise}\end{cases} \quad \text{for } n > m \ge 16.
\end{aligned}$$

In addition to the notation introduced above, throughout the paper we also use the following symbols: $[x]$ is the greatest integer not exceeding x, $d(v)$ is the degree of the vertex v in a graph, $\Gamma(v)$ is the set of vertices adjacent to the vertex v, $d(u,v)$ is the distance between the two vertices u and v in a graph, K_n is the complete graph on n vertices, $G[V_0]$ is the subgraph of G induced by vertices in the set V_0 (we write $G[v_1,\dots,v_m]$ instead of $G[\{v_1,\dots,v_m\}]$), $G-V_0$ is the subgraph of G obtained by deleting the vertices in V_0 and all edges incident to them, and finally $e(V_1V_1')$ is the number of edges with one endpoint in V_1 and another endpoint in V_1'.

2. Evaluation of $\mathrm{ex}(p;T_n^1)$

LEMMA 2.1. *Let $p,n\in\mathbb{N}$ with $p\ge n-1\ge 1$. Then* $\mathrm{ex}(p;K_{1,n-1}) = [(n-2)p/2]$.

This is a known result. See for example [10, Theorem 2.1].

LEMMA 2.2. *Let $p,n\in\mathbb{N}$, $p\ge n\ge 7$ and $G\in \mathrm{Ex}(p;T_n^1)$. Suppose that G is connected. Then $\Delta(G)=n-4$ and $e(G)=[(n-4)p/2]$.*

Proof. Since a graph not containing $K_{1,n-3}$ as a subgraph implies that the graph does not contain T_n^1 as a subgraph, by Lemma 2.1 we have

$$e(G)=\mathrm{ex}(p;T_n^1)\ge \mathrm{ex}(p;K_{1,n-3})=[(n-4)p/2]. \tag{2.1}$$

If $\Delta(G)\le n-5$, using Euler's theorem we see that $e(G)=\frac12\sum_{v\in V(G)}d(v)\le (n-5)p/2$, which together with (2.1) yields $((n-4)p-1)/2\le[(n-4)p/2]$ $\le e(G)\le (n-5)p/2$, which is impossible. Hence $\Delta(G)\ge n-4$. Now we show that $\Delta(G)=n-4$.

Suppose $q\ge n$ and $q=k(n-1)+r$ with $k\in\mathbb{N}$ and $r\in\{0,1,\dots,n-2\}$. Then clearly $kK_{n-1}\cup K_r$ does not contain any copies of T_n^1 and so $\mathrm{ex}(q;T_n^1)\ge e(kK_{n-1}\cup K_r)$. For $q=n$ we see that $e(kK_{n-1}\cup K_r)=e(K_{n-1}\cup K_1)=(n-1)(n-2)/2>2n-1$. For $q\ge n+1$ we have $(n-6)q\ge(n-6)(n+1)>\left(\frac{n-1}{2}\right)^2-2$ and so

$$\begin{aligned}e(kK_{n-1}\cup K_r)&=\frac{k(n-1)(n-2)}{2}+\frac{r(r-1)}{2}=\frac{(n-2)q-r(n-1-r)}{2}\\&\ge\frac{(n-2)q-\left(\frac{n-1}{2}\right)^2}{2}>2q-1.\end{aligned}$$

Hence

$$\mathrm{ex}(q;T_n^1)\ge e(kK_{n-1}\cup K_r)>2q-1\quad\text{ for } q\ge n. \tag{2.2}$$

Suppose $v_0\in V(G), d(v_0)=\Delta(G)=m$ and $\Gamma(v_0)=\{v_1,\dots,v_m\}$. If $m=p-1$, then since G does not contain T_n^1 as a subgraph, $G[v_1,\dots,v_m]$ does not contain $2K_2$ as a subgraph and hence $e(G[v_1,\dots,v_m])\le m-1$. Therefore

$$e(G)=d(v_0)+e(G[v_1,\dots,v_m])\le m+m-1=2p-3. \tag{2.3}$$

By (2.2), we have $e(G)=\mathrm{ex}(p;T_n^1)>2p-1$ and we get a contradiction. Hence $m<p-1$. Suppose that $u_1,\dots,u_t$ are all the vertices in G such

that $d(u_1,v_0)=\cdots=d(u_t,v_0)=2$. Then $t\ge 1$. Assume $u_1v_1\in E(G)$ with no loss of generality. If $m=p-2$, then $V(G)=\{v_0,v_1,\dots,v_m,u_1\}$ and $v_iv_j\notin E(G)$ for $2\le i<j\le m$. If $v_1v_i\in E(G)$ for some $i\in\{2,\dots,m\}$, then $u_1v_j\notin E(G)$ for all $j\ne 1,i$. So $\mathrm{ex}(p;T_n^1)=e(G)\le\max\{2m,m+3\}\le 2m=2p-4$, which contradicts (2.2).

By the above, $m<p-2$. We first assume $m\ge n-2$. As G does not contain any copies of T_n^1, we see that $\{v_2,\dots,v_m\}$ is an independent set, $u_iv_j\notin E(G)$ for any $i\in\{2,\dots,t\}$ and $j\in\{2,\dots,m\}$, and $u_iv_1\in E(G)$ for any $i=1,\dots,t$. Set $V_1=\{v_0,v_2,v_3,\dots,v_m\}$. Then $e(G[V_1])=m-1$. If u_1 is adjacent to at least two vertices in $\{v_2,\dots,v_m\}$, then $v_1v_j\notin E(G)$ for any $j=2,\dots,m$. If v_1 is adjacent to at least two vertices in $\{v_2,\dots,v_m\}$, then $u_1v_j\notin E(G)$ for any $j=2,\dots,m$. Hence there are at most m edges with one endpoint in V_1 and the other in $G-V_1$. Therefore,

$$e(G)\le e(G[V_1])+m+e(G-V_1)=2m-1+e(G-V_1). \tag{2.4}$$

For $m\in\{n-2,n-1\}$ let $G_1=K_m$. Then clearly $e(G_1)=m(m-1)/2>2m-1$. For $m=k(n-1)+r\ge n$ with $k\in\mathbb{N}$ and $0\le r\le n-2$ let $G_1=kK_{n-1}\cup K_r$. Then G_1 does not contain any copies of T_n^1 and $e(G_1)>2m-1$ by (2.2). Thus, by (2.4) we have $e(G)\le 2m-1+e(G-V_1)<e(G_1\cup(G-V_1))$ for $m\ge n-2$. This contradicts the fact that $G\in\mathrm{Ex}(p;T_n^1)$.

Suppose that $m=n-3$ and $d(v_1)=n-3$. Then $v_1v_s\notin E(G)$ for some $s\in\{2,\dots,n-3\}$. We claim that $V(G)=\{v_0,v_1,\dots,v_m,u_1,\dots,u_t\}$. Otherwise, there exists $w\in V(G)$ such that $d(v_0,w)=3$. As $d(v_1)=n-3$, we see that the subgraph induced by $\{v_1,v_s,w\}\cup\Gamma(v_1)$ contains a copy of T_n^1. This contradicts the assumption $G\in\mathrm{Ex}(p;T_n^1)$. Hence the claim is true and so $|V(G)|=p=n-2+t$. Since $p\ge n$ we have $t\ge 2$. For $i=1,\dots,t$ and $j=2,\dots,n-3$ we have $u_iv_j\notin E(G)$, $u_iv_1\in E(G)$ and so $t+1\le d(v_1)=n-3$. Therefore $2\le t\le n-4$ and hence

$$\begin{aligned}e(G)&=e(G[v_0,v_2,v_3,\dots,v_{n-3}])+d(v_1)+e(G[u_1,\dots,u_t])\\&\le\binom{n-3}{2}+n-3+\binom{t}{2}=\binom{n-2}{2}+\binom{t}{2}.\end{aligned}$$

Clearly $K_{n-1}\cup K_{t-1}$ does not contain T_n^1 and

$$\begin{aligned}e(K_{n-1}\cup K_{t-1})&=\binom{n-1}{2}+\binom{t-1}{2}\\&=\binom{n-2}{2}+\binom{t}{2}+n-1-t>e(G).\end{aligned}$$

This contradicts the assumption $G\in\mathrm{Ex}(n-2+t;T_n^1)$.

Now suppose $m=n-3$ and $d(v_1)\le n-4$. If $t=1$, setting $V_2=\{v_0,v_1,\dots,v_{n-3},u_1\}$ we see that

$$
\begin{aligned}
e(G) &= e(G[v_0, v_2, v_3, \ldots, v_{n-3}]) + d(v_1) + d(u_1) - 1 + e(G - V_2) \\
&\le \binom{n-3}{2} + n - 4 + n - 4 + e(G - V_2) = \frac{n^2 - 3n - 4}{2} + e(G - V_2) \\
&< e(K_{n-1} \cup (G - V_2)).
\end{aligned}
$$

This contradicts the assumption $G \in \mathrm{Ex}(p; T_n^1)$. Hence $t \ge 2$. For $i = 1, \ldots, t$ and $j = 2, \ldots, n-3$, we see that $u_i v_j \notin E(G)$ and $u_i v_1 \in E(G)$. Let $V_3 = \{v_0, v_1, \ldots, v_{n-3}\}$. Then

$$
\begin{aligned}
e(G) &= d(v_1) + e(G[v_0, v_2, v_3, \ldots, v_{n-3}]) + e(G - V_3) \\
&\le n - 4 + \binom{n-3}{2} + e(G - V_3) = \frac{n^2 - 5n + 4}{2} + e(G - V_3) \\
&< e(K_{n-2} \cup (G - V_3)).
\end{aligned}
$$

Since G is an extremal graph, we get a contradiction.

Summarizing all the above we obtain $\Delta(G) = n - 4$ and so $e(G) = \sum_{v \in V(G)} d(v) \le (n-4)p/2$. This together with (2.1) yields $e(G) = [(n-4)p/2]$, which completes the proof.

LEMMA 2.3. *Let $n, n_1, n_2 \in \mathbb{N}$ with $n_1 < n - 1$ and $n_2 < n - 1$. Then*

$$
\binom{n_1}{2} + \binom{n_2}{2} < \min\left\{\binom{n_1 + n_2}{2}, \binom{n-1}{2} + \binom{n_1 + n_2 - n + 1}{2}\right\}.
$$

Proof. It is clear that

$$
\binom{n_1}{2} + \binom{n_2}{2} = \frac{(n_1 + n_2)(n_1 + n_2 - 1) - 2n_1 n_2}{2} < \binom{n_1 + n_2}{2}
$$

and

$$
\begin{aligned}
\binom{n-1}{2} &+ \binom{n_1 + n_2 - n + 1}{2} - \binom{n_1}{2} - \binom{n_2}{2} \\
&= \frac{(n-1)(n-2) + (n_1 + n_2 - n + 1)(n_1 + n_2 - n)}{2} \\
&\quad - \frac{(n_1 + n_2)(n_1 + n_2 - 1) - 2n_1 n_2}{2} \\
&= (n - 1 - n_1)(n - 1 - n_2) > 0.
\end{aligned}
$$

Thus the lemma is proved.

LEMMA 2.4. *Suppose that $p \in \mathbb{N}$, $p \ge 6$, and G is a connected graph of order p that does not contain any copies of T_6^1. Then $e(G) \le 2p - 3$.*

Proof. Clearly $\Delta(T_6^1) = 3$. Suppose $v_0 \in V(G)$, $d(v_0) = \Delta(G) = m$ and $\Gamma(v_0) = \{v_1, \ldots, v_m\}$. If $\Delta(G) = m \le 3$, using Euler's theorem we see that $e(G) \le 3p/2 \le 2p - 3$. From now on we assume $\Delta(G) = m \ge 4$. If $d(v) \le 2$

for all $v \in V(G) - \{v_0\}$, then

$$e(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \le \frac{1}{2}(m + 2(p-1)) \le \frac{3(p-1)}{2} < 2p - 3.$$

So the result is true. Now we assume $d(v) \ge 3$ for some $v \in V(G) - \{v_0\}$. We may choose a vertex $u_0 \in V(G)$ so that $u_0 \ne v_0$, $d(u_0) \ge 3$ and $d(u_0, v_0)$ is as small as possible.

We first assume $d(u_0, v_0) = 1$ and $u_0 = v_1$ with no loss of generality. That is, $d(v_1) \ge 3$. Suppose $\Gamma(v_1) \subset \{v_0, v_1, \dots, v_m\}$. Since $d(v_1) \ge 3$ and G does not contain any copies of T_6^1, we see that $V(G) = \{v_0, v_1, \dots, v_m\}$, $m = p - 1 \ge 5$ and $G[v_1, \dots, v_m]$ does not contain any copies of $2K_2$. Thus $e(G) \le d(v_0) + m - 1 = 2m - 1 \le 2(m+1) - 3 = 2p - 3$. Now assume $\Gamma(v_1) - \{v_0, v_1, \dots, v_m\} = \{w_1, \dots, w_t\}$. Since $d(v_0) = m \ge 5$, $d(v_1) \ge 3$ and G does not contain any copies of T_6^1, we see that $V(G) = \{v_0, v_1, \dots, v_m, w_1, \dots, w_t\}$ and $\{v_2, \dots, v_m\}$ is an independent set. For $t \ge 2$, we have $e(G[w_1, \dots, w_t]) \le 1$, and $v_i w_j \notin E(G)$ for any $i \in \{2, \dots, m\}$ and $j \in \{1, \dots, t\}$. Therefore $e(G) \le d(v_0) + d(v_1) - 1 + 1 \le 2m < 2(m + 1 + t) - 3 = 2p - 3$. Now assume $t = 1$. Then $v_1 v_i \in E(G)$ for some $i \in \{2, \dots, m\}$ and $v_j w_1 \notin E(G)$ for $j \in \{2, \dots, m\} - \{i\}$. Hence $e(G) \le d(v_0) + d(v_1) - 1 + 1 \le 2m < 2(m+2) - 3 = 2p - 3$.

Next we assume $d(u_0, v_0) = 2$. Then $\{v_1, \dots, v_m\}$ is an independent set. If $\Gamma(u_0) \subseteq \{v_1, \dots, v_m\}$, then $V(G) = \{v_0, v_1, \dots, v_m, u_0\}$ and so $e(G) = d(v_0) + d(u_0) \le m + m < 2(m+2) - 3 = 2p - 3$. If $\Gamma(u_0) - \{v_2, \dots, v_m\} = \{v_1, w_1, \dots, w_t\}$, we see that $V(G) = \{v_0, v_1, \dots, v_m, u_0, w_1, \dots, w_t\}$ and so $e(G) = d(v_0) + d(u_0) + e(G[w_1, \dots, w_t]) \le m + m + 1 < 2(m+2+t) - 3 = 2p - 3$.

Finally we assume $d(u_0, v_0) \ge 3$. Suppose that $v_0 v_1 u_1 u_2 \dots u_k u_0$ is the shortest path in G between v_0 and u_0, and $\Gamma(u_0) = \{w_1, \dots, w_t, u_k\}$. Since G is connected and G does not contain any copies of T_6^1, it is easily seen that $V(G) = \{v_0, v_1, \dots, v_m, u_1, \dots, u_k, u_0, w_1, \dots, w_t\}$, $d(v_2) = \dots = d(v_m) = 1$, $d(v_1) = d(u_1) = \dots = d(u_k) = 2$ and $e(G[w_1, \dots, w_t]) \le 1$. Clearly, G is a tree or a graph obtained by adding an edge to a tree. Hence $e(G) \le p < 2p - 3$.

Summarizing all the above proves the lemma.

THEOREM 2.1. *Suppose $p, n \in \mathbb{N}$, $p \ge n - 1 \ge 4$ and $p = k(n-1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then*

$$\begin{aligned} \mathrm{ex}(p; T_n^1) &= \max\left\{ \left[\frac{(n-2)p}{2}\right] - (n-1+r), \frac{(n-2)p - r(n-1-r)}{2} \right\} \\ &= \begin{cases} \left[\frac{(n-2)p}{2}\right] - (n-1+r) & \text{if } n \ge 16 \text{ and } 3 \le r \le n-6, \text{ or if } \\ & \quad 13 \le n \le 15 \text{ and } 4 \le r \le n-7, \\ \frac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Clearly, $\mathrm{ex}(n-1;T_n^1) = e(K_{n-1}) = (n-2)(n-1)/2$. Thus the result is true for $p = n-1$. From now on we assume $p \ge n$. Since $T_5^1 \cong P_5$, by (1.1) we obtain the result in the case $n = 5$. Now we assume $n \ge 6$. Suppose $G \in \mathrm{Ex}(p;T_n^1)$ and $G_1, \ldots, G_t$ are all components of G with $|V(G_i)| = p_i$ and $p_1 \le \cdots \le p_t$. Then clearly $G_i \in \mathrm{Ex}(p_i;T_n^1)$ for $i = 1, \ldots, t$.

We first consider the case $n = 6$. If $p_i \le 5$, then clearly $G_i \cong K_{p_i}$ and $e(G_i) = \binom{p_i}{2}$. If $p_i \ge 6$ and $p_i = 5k_i + r_i$ with $k_i \in \mathbb{N}$ and $0 \le r_i \le 4$, from Lemma 2.4 we have $e(G_i) \le 2p_i - 3 \le 2p_i - r_i(5-r_i)/2 = e(k_iK_5 \cup K_{r_i})$. Since $k_iK_5 \cup K_{r_i}$ does not contain any copies of T_6^1, and $G_i \in \mathrm{Ex}(p_i;T_6^1)$, we see that $e(G_i) \ge e(k_iK_5 \cup K_{r_i})$ and so $e(G_i) = e(k_iK_5 \cup K_{r_i})$. Therefore, there is a graph $G' \in \mathrm{Ex}(p;T_6^1)$ such that $G' = a_1K_1 \cup a_2K_2 \cup a_3K_3 \cup a_4K_4 \cup a_5K_5$, where $a_1, \ldots, a_5$ are nonnegative integers. If $a_1 + a_2 + a_3 + a_4 \le 1$, then $\mathrm{ex}(p;T_6^1) = e(G') = e(a_5K_5 \cup K_r) = k\binom{5}{2} + \binom{r}{2}$. If $a_1 + a_2 + a_3 + a_4 > 1$, then $2a_1 + 3a_2 + 3a_3 + 2a_4 > 3 \ge r(5-r)/2$ and so

$$\begin{aligned} e(a_1K_1 \cup a_2K_2 \cup a_3K_3 \cup a_4K_4) &= a_2 + 3a_3 + 6a_4 \\ &< 2(a_1 + 2a_2 + 3a_3 + 4a_4) - \frac{r(5-r)}{2} = (k - a_5)\binom{5}{2} + \binom{r}{2}. \end{aligned}$$

Thus, $\mathrm{ex}(p;T_6^1) = e(G') = e(a_1K_1 \cup a_2K_2 \cup a_3K_3 \cup a_4K_4) + e(a_5K_5) < k\binom{5}{2} + \binom{r}{2}$. Since $kK_5 \cup K_r$ does not contain any copies of T_6^1, we get a contradiction. Thus $\mathrm{ex}(p;T_6^1) = e(kK_5 \cup K_r) = k\binom{5}{2} + \binom{r}{2} = 2p - r(5-r)/2$. This proves the result for $n = 6$.

From now on we assume $n \ge 7$. If $t = 1$, then G is connected. Thus, by Lemma 2.2 we have

$$e(G) = \left[\frac{(n-4)p}{2}\right] \quad \text{for } t = 1. \tag{2.5}$$

Now we assume $t \ge 2$. We claim that $p_i \ge n-1$ for $i \ge 2$. Otherwise, $p_1 \le p_2 < n-1$ and so $G_1 \cup G_2 \cong K_{p_1} \cup K_{p_2}$. If $p_1 + p_2 < n$, by Lemma 2.3 we have $e(G_1 \cup G_2) = e(K_{p_1} \cup K_{p_2}) = \binom{p_1}{2} + \binom{p_2}{2} < \binom{p_1+p_2}{2} = e(K_{p_1+p_2})$. Since $K_{p_1+p_2}$ does not contain T_n^1, and $G_1 \cup G_2 \in \mathrm{Ex}(p_1+p_2;T_n^1)$, we get a contradiction. Hence $p_1 + p_2 \ge n$. Using Lemma 2.3 again we see that

$$\begin{aligned} e(G_1 \cup G_2) = e(K_{p_1} \cup K_{p_2}) &= \binom{p_1}{2} + \binom{p_2}{2} \\ &< \binom{n-1}{2} + \binom{p_1+p_2-n+1}{2} = e(K_{n-1} \cup K_{p_1+p_2-n+1}). \end{aligned}$$

Since $p_1 \le p_2 < n-1$, we have $p_1 + p_2 - n + 1 < n - 1$. Therefore $K_{n-1} \cup K_{p_1+p_2-n+1}$ does not contain T_n^1. As $G_1 \cup G_2$ is an extremal graph without T_n^1, we also get a contradiction. Thus, the claim is true.

Next we claim that $p_i \le n-1$ for all $i = 1, \dots, t-1$. If $p_{t-1} \ge n$, by Lemma 2.2 we have

$$\begin{aligned} e(G_{t-1} \cup G_t) &= e(G_{t-1}) + e(G_t) \\ &= \left[\frac{(n-4)p_{t-1}}{2}\right] + \left[\frac{(n-4)p_t}{2}\right] \le \left[\frac{(n-4)(p_{t-1}+p_t)}{2}\right]. \end{aligned}$$

Let $H \in \mathrm{Ex}(p_{t-1}+p_t-n+1; K_{1,n-3})$. As $p_{t-1}+p_t-n+1 \ge p_t+1 \ge n+1$, we have $e(H) = [(n-4)(p_{t-1}+p_t-n+1)/2]$ by Lemma 2.1. Clearly $K_{n-1} \cup H$ does not contain any copies of T_n^1, and

$$\begin{aligned} e(K_{n-1} \cup H) = e(K_{n-1}) + e(H) &= \binom{n-1}{2} + \left[\frac{(n-4)(p_{t-1}+p_t-n+1)}{2}\right] \\ &= \left[\frac{(n-4)(p_{t-1}+p_t)}{2}\right] + n - 1 > e(G_{t-1} \cup G_t). \end{aligned}$$

Since $G_{t-1} \cup G_t \in \mathrm{Ex}(p_{t-1}+p_t; T_n^1)$, we get a contradiction. Hence $p_1 \le \cdots \le p_{t-1} \le n-1$. Combining this with the previous assertion that $p_t \ge \cdots \ge p_2 \ge n-1$ we obtain

$$p_1 \le n-1, \quad p_2 = \cdots = p_{t-1} = n-1 \quad \text{and} \quad p_t \ge n-1. \tag{2.6}$$

As G is an extremal graph, we must have

$$G_1 \cong K_{p_1}, \quad G_2 \cong K_{n-1}, \quad \dots, \quad G_{t-1} \cong K_{n-1}. \tag{2.7}$$

If $p_t = n-1$, then $G_t \cong K_{n-1}$. By (2.7), $G \cong K_{p_1} \cup (t-1)K_{n-1} \cong kK_{n-1} \cup K_r$. Thus,

$$e(G) = k\binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2} \quad \text{for } t \ge 2 \text{ and } p_t = n-1. \tag{2.8}$$

Now we assume that $p_t \ge n$. By Lemma 2.2, $e(G_t) = [(n-4)p_t/2]$. Since $p_1 \le n-1$, we have $G_1 \cong K_{p_1}$ and so $e(G_1) = e(K_{p_1}) = \binom{p_1}{2}$. Let $H_1 \in \mathrm{Ex}(p_1+p_t; K_{1,n-3})$. Then H_1 does not contain T_n^1 as a subgraph. By Lemma 2.1, for $p_1 \le n-4$ we have

$$\begin{aligned} e(H_1) = \left[\frac{(n-4)(p_1+p_t)}{2}\right] &\ge \left[\frac{(n-4)p_t}{2}\right] + \left[\frac{(n-4)p_1}{2}\right] \\ &\ge \left[\frac{(n-4)p_t}{2}\right] + \frac{(n-4)(p_1-1)}{2} + 1 \\ &> \left[\frac{(n-4)p_t}{2}\right] + \frac{p_1(p_1-1)}{2} = e(G_1 \cup G_t). \end{aligned}$$

This contradicts $G_1 \cup G_t \in \mathrm{Ex}(p_1+p_t; T_n^1)$. Hence $n-3 \le p_1 \le n-1$.

For $p_1 \in \{n-3, n-2\}$ and $p_t \ge n$, we have $p_1(p_1-(n-3)) \le 2n-4$ and so

$$\begin{aligned}
e(G_1 \cup G_t) &= e(G_1) + e(G_t) = \binom{p_1}{2} + \left[\frac{(n-4)p_t}{2}\right] \\
&\le \frac{p_1(p_1-1)+(n-4)p_t}{2} = \frac{p_1(p_1-(n-3))+(n-4)(p_1+p_t)}{2} \\
&\le \frac{2n-4+(n-4)(p_1+p_t)}{2} = \binom{n-1}{2} + \frac{(n-4)(p_1+p_t-n+1)-2}{2} \\
&< \binom{n-1}{2} + \left[\frac{(n-4)(p_1+p_t-n+1)}{2}\right].
\end{aligned}$$

Let $H_2 \in \mathrm{Ex}(p_1+p_t-n+1; K_{1,n-3})$. Then $K_{n-1} \cup H_2$ does not contain any copies of T_n^1. Since $p_1+p_t-n+1 \ge p_1+1 \ge n-2$, applying Lemma 2.1 we have $e(H_2) = [(n-4)(p_1+p_t-n+1)/2]$. Thus, we have $e(K_{n-1} \cup H_2) = \binom{n-1}{2} + [(n-4)(p_1+p_t-n+1)/2] > e(G_1 \cup G_t)$. This contradicts $G_1 \cup G_t \in \mathrm{Ex}(p_1+p_t; T_n^1)$.

By the above, for $t \ge 2$ and $p_t \ge n$ we have $p_1 = \cdots = p_{t-1} = n-1$. If $p_t \ge 2n-2$, setting $H_3 \in \mathrm{Ex}(p_t-(n-1); K_{1,n-3})$ and then applying Lemmas 2.1 and 2.2 we find that

$$e(G_t) = \left[\frac{(n-4)p_t}{2}\right] < \binom{n-1}{2} + \left[\frac{(n-4)(p_t-(n-1))}{2}\right] = e(K_{n-1} \cup H_3).$$

This contradicts the fact that $G_t \in \mathrm{Ex}(p_t; T_n^1)$. Hence $n \le p_t < 2n-2$ and so $r \ge 1$. Note that $p = k(n-1)+r = (k-1)(n-1)+n-1+r$ and $n \le n-1+r < 2n-2$. Hence $t=k$, $p_t = n-1+r$ and therefore

$$\begin{aligned}
(2.9) \qquad e(G) &= e((k-1)K_{n-1}) + e(G_t) \\
&= (k-1)\binom{n-1}{2} + \left[\frac{(n-4)(n-1+r)}{2}\right] \\
&= \left[\frac{(n-2)p}{2}\right] - (n-1+r) \qquad \text{for } t \ge 2 \text{ and } p_t \ge n.
\end{aligned}$$

Since $G \in \mathrm{Ex}(p; T_n^1)$, by comparing (2.5), (2.8) and (2.9) we get

$$e(G) = \max\left\{\left[\frac{(n-4)p}{2}\right], \frac{(n-2)p-r(n-1-r)}{2}, \left[\frac{(n-2)p}{2}\right] - (n-1+r)\right\}.$$

Observe that $p = k(n-1)+r \ge n-1+r$. We see that $[(n-4)p/2] = [(n-2)p/2] - p \le [(n-2)p/2] - (n-1+r)$ and therefore

$$\begin{aligned}
(2.10) \qquad & \mathrm{ex}(p; T_n^1) \\
&= e(G) = \max\left\{\frac{(n-2)p-r(n-1-r)}{2}, \left[\frac{(n-2)p}{2}\right] - (n-1+r)\right\} \\
&= \frac{(n-2)p-r(n-1-r)}{2} + \max\left\{0, \left[\frac{r(n-3-r)-2(n-1)}{2}\right]\right\}.
\end{aligned}$$

For $7 \le n \le 12$ we have $r(n-3-r)-2(n-1) \le (n-3)^2/4-2(n-1) = ((n-7)^2-32)/4 < 0$. For $r \in \{0,1,2,n-5,n-4,n-3,n-2\}$ we see that $r(n-3-r)-2(n-1) < 0$. Suppose $n \ge 13$ and $3 \le r \le n-6$. For $4 \le r \le n-7$ we have $|r-(n-3)/2| \le (n-11)/2$ and so

$$\begin{aligned} r(n-3-r)-2(n-1) &= \frac{n^2-14n+17}{4} - \Big(r-\frac{n-3}{2}\Big)^2 \\ &\ge \frac{n^2-14n+17}{4} - \Big(\frac{n-11}{2}\Big)^2 = 2n-26 \ge 0. \end{aligned}$$

For $r \in \{3, n-6\}$ we have $r(n-3-r)-2(n-1) = 3(n-6)-2(n-1) = n-16$. Now combining the above with (2.10) we deduce the result.

COROLLARY 2.1. *Suppose $p, n \in \mathbb{N}$, $p \ge n \ge 5$ and $n-1 \nmid p$. Then*

$$\frac{(n-2)p}{2} - \frac{(n-1)^2}{8} \le \mathrm{ex}(p; T_n^1) \le \frac{(n-2)(p-1)}{2}.$$

Proof. Suppose $p = k(n-1)+r$ with $k \in \mathbb{N}$ and $r \in \{0,1,\dots,n-2\}$. Then $r \ge 1$. Clearly

$$\begin{aligned} \frac{(n-1)^2}{4} &\ge r(n-1-r) = \Big(\frac{n-1}{2}\Big)^2 - \Big(\frac{n-1}{2}-r\Big)^2 \\ &\ge \Big(\frac{n-1}{2}\Big)^2 - \Big(\frac{n-1}{2}-1\Big)^2 = n-2 \end{aligned}$$

and $n-1+r > (n-2)/2$. Thus, from Theorem 2.1 we deduce that

$$\mathrm{ex}(p; T_n^1) \le \frac{(n-2)p-(n-2)}{2}$$

and

$$\mathrm{ex}(p; T_n^1) \ge \frac{(n-2)p - r(n-1-r)}{2} \ge \frac{(n-2)p-(n-1)^2/4}{2}.$$

This proves the corollary.

3. Evaluation of $\mathrm{ex}(p; T_n^2)$

LEMMA 3.1. *Let $p, n \in \mathbb{N}$, $p \ge n \ge 7$ and $G \in \mathrm{Ex}(p; T_n^2)$. Suppose that G is connected. Then $\Delta(G) \le n-3$. Moreover, for $p < 2n-2$ we have $\Delta(G) \le n-4$.*

Proof. Since a graph not containing $K_{1,n-3}$ implies that the graph does not contain T_n^2, by Lemma 2.1 we have

$$e(G) = \mathrm{ex}(p; T_n^2) \ge \mathrm{ex}(p; K_{1,n-3}) = \Big[\frac{(n-4)p}{2}\Big]. \tag{3.1}$$

Suppose that $v_0 \in V(G)$, $d(v_0) = \Delta(G) = m$ and $\Gamma(v_0) = \{v_1, \dots, v_m\}$. If $V(G) = \{v_0, v_1, \dots, v_m\}$, then $m = p-1 \ge n-1$. Since G does not contain T_n^2, we see that $G[v_1,\dots,v_m]$ does not contain $K_{1,2}$ and hence $e(G[v_1,\dots,v_m]) \le m/2$. Therefore $e(G) = d(v_0) + e(G[v_1,\dots,v_m]) \le m + m/2 = 3(p-1)/2 \le ((n-4)p-3)/2 < [(n-4)p/2]$. This contradicts (3.1). Thus $p > m+1$. Suppose that $u_1,\dots,u_t$ are all vertices such that $d(u_1,v_0) = \dots = d(u_t,v_0) = 2$. Then $t \ge 1$. We may assume without loss of generality that $v_1,\dots,v_s$ are all vertices in $\Gamma(v_0)$ adjacent to some vertex in the set $\{u_1,\dots,u_t\}$. Then $1 \le s \le m$. Let $V_1 = \{v_0,v_1,\dots,v_m\}$, $V_1' = V(G) - V_1$ and let $e(V_1V_1')$ be the number of edges with one endpoint in V_1 and the other in V_1'. Since G does not contain T_n^2, for $m \ge n-3$ each v_i $(1 \le i \le s)$ has one and only one adjacent vertex in the set $\{u_1,\dots,u_t\}$. Thus, for $m \ge n-3$ we must have $e(V_1V_1') = s \ge t$.

If $m \ge n-1$, since G does not contain T_n^2 as a subgraph, we see that $d(v_i) \le 2$ for $i = 1,\dots,m$ and so $e(G[V_1]) = d(v_0) + e(G[v_{s+1},\dots,v_m]) \le m + (m-s)/2$. Hence

$$\begin{aligned} e(G) &= e(G[V_1]) + e(V_1V_1') + e(G-V_1) \\ &\le \frac{3m-s}{2} + s + e(G-V_1) \le 2m + e(G-V_1). \end{aligned}$$

Suppose $m+1 = k(n-1)+r$ with $k \in \mathbb{N}$ and $0 \le r \le n-2$. Set $G_1 = kK_{n-1} \cup K_r$. Since $m+1 \ge n$, by (2.2) we have $e(G_1) > 2(m+1)-1 > 2m$. Thus, $e(G_1 \cup (G-V_1)) = e(G_1) + e(G-V_1) > 2m + e(G-V_1) \ge e(G)$. As G_1 does not contain any copies of T_n^2 and G is an extremal graph, we get a contradiction. Hence $\Delta(G) = m \le n-2$.

Suppose $m = n-2$. As G does not contain T_n^2 as a subgraph, we see that $d(v_1) = \dots = d(v_s) = 2$ and so $e(G[V_1]) \le n-2+\binom{n-2-s}{2}$. Since $1 \le s \le m = n-2 \le 2n-8$, we have

$$\begin{aligned} e(G) &= e(G[V_1]) + e(V_1V_1') + e(G-V_1) \\ &\le \binom{n-2-s}{2} + n-2+s+e(G-V_1) \\ &= \frac{(n-2)(n-1)-s(2n-7-s)}{2} + e(G-V_1) \\ &< \binom{n-1}{2} + e(G-V_1) = e(K_{n-1} \cup (G-V_1)). \end{aligned}$$

This is impossible since G is an extremal graph.

By the above, $\Delta(G) \le n-3$. We first assume $\Delta(G) = n-3$. We claim that $d(v_i) \le n-4$ for $i = 1,\dots,s$. If $i \in \{1,\dots,s\}$ and $d(v_i) = n-3$, let u_j be the unique adjacent vertex of v_i in $\{u_1,\dots,u_t\}$ and let $V_2 = \{v_0,v_1,\dots,v_{n-3},u_j\}$. Then there is at most one vertex adjacent to u_j in $G-V_2$. Hence $e(G-V_1) \le 1 + e(G-V_2)$. Since each v_r $(1 \le r \le s)$ is adjacent to one and only one vertex in $\{u_1,\dots,u_t\}$ and $\Delta(G[V_1]) \le n-3$, we see that

$$\begin{aligned} e(G[V_1]) &= \frac{1}{2}\sum_{r=0}^{n-3} d_{G[V_1]}(v_r) \\ &\le \frac{s(n-4)+(n-2-s)(n-3)}{2} = \frac{(n-2)(n-3)-s}{2}. \end{aligned}$$

Note that $s \le \Delta(G) = n-3$. From the above we deduce that

$$\begin{aligned} e(G) &= e(G[V_1])+e(V_1V_1')+e(G-V_1) = e(G[V_1])+s+e(G-V_1) \\ &\le e(G[V_1])+s+1+e(G-V_2) \le \frac{(n-2)(n-3)-s}{2}+s+1+e(G-V_2) \\ &= \frac{(n-2)(n-3)+s+2}{2}+e(G-V_2) \le \frac{(n-2)(n-3)+n-1}{2}+e(G-V_2) \\ &< \frac{(n-1)(n-2)}{2}+e(G-V_2) = e(K_{n-1}\cup(G-V_2)). \end{aligned}$$

Since $K_{n-1}\cup(G-V_2)$ does not contain T_n^2 and G is an extremal graph, we get a contradiction. Hence the claim is true. Thus, for $\Delta(G) = n-3$ we have $d_{G[V_1]}(v_i) \le n-5$ for $i = 1,\dots,s$ and so

$$\begin{aligned} e(G[V_1]) &= \frac{1}{2}\sum_{i=0}^{n-3} d_{G[V_1]}(v_i) \\ &\le \frac{s(n-5)+(n-2-s)(n-3)}{2} = \frac{(n-2)(n-3)}{2} - s. \end{aligned} \tag{3.2}$$

Now assume $p < 2n-2$ and $p = n-1+r$. Then $1 \le r < n-1$. By the above, $\Delta(G) \le n-3$. Assume $\Delta(G) = n-3$. Then $|V(G-V_1)| = p-(n-2) = r+1 < n$, $\Delta(G-V_1) \le n-3$ and so $e(G-V_1) \le \min\{\binom{r+1}{2}, (r+1)(n-3)/2\}$. Since $e(G[V_1]) \le (n-2)(n-3)/2 - s$ by (3.2), we deduce that

$$\begin{aligned} e(G) &= e(G[V_1]) + e(V_1V_1') + e(G-V_1) \\ &\le \frac{(n-2)(n-3)}{2} - s + s + \min\left\{\frac{r(r+1)}{2}, \frac{(r+1)(n-3)}{2}\right\} \\ &= \begin{cases} \dfrac{(n-2)(n-3)}{2} + \dbinom{r+1}{2} & \text{if } r \le n-3, \\ \dfrac{(n-2)(n-3)}{2} + \dfrac{(n-3)(n-1)}{2} & \text{if } r = n-2 \end{cases} \\ &< \binom{n-1}{2} + \binom{r}{2} = e(K_{n-1}\cup K_r). \end{aligned}$$

This is impossible since G is an extremal graph. Thus, $\Delta(G) \le n-4$ for $p < 2n-2$. Now the proof is complete.

LEMMA 3.2. *Let $p, n \in \mathbb{N}$, $p \ge n \ge 7$ and $G \in \mathrm{Ex}(p; T_n^2)$. Suppose that G is connected. Then $p < 2n-2$.*

Proof. By Lemma 3.1, we have $\Delta(G) \le n-3$ and so $e(G) \le \frac{(n-3)p}{2}$. Assume that $p = k(n-1)+r$ with $k \in \mathbb{N}$ and $r \in \{0,1,\dots,n-2\}$. Let $G_1 \in \mathrm{Ex}(n-1+r; K_{1,n-3})$. Then $e(G_1) = [(n-4)(n-1+r)/2]$ by Lemma 2.1. Hence, if $(k-2)(n-1)-r \ge 2$, then

$$\begin{aligned}
e((k-1)K_{n-1} \cup G_1) &= (k-1)\binom{n-1}{2} + \left[\frac{(n-4)(n-1+r)}{2}\right] \\
&= \frac{(p-r-(n-1))(n-2)}{2} + \left[\frac{(n-4)(n-1+r)}{2}\right] \\
&= \left[\frac{(n-3)p}{2} + \frac{p-2r-2(n-1)}{2}\right] \\
&= \left[\frac{(n-3)p}{2} + \frac{(k-2)(n-1)-r}{2}\right] > \left[\frac{(n-3)p}{2}\right] \ge e(G).
\end{aligned}$$

This is impossible since $(k-1)K_{n-1} \cup G_1$ does not contain T_n^2 as a subgraph and $G \in \mathrm{Ex}(p; T_n^2)$. Thus $(k-2)(n-1)-r \le 1$. If $k=3$, then $r = n-2$ and $p = 3(n-1)+n-2 = 4n-5$ and so

$$\begin{aligned}
e(G) &\le \left[\frac{(n-3)p}{2}\right] \le \frac{(n-3)(4n-5)}{2} = \frac{4n^2-17n+15}{2} \\
&< \frac{4n^2-14n+12}{2} = 3\binom{n-1}{2} + \binom{n-2}{2} = e(3K_{n-1} \cup K_{n-2}).
\end{aligned}$$

Since $3K_{n-1} \cup K_{n-2}$ does not contain T_n^2 and $G \in \mathrm{Ex}(p; T_n^2)$, we get a contradiction. Thus $k \le 2$.

For $p = 2(n-1)+r$ with $r \in \{0,1,2,n-4,n-3,n-2\}$ we see that $r(n-2-r) < 2n-2$ and so

$$e(2K_{n-1} \cup K_r) = \frac{2(n-1)(n-2)+r(r-1)}{2} > \frac{(n-3)(2n-2+r)}{2} \ge e(G).$$

This contradicts the assumption $G \in \mathrm{Ex}(p; T_n^2)$. Now suppose $p = 2(n-1)+r$ with $3 \le r \le n-5$. If $\Delta(G) \le n-4$, then $e(G) \le (n-4)p/2$. From the previous argument we have

$$\begin{aligned}
e(K_{n-1} \cup G_1) &= \binom{n-1}{2} + \left[\frac{(n-4)(n-1+r)}{2}\right] = \left[\frac{(n-3)p-r}{2}\right] \\
&= \left[\frac{(n-4)p}{2}\right] + n - 1 > \frac{(n-4)p}{2} \ge e(G).
\end{aligned}$$

Since $K_{n-1} \cup G_1$ does not contain T_n^2 as a subgraph and $G \in \mathrm{Ex}(p; T_n^2)$, we get a contradiction. Hence $\Delta(G) = n-3$. Suppose $v_0 \in V(G)$, $d(v_0) = n-3$, $\Gamma(v_0) = \{v_1,\dots,v_{n-3}\}$, $V_1 = \{v_0,v_1,\dots,v_{n-3}\}$ and $V_1' = V(G) - V_1$. Suppose also there are exactly s vertices in $\Gamma(v_0)$ adjacent to some vertex in V_1'. Then $1 \le s \le n-3$. By (3.2), $e(G[V_1]) \le (n-2)(n-3)/2 - s$.

As G does not contain any copies of T_n^2, we find that $e(V_1V_1') = s$. Since $|V(G-V_1)| = |V_1'| = p-(n-2) = n+r$ and $G-V_1$ does not contain any copies of T_n^2, we see that $e(G-V_1) \le \mathrm{ex}(n+r;T_n^2)$.

We claim that

$$\mathrm{ex}(n+r;T_n^2) \le \max\left\{\frac{(n-4)(n+r)}{2}, \frac{(n-1)(n-2)+r(r+1)}{2}\right\} \quad \text{for } 3 \le r \le n-5.$$

Let $G' \in \mathrm{Ex}(n+r;T_n^2)$. If G' is connected, using Lemma 3.1 we have $\Delta(G') \le n-4$ and so $e(G') \le (n-4)(n+r)/2$. Now suppose that G' is not connected. If $n_1, n_2 \in \{1,\dots,n-2\}$, from Lemma 2.3 we have $e(K_{n_1}\cup K_{n_2}) < e(K_{n_1+n_2})$ for $n_1+n_2 < n$, and $e(K_{n_1}\cup K_{n_2}) < e(K_{n-1}\cup K_{n_1+n_2-(n-1)})$ for $n_1+n_2 \ge n$. Thus, $G' = G_1'\cup G_2'$, where G_1' and G_2' are components of G' with $|V(G_1')| = p_1' < n-1$ and $|V(G_2')| = p_2' \ge n-1$. For $p_2' \ge n$ we have $p_1' \le r \le n-3$ and so $e(G_1') = p_1'(p_1'-1)/2 \le (n-4)p_1'/2$. For $p_2' \ge n$ we also have $\Delta(G_2') \le n-4$ and so $e(G_2') \le (n-4)p_2'/2$ by Lemma 3.1. Hence for $p_2' \ge n$ we find that $e(G') = e(G_1')+e(G_2') \le (n-4)p_1'/2+(n-4)p_2'/2 = (n-4)(n+r)/2$. Now assume $p_2' = n-1$. Then $p_1' = r+1$ and

$$e(G') = e(K_{n-1}\cup K_{r+1}) = \frac{(n-1)(n-2)+r(r+1)}{2}.$$

Hence the claim is true and so

$$\begin{aligned} e(G-V_1) &\le \mathrm{ex}(n+r;T_n^2) \\ &\le \max\left\{\frac{(n-4)(n+r)}{2}, \frac{(n-1)(n-2)+r(r+1)}{2}\right\}. \end{aligned}$$

Thus,

$$\begin{aligned} e(G) &= e(G[V_1])+e(V_1V_1')+e(G-V_1) \\ &\le \frac{(n-2)(n-3)}{2}-s+s+\max\left\{\frac{(n-4)(n+r)}{2}, \frac{(n-1)(n-2)+r(r+1)}{2}\right\} \\ &= \binom{n-1}{2} \\ &\quad +\max\left\{\frac{(n-4)(n-1+r)-n}{2}, \frac{(n-1)(n-2)+r(r-1)}{2}-(n-2-r)\right\} \\ &< \binom{n-1}{2}+\max\left\{\left[\frac{(n-4)(n-1+r)}{2}\right], \frac{(n-1)(n-2)+r(r-1)}{2}\right\} \\ &= \max\{e(K_{n-1}\cup G_1), e(2K_{n-1}\cup K_r)\}. \end{aligned}$$

This is impossible since G is an extremal graph.

By the above we must have $k=1$ and so $p = k(n-1)+r < 2n-2$ as asserted.

LEMMA 3.3. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 7$ and $G \in \mathrm{Ex}(p; T_n^2)$. Suppose that G is connected. Then $\Delta(G) = n-4$ and $e(G) = [(n-4)p/2]$.*

Proof. By (3.1), $e(G) \geq [(n-4)p/2]$. If $\Delta(G) \leq n-5$, using Euler's theorem we see that $e(G) = \frac{1}{2}\sum_{v\in V(G)} d(v) \leq (n-5)p/2$. Hence $((n-4)p-1)/2 \leq [(n-4)p/2] \leq e(G) \leq (n-5)p/2$. This is impossible. Thus $\Delta(G) \geq n-4$. By Lemmas 3.1 and 3.2, $\Delta(G) \leq n-4$. Therefore $\Delta(G) = n-4$ and so $e(G) = \frac{1}{2}\sum_{v\in V(G)} d(v) \leq (n-4)p/2$. Recall that $e(G) \geq [(n-4)p/2]$. Then $e(G) = [(n-4)p/2]$ as asserted.

LEMMA 3.4. *Let p and k be nonnegative integers, $p = 5k+r$ and $r \in \{0,1,2,3,4\}$. Suppose that G is a graph of order p without T_6^2. Then $e(G) \leq 2p - r(5-r)/2$.*

Proof. Clearly $\Delta(T_6^2) = 3$. We prove the lemma by induction on p. For $p \leq 5$ we have $e(G) \leq p(p-1)/2 = 2p - r(5-r)/2$. Now suppose that $p \geq 6$ and the lemma is true for all graphs of order $p_0 < p$ without T_6^2. If $\Delta(G) \leq 3$, then $e(G) = \frac{1}{2}\sum_{v\in V(G)} d(v) \leq 3p/2 \leq 2p-3 \leq 2p - r(5-r)/2$.

Suppose $\Delta(G) = m \geq 4$, $v_0 \in V(G)$, $d(v_0) = m$, $\Gamma(v_0) = \{v_1, \ldots, v_m\}$, $V_1 = \{v_0, v_1, \ldots, v_m\}$ and $V_1' = V(G) - V_1$. If $G[V_1]$ is a component of G, then $e(G[V_1]) = e(K_5) = 10$ for $m = 4$, and $e(G[V_1]) \leq m + m/2 = 3m/2$ for $m \geq 5$ since $d(v_i) \leq 2$ for $i = 1, \ldots, m$. By the inductive hypothesis, $e(G[V_1']) \leq 2(p-m-1) - r_1(5-r_1)/2$, where $r_1 \in \{0,1,2,3,4\}$ is given by $p - m - 1 \equiv r_1 \pmod 5$. Thus, for $m = 4$ we have

$$e(G) = e(G[V_1]) + e(G[V_1']) \leq 10 + 2(p-5) - \frac{r(5-r)}{2} = 2p - \frac{r(5-r)}{2},$$

and for $m \geq 5$ we have

$$\begin{aligned} e(G) = e(G[V_1]) + e(G[V_1']) &\leq \frac{3m}{2} + 2(p-m-1) - \frac{r_1(5-r_1)}{2} \\ &\leq 2p - 2 - \frac{m}{2} \leq 2p - 3 \leq 2p - \frac{r(5-r)}{2}. \end{aligned}$$

From now on we assume that $G[V_1]$ is not a component of G and $m = \Delta(G) \geq 4$. Hence there is a vertex u_1 such that $d(u_1, v_0) = 2$ and $u_1v_1 \in E(G)$ with no loss of generality. Then $v_1v_i \notin E(G)$ for $i = 2, \ldots, m$. For $m = 4$ we see that $e(G[V_1]) + e(V_1V_1') \leq 4 + 4 = 8$. For $m \geq 5$ we find that $d(v_i) \leq 2$ for $i = 1, \ldots, m$ and so $e(G[V_1]) + e(V_1V_1') \leq \sum_{i=1}^m d(v_i) \leq 2m$. Hence, for $m \geq 4$ we have $e(G) = e(G[V_1]) + e(V_1V_1') + e(G[V_1']) \leq 2m + e(G[V_1'])$. By the inductive hypothesis, $e(G[V_1']) \leq 2(p-m-1) - r_1(5-r_1)/2$, where $r_1 \in \{0,1,2,3,4\}$ is given by $p - m - 1 \equiv r_1 \pmod 5$. Thus, $e(G) \leq 2m + 2(p-m-1) - r_1(5-r_1)/2 = 2p - 2 - r_1(5-r_1)/2$. For $r_1 \geq 1$ we have $e(G) \leq 2p - 2 - 2 < 2p - r(5-r)/2$. For $r_1 = 0$ and $r = 0, 1, 4$ we have $e(G) \leq 2p - 2 \leq 2p - r(5-r)/2$. Therefore, we only need to consider the case $p \equiv m + 1 \equiv 2, 3 \pmod 5$.

Now assume $p \equiv m+1 \equiv 2,3 \pmod 5$ and $\Gamma(u_1)-\{v_1,\dots,v_m\}=\{w_1,\dots,w_t\}$. As $m\ge 4$ we have $m\ge 6$. Set $V_2=\{v_0,v_1,\dots,v_m,u_1\}$ and $V_2'=V(G)-V_2$. Since $d(v_i)\le 2$ for $i=1,\dots,m$, we see that

$$\begin{aligned}e(G)&=e(G[V_2])+e(V_2V_2')+e(G[V_2'])\\&\le \sum_{i=1}^m d(v_i)+t+e(G[V_2'])\le 2m+t+e(G[V_2']).\end{aligned}$$

Note that $p-m-2\equiv 4 \pmod 5$ and $e(G[V_2'])\le 2(p-m-2)-4(5-4)/2$ by the inductive hypothesis. We then have $e(G)\le 2m+t+2(p-m-2)-2=2p+t-6$. For $t\le 3$ we get $e(G)\le 2p+t-6\le 2p-3=2p-r(5-r)/2$. For $t\ge 4$ set $V_3=\{v_0,v_1,\dots,v_m,u_1,w_1,\dots,w_t\}$ and $V_3'=V(G)-V_3$. Since $d(v_i)\le 2$ for $i=1,\dots,m$ and $d(w_j)\le 2$ for $j=1,\dots,t$, using the inductive hypothesis we see that

$$\begin{aligned}e(G)&=e(G[V_3])+e(V_3V_3')+e(G[V_3'])\\&\le \sum_{i=1}^m d(v_i)+\sum_{j=1}^t d(w_j)+e(G[V_3'])\\&\le 2m+2t+e(G[V_3'])\le 2m+2t+2(p-m-2-t)=2p-4\\&<2p-\frac{r(5-r)}{2}.\end{aligned}$$

By the above, the lemma has been proved by induction.

THEOREM 3.1. *Let $p,n\in\mathbb{N}$, $p\ge n-1\ge 4$ and $p=k(n-1)+r$, where $k\in\mathbb{N}$ and $r\in\{0,1,\dots,n-2\}$. Then*

$$\begin{aligned}\mathrm{ex}(p;T_n^2)&=\max\left\{\left[\frac{(n-2)p}{2}\right]-(n-1+r),\ \frac{(n-2)p-r(n-1-r)}{2}\right\}\\&=\begin{cases}\left[\frac{(n-2)p}{2}\right]-(n-1+r) & \text{if } n\ge 16 \text{ and } 3\le r\le n-6, \text{ or if}\\ & \quad 13\le n\le 15 \text{ and } 4\le r\le n-7,\\ \frac{(n-2)p-r(n-1-r)}{2} & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. Clearly $\mathrm{ex}(n-1;T_n^2)=e(K_{n-1})=(n-2)(n-1)/2$. Thus the result is true for $p=n-1$. Now we assume $p\ge n$. Since $T_5^2\cong T_5'$, taking $n=5$ in [10, Theorem 3.1] we obtain the result in the case $n=5$. For $n=6$ we see that $\mathrm{ex}(p;T_6^2)\ge e(kK_5\cup K_r)=10k+r(r-1)/2=2p-r(5-r)/2$. This together with Lemma 3.4 gives the result in this case. Applying Lemmas 3.3, 2.3 and replacing T_n^1 with T_n^2 in the proof of Theorem 2.1 we deduce the result for $n\ge 7$.

COROLLARY 3.1. *Suppose $p, n \in \mathbb{N}$, $p \geq n \geq 5$ and $n-1 \nmid p$. Then*

$$\frac{(n-2)p}{2} - \frac{(n-1)^2}{8} \leq \mathrm{ex}(p; T_n^2) \leq \frac{(n-2)(p-1)}{2}.$$

4. The Ramsey number $r(T_n^i, T_n)$

LEMMA 4.1 ([9, Lemma 2.1]). *Let G_1 and G_2 be two graphs. Suppose $p \in \mathbb{N}$, $p \geq \max\{|V(G_1)|, |V(G_2)|\}$ and $\mathrm{ex}(p; G_1) + \mathrm{ex}(p; G_2) < \binom{p}{2}$. Then $r(G_1, G_2) \leq p$.*

Proof. Let G be a graph of order p. If $e(G) \leq \mathrm{ex}(p; G_1)$ and $e(\overline{G}) \leq \mathrm{ex}(p; G_2)$, then $\mathrm{ex}(p; G_1) + \mathrm{ex}(p; G_2) \geq e(G) + e(\overline{G}) = \binom{p}{2}$. This contradicts the assumption. Hence, either $e(G) > \mathrm{ex}(p; G_1)$ or $e(\overline{G}) > \mathrm{ex}(p; G_2)$. Therefore, G contains a copy of G_1 or $\overline{G}$ contains a copy of G_2. This shows that $r(G_1, G_2) \leq |V(G)| = p$. So the lemma is proved.

LEMMA 4.2 ([9, Lemma 2.3]). *Let G_1 and G_2 be two graphs with $\Delta(G_1) = d_1 \geq 2$ and $\Delta(G_2) = d_2 \geq 2$. Then*

(i) $r(G_1, G_2) \geq d_1 + d_2 - (1 - (-1)^{(d_1-1)(d_2-1)})/2$.
(ii) *Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \leq m$.*

Then $r(G_1, G_2) \geq 2d_2 - 1 \geq d_1 + d_2$.

(iii) *If G_1 is a connected graph of order m, $d_1 \neq m-1$ and $d_2 > m$, then $r(G_1, G_2) \geq d_1 + d_2$.*

THEOREM 4.1. *Let $n \in \mathbb{N}$ and $i, j \in \{1, 2\}$.*

(i) *If n is odd with $n \geq 17$, then $r(T_n^i, T_n^j) = 2n - 7$.*
(ii) *If n is even with $n \geq 12$, then $r(T_n^i, T_n^j) = 2n - 6$.*

Proof. Suppose $n \geq 12$. Since $\Delta(T_n^i) = \Delta(T_n^j) = n - 3$, from Lemma 4.2 we know that $r(T_n^i, T_n^j) \geq 2n - 7$ for n odd, and $r(T_n^i, T_n^j) \geq 2n - 6$ for n even. If n is odd with $n \geq 17$, using Theorems 2.1 and 3.1 (with $k = 1$ and $r = n - 6$) we see that

$$\begin{aligned}\mathrm{ex}(2n-7; T_n^i) &= \frac{(n-2)(2n-7)-1}{2} - (2n-7) \\ &< \frac{(n-4)(2n-7)}{2} = \frac{1}{2}\binom{2n-7}{2}\end{aligned}$$

and so $\mathrm{ex}(2n-7; T_n^i) + \mathrm{ex}(2n-7; T_n^j) < \binom{2n-7}{2}$. Thus, by Lemma 4.1 we have $r(T_n^i, T_n^j) \leq 2n - 7$. Hence (i) is true. From Theorems 2.1 and 3.1 (with

$k = 1$ and $r = n-5$) we see that for $n \geq 12$,

$$\begin{aligned} \mathrm{ex}(2n-6; T_n^i) &= \frac{(n-2)(2n-6)-4(n-5)}{2} = n^2 - 7n + 16 \\ &< n^2 - \frac{13}{2}n + \frac{21}{2} = \frac{1}{2}\binom{2n-6}{2} \end{aligned}$$

and so $\mathrm{ex}(2n-6; T_n^i) + \mathrm{ex}(2n-6; T_n^j) < \binom{2n-6}{2}$. Thus, by Lemma 4.1 we have $r(T_n^i, T_n^j) \leq 2n-6$. Hence $r(T_n^i, T_n^j) = 2n-6$ for n even, proving (ii).

LEMMA 4.3. *Let $n \in \mathbb{N}$, $n \geq 5$ and $i \in \{1, 2\}$. Let G_n be a connected graph of order n such that* $\mathrm{ex}(2n-5; G_n) < n^2 - 5n + 4$. *Then* $r(T_n^i, G_n) \leq 2n-5$.

Proof. By Theorems 2.1 and 3.1,

$$\mathrm{ex}(2n-5; T_n^i) = \frac{(n-2)(2n-5)-3(n-4)}{2} = n^2 - 6n + 11.$$

Thus,

$$\mathrm{ex}(2n-5; G_n) + \mathrm{ex}(2n-5; T_n^i) < n^2 - 5n + 4 + n^2 - 6n + 11 = \binom{2n-5}{2}.$$

Appealing to Lemma 4.1 we obtain $r(T_n^i, G_n) \leq 2n-5$.

LEMMA 4.4 ([10, Theorem 3.1]). *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 5$. Let $r \in \{0, 1, \ldots, n-2\}$ be given by $p \equiv r$* (mod $n-1$). *Then*

$$\mathrm{ex}(p; T_n') = \begin{cases} \left[\dfrac{(n-2)(p-1)-r-1}{2}\right] & \text{if } n \geq 7 \text{ and } 2 \leq r \leq n-4, \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

THEOREM 4.2. *Let $n \in \mathbb{N}$, $n \geq 8$ and $i \in \{1, 2\}$. Then* $r(T_n^i, T_n') = r(T_n^i, T_n^*) = 2n-5$.

Proof. Let $T_n \in \{T_n', T_n^*\}$. As $2K_{n-3}$ does not contain any copies of T_n^i, and $\overline{2K_{n-3}} = K_{n-3,n-3}$ does not contain any copies of T_n, we see that $r(T_n^i, T_n) \geq 1 + 2(n-3) = 2n-5$. Taking $p = 2n-5$ and $r = n-4$ in Lemma 4.4 we find that

$$\begin{aligned} \mathrm{ex}(2n-5; T_n') &= \left[\frac{(n-2)(2n-6)-(n-4)-1}{2}\right] \leq n^2 - \frac{11}{2}n + \frac{15}{2} \\ &< n^2 - 5n + 4. \end{aligned}$$

By [10, Theorem 4.1],

$$\mathrm{ex}(2n-5; T_n^*) = \frac{(n-2)(2n-5)-3(n-4)}{2} = n^2 - 6n + 11 < n^2 - 5n + 4.$$

Thus, applying Lemma 4.3 we obtain $r(T_n^i, T_n) \leq 2n-5$. Hence $r(T_n^i, T_n) = 2n-5$ as asserted.

Remark 4.1. Let $n \in \mathbb{N}$, $n \geq 5$ and $i \in \{1,2\}$. From [5, Theorem 3.1(ii)] we know that $r(K_{1,n-1}, T_n^i) = 2n-3$.

Theorem 4.3. *Let $n \in \mathbb{N}$ and $i \in \{1,2\}$. Then $r(P_n, T_n^i) = 2n-7$ for $n \geq 17$, $r(P_{n-1}, T_n^i) = 2n-7$ for $n \geq 13$, $r(P_{n-2}, T_n^i) = 2n-7$ for $n \geq 11$, and $r(P_{n-3}, T_n^i) = 2n-7$ for $n \geq 8$.*

Proof. Suppose $n \geq 8$ and $s \in \{0,1,2,3\}$. From Lemma 4.2(ii) we have $r(P_{n-s}, T_n^i) \geq 2(n-3)-1 = 2n-7$. By (1.1),

$$\begin{aligned}
&\mathrm{ex}(2n-7; P_{n-s}) \\
&\quad = \begin{cases}
\frac{(n-2)(2n-7)-5(n-6)}{2} = \frac{(n-4)(2n-7)+16-n}{2} & \text{if } s=0, \\
\frac{(n-3)(2n-7)-3(n-5)}{2} = \frac{(n-4)(2n-7)+8-n}{2} & \text{if } s=1, \\
\frac{(n-4)(2n-7)-(n-4)}{2} & \text{if } s=2, \\
\frac{(n-5)(2n-7)-(n-5)}{2} = \frac{(n-4)(2n-7)+12-3n}{2} & \text{if } s=3.
\end{cases}
\end{aligned}$$

By Theorems 2.1 and 3.1,

$$\begin{aligned}
&\mathrm{ex}(2n-7; T_n^i) \\
&\quad = \begin{cases}
\left[\frac{(n-4)(2n-7)}{2}\right] & \text{if } n \geq 16, \\
\frac{(n-2)(2n-7)-5(n-6)}{2} = \frac{(n-4)(2n-7)+16-n}{2} & \text{if } n < 16.
\end{cases}
\end{aligned}$$

For $n \geq 17, 13, 11$ or 8 according as $s = 0, 1, 2$ or 3, from the above we find $\mathrm{ex}(2n-7; P_{n-s}) + \mathrm{ex}(2n-7; T_n^i) < \binom{2n-7}{2}$ and so $r(P_{n-s}, T_n^i) \leq 2n-7$ by Lemma 4.1. This completes the proof.

5. The Ramsey number $r(T_m^i, T_n)$ for $m < n$

Proposition 5.1 (Burr [1]). *Let $m, n \in \mathbb{N}$ with $m \geq 3$ and $m-1 \mid n-2$. Let T_m be a tree on m vertices. Then $r(T_m, K_{1,n-1}) = m+n-2$.*

Proposition 5.2 (Guo and Volkmann [5, Theorem 3.1]). *Let $m, n \in \mathbb{N}$, $m \geq 3$ and $n = k(m-1)+b$ with $k \in \mathbb{N}$ and $b \in \{0,1,\dots,m-2\} \setminus \{2\}$. Let $T_m \neq K_{1,m-1}$ be a tree on m vertices. Then $r(T_m, K_{1,n-1}) \leq m+n-3$. Moreover, if $k \geq m-b$, then $r(T_m, K_{1,n-1}) = m+n-3$.*

Lemma 5.1 ([6, Theorem 8.3, pp. 11–12]). *Let $a, b, n \in \mathbb{N}$. If a is coprime to b and $n \geq (a-1)(b-1)$, then there are two nonnegative integers x and y such that $n = ax+by$.*

Theorem 5.1. *Let $m, n \in \mathbb{N}$, $n > m \geq 5$, $m-1 \nmid n-2$ and $i \in \{1,2\}$. Then $r(T_m^i, K_{1,n-1}) = m+n-3$ or $m+n-4$. Moreover, if $n \geq (m-3)^2+1$*

or $m+n-4=(m-1)x+(m-2)y$ *for some nonnegative integers* x *and* y, *then* $r(T_m, K_{1,n-1}) = m+n-3$ *for any tree* $T_m \neq K_{1,m-1}$ *of order* m.

Proof. Let $T_m \neq K_{1,m-1}$ be a tree on m vertices. From Proposition 5.2 we know that $r(T_m, K_{1,n-1}) \le m+n-3$. By Lemma 4.2(iii), $r(T_m^i, K_{1,n-1}) \ge m-3+n-1$. Thus, $r(T_m^i, K_{1,n-1}) = m+n-3$ or $m+n-4$. If $n \ge (m-3)^2+1$, then $m+n-4 \ge (m-2)(m-3)$ and so $m+n-4=(m-1)x+(m-2)y$ for some nonnegative integers x and y by Lemma 5.1. If $m+n-4=(m-1)x+(m-2)y$ for $x, y \in \{0,1,2,\ldots\}$, setting $G = xK_{m-1} \cup yK_{m-2}$ we see that G does not contain any copies of T_m, and $\overline{G}$ does not contain any copies of $K_{1,n-1}$. Thus $r(T_m, K_{1,n-1}) \ge 1+|V(G)| = m+n-3$. Now putting all the above together we obtain the theorem.

Theorem 5.2. *Let* $m,n \in \mathbb{N}$, $n>m\ge 6$, $m-1 \mid n-3$ *and* $i \in \{1,2\}$. *Then* $r(T_m^i, T_n') = m+n-3$.

Proof. By Theorems 2.1 and 3.1,

$$\mathrm{ex}(m+n-3; T_m^i) = \frac{(m-2)(m+n-3)-(m-2)}{2} < \frac{(m-2)(m+n-3)}{2}.$$

Thus applying [9, Theorem 5.1] we obtain the conclusion.

Theorem 5.3. *Suppose that* $i \in \{1,2\}$, $m,n \in \mathbb{N}$, $n>m\ge 7$ *and* $m-1 \nmid n-3$. *Then* $m+n-5 \le r(T_m^i, T_n') \le m+n-4$ *and* $m+n-6 \le r(T_m^i, T_n^*) \le m+n-4$. *Moreover, if* $n = k(m-1)+b = q(m-2)+a$, $k,q \in \mathbb{N}$, $a \in \{0,1,\ldots,m-3\}$, $b \in \{0,1,\ldots,m-2\}$, *and one of the following conditions holds:*

(1) $b \in \{1,2,4\}$,
(2) $b=0$ *and* $k \ge 3$,
(3) $n \ge (m-3)^2+2$,
(4) $n \ge m^2-1-b(m-2)$,
(5) $a \ge 3$ *and* $n \ge (a-4)(m-1)+4$,

then $r(T_m^i, T_n^*) = r(T_m^i, T_n') = m+n-4$.

Proof. By Lemma 4.2 we have $r(T_m^i, T_n') \ge m-3+n-2$ and $r(T_m^i, T_n^*) \ge m-3+n-3$. Since $m-1 \nmid n-3$, we see that $m-1 \nmid m+n-4$. From Corollaries 2.1 and 3.1 we have $\mathrm{ex}(m+n-4; T_m^i) \le (m-2)(m+n-5)/2$. Hence, by [9, Lemma 5.2] we find $r(T_m^i, T_n') \le m+n-4$, and by [9, Lemma 4.2] we have $r(T_m^i, T_n^*) \le m+n-4$. Now applying [9, Theorems 4.4 and 5.4] we deduce the remaining assertion.

6. The Ramsey number $r(G_m, T_n^j)$ for $m<n$

Theorem 6.1. *Let* $m,n \in \mathbb{N}$, $m \ge 5$, $n \ge 8$, $n>m$ *and* $j \in \{1,2\}$. *Then* $r(K_{1,m-1}, T_n^j) = m+n-4$ *or* $m+n-5$. *Moreover, if* $2 \mid mn$, *then* $r(K_{1,m-1}, T_n^j) = m+n-4$.

Proof. From Lemma 4.2 we deduce that $r(K_{1,m-1}, T_n^j) \geq m-1+n-3-(1-(-1)^{(m-2)(n-4)})/2 = m+n-4-(1-(-1)^{mn})/2$. So, it suffices to prove that $r(K_{1,m-1}, T_n^j) \leq m+n-4$. By Lemma 2.1, $\mathrm{ex}(m+n-4; K_{1,m-1}) = [(m-2)(m+n-4)/2]$. By Theorems 2.1 and 3.1, we have

$$\mathrm{ex}(m+n-4; T_n^j) = \left[\frac{(n-4)(m+n-4)}{2}\right] \quad \text{or} \quad \frac{(n-2)(m+n-4)-(m-3)(n-m+2)}{2}.$$

Since

$$\left[\frac{(m-2)(m+n-4)}{2}\right] + \left[\frac{(n-4)(m+n-4)}{2}\right] \leq \frac{(m+n-6)(m+n-4)}{2} < \binom{m+n-4}{2}$$

and

$$\frac{(m-2)(m+n-4)}{2} + \frac{(n-2)(m+n-4)-(m-3)(n-m+2)}{2} = \frac{(m+n-4)(m+n-5)-(m-4)\big(n-m-\frac{2}{m-4}\big)}{2} < \binom{m+n-4}{2},$$

we see that $\mathrm{ex}(m+n-4; K_{1,m-1}) + \mathrm{ex}(m+n-4; T_n^j) < \binom{m+n-4}{2}$ and so $r(K_{1,m-1}, T_n^j) \leq m+n-4$ by Lemma 4.1. This completes the proof.

Theorem 6.2. *Let $m, n \in \mathbb{N}$, $m \geq 4$, $n \geq 7$, $m-1 \mid n-4$ and $j \in \{1, 2\}$.*

(i) *If G_m is a connected graph of order m with $\mathrm{ex}(m+n-4; G_m) \leq (m-2)(m+n-5)/2$, then $r(G_m, T_n^j) = m+n-4$.*
(ii) *$r(T'_m, T_n^j) = r(T_m^1, T_n^j) = r(T_m^2, T_n^j) = m+n-4$ for $m \geq 5$, $r(T_m^*, T_n^j) = m+n-4$ for $m \geq 6$, and $r(P_m, T_n^j) = m+n-4$.*

Proof. Set $t = (n-4)/(m-1)$. Suppose that G_m is a connected graph of order m with $\mathrm{ex}(m+n-4; G_m) \leq (m-2)(m+n-5)/2$. Then clearly $\Delta(\overline{(t+1)K_{m-1}}) = t(m-1) = n-4$. Thus, $(t+1)K_{m-1}$ does not contain any copies of G_m, and $\overline{(t+1)K_{m-1}}$ does not contain any copies of T_n^j. Hence $r(G_m, T_n^j) \geq 1+(t+1)(m-1) = m+n-4$. By Theorems 2.1 and 3.1,

$$\mathrm{ex}(m+n-4; T_n^j) = \left[\frac{(n-4)(m+n-4)}{2}\right] \quad \text{or} \quad \frac{(n-2)(m+n-4)-(m-3)(n-m+2)}{2}.$$

If $\mathrm{ex}(m+n-4; T_n^j) = [(n-4)(m+n-4)/2]$, then

$$\mathrm{ex}(m+n-4; G_m) + \mathrm{ex}(m+n-4; T_n^j) \leq \frac{(m-2)(m+n-5)+(n-4)(m+n-4)}{2} < \binom{m+n-4}{2}.$$

If $\mathrm{ex}(m+n-4;T_n^j)=((n-2)(m+n-4)-(m-3)(n-m+2))/2$, then

$$\begin{aligned}\mathrm{ex}(m&+n-4;G_m)+\mathrm{ex}(m+n-4;T_n^j)\\&\le\frac{(m-2)(m+n-5)+(n-2)(m+n-4)-(m-3)(n-m+2)}{2}\\&=\binom{m+n-4}{2}-\frac{(m-4)(n-m+1)}{2}<\binom{m+n-4}{2}.\end{aligned}$$

Therefore, by Lemma 4.1 we always have $r(G_m,T_n^j)\le m+n-4$ and hence $r(G_m,T_n^j)=m+n-4$. This proves (i).

Now consider (ii). Note that $m+n-4\equiv 1 \pmod{m-1}$. By (1.1), we have $\mathrm{ex}(m+n-4;P_m)=(m-2)(m+n-5)/2$. By Lemma 4.4, $\mathrm{ex}(m+n-4;T'_m)=(m-2)(m+n-5)/2$ for $m\ge 5$. By [10, Theorem 4.2], $\mathrm{ex}(m+n-4;T^*_m)=(m-2)(m+n-5)/2$ for $m\ge 6$. By Theorems 2.1 and 3.1, $\mathrm{ex}(m+n-4;T^i_m)=(m-2)(m+n-5)/2$ for $i\in\{1,2\}$ and $m\ge 5$. Thus from (i) and the above we deduce (ii). The proof is complete.

Lemma 6.1. *Let $j\in\{1,2\}$, $m,n\in\mathbb{N}$, $m\ge 7$ and $m-1\nmid n-4$. Assume $n=m+1\ge 12$ or $n\ge\max\{m+2,19-m\}$.*

- (i) *If G_m is a connected graph of order m with $\mathrm{ex}(m+n-5;G_m)\le(m-2)(m+n-6)/2$, then $r(G_m,T_n^j)\le m+n-5$.*
- (ii) *For $T_m\in\{P_m,T'_m,T^*_m,T^1_m,T^2_m\}$ we have $r(T_m,T_n^j)\le m+n-5$.*

Proof. Since $m+n-5=n-1+m-4$, by Theorems 2.1 and 3.1 we have

$$\begin{aligned}\mathrm{ex}(m+n-5;T_n^j)&=\left[\frac{(n-4)(m+n-5)}{2}\right]\\&\quad\text{or}\quad\frac{(n-2)(m+n-5)-(m-4)(n-1-(m-4))}{2}.\end{aligned}$$

If $n=m+1$, then $(m-4)(n-3-(m-4))=2(n-5)$. If $n\ge m+2$, then $3\le m-4\le n-6$ and so $(m-4)(n-3-(m-4))=\left(\frac{n-3}{2}\right)^2-\left(m-4-\frac{n-3}{2}\right)^2\ge\left(\frac{n-3}{2}\right)^2-\left(n-6-\frac{n-3}{2}\right)^2=3(n-6)$. Thus,

$$\begin{aligned}&\frac{(n-4)(m+n-5)+m-2}{2}\\&\qquad-\frac{(n-2)(m+n-5)-(m-4)(n-1-(m-4))}{2}\\&=\frac{(m-4)(n-3-(m-4))-2n+m}{2}\\&\ge\begin{cases}\dfrac{2(n-5)-2n+m}{2}=\dfrac{m-10}{2}>0 & \text{if } n=m+1\ge 12,\\[2mm] \dfrac{3(n-6)-2n+m}{2}=\dfrac{n-10+m-8}{2}>0 & \text{if } n\ge\max\{m+2,19-m\}.\end{cases}\end{aligned}$$

Therefore, from the above we deduce that

$$\text{ex}(m+n-5;T_n^j) < \frac{(n-4)(m+n-5)+m-2}{2}. \tag{6.1}$$

Hence, if G_m is a connected graph of order m with $\text{ex}(m+n-5;G_m) \le (m-2)(m+n-6)/2$, then

$$\begin{aligned}&\text{ex}(m+n-5;G_m)+\text{ex}(m+n-5;T_n^j)\\&\quad< \frac{(m-2)(m+n-6)}{2}+\frac{(n-4)(m+n-5)+m-2}{2}=\binom{m+n-5}{2}.\end{aligned}$$

Applying Lemma 4.1 we obtain (i).

Now we consider (ii). Since $m-1 \nmid m+n-5$, by Corollaries 2.1 and 3.1 we have $\text{ex}(m+n-5;T_m^i) \le (m-2)(m+n-6)/2$ for $i \in \{1,2\}$. By (1.1), $\text{ex}(m+n-5;P_m) \le (m-2)(m+n-6)/2$. By Lemma 4.4, $\text{ex}(m+n-5;T'_m) \le (m-2)(m+n-6)/2$. By [10, Theorems 4.1–4.5], $\text{ex}(m+n-5;T^*_m) \le (m-2)(m+n-6)/2$. Thus, from the above and (i) we deduce (ii). This proves the lemma.

Theorem 6.3. *Let $m \in \mathbb{N}$ and $j \in \{1,2\}$.*

(i) *We have*

$$r(T'_m,T_{m+1}^j)=\begin{cases}2m-4 & \text{if } 2\nmid m \text{ and } m\ge 9,\\ 2m-5 & \text{if } 2\mid m \text{ and } m \ge 16.\end{cases}$$

(ii) *If $n\in\mathbb{N}$, $m\ge 7$, $n \ge \max\{m+2,19-m\}$ and $m-1\nmid n-4$, then $r(T'_m,T_n^j)=m+n-5$.*

Proof. We first assume $2\nmid m$ and $m\ge 9$. By Lemma 4.2(i), we have $r(T'_m,T_{m+1}^j)\ge m-2+m-2=2m-4$. By Lemma 4.4,

$$\text{ex}(2m-4;T'_m)=\frac{(m-2)(2m-4)-2(m-3)}{2}=m^2-5m+7.$$

By Theorems 2.1 and 3.1,

$$\text{ex}(2m-4;T_{m+1}^j)=\frac{(m-1)(2m-4)-4(m-4)}{2}=m^2-5m+10.$$

Thus,

$$\begin{aligned}\text{ex}(2m-4;T'_m)+\text{ex}(2m-4;T_{m+1}^j)&=m^2-5m+7+m^2-5m+10\\&=2m^2-10m+17<2m^2-9m+10=\binom{2m-4}{2}.\end{aligned}$$

Hence, by Lemma 4.1 we obtain $r(T'_m,T_{m+1}^j)\le 2m-4$ and so $r(T'_m,T_{m+1}^j) = 2m-4$.

Now we assume $2\,|\,m$ and $m \geq 16$. By Lemma 4.2(i), $r(T'_m, T^j_{m+1}) \geq m-2+m-2-1 = 2m-5$. By Lemma 4.4,

$$\mathrm{ex}(2m-5; T'_m) = \left[\frac{(m-2)(2m-6)-(m-3)}{2}\right] = \frac{2m^2-11m+14}{2}.$$

By Theorems 2.1 and 3.1,

$$\mathrm{ex}(2m-5; T^j_{m+1}) = \left[\frac{(m-1)(2m-5)}{2}\right] - (2m-5) = \frac{2m^2-11m+14}{2}.$$

Thus,

$$\begin{aligned}\mathrm{ex}(2m-5; T'_m) + \mathrm{ex}(2m-5; T^j_{m+1}) &= 2m^2-11m+14 \\ &< 2m^2-11m+15 = \binom{2m-5}{2}.\end{aligned}$$

Hence, by Lemma 4.1 we obtain $r(T'_m, T^j_{m+1}) \leq 2m-5$ and so $r(T'_m, T^j_{m+1}) = 2m-5$. This proves (i).

Now we consider (ii). Suppose $n \in \mathbb{N}$, $m \geq 7$ and $n \geq \max\{m+2, 19-m\}$. By Lemma 6.1(ii), $r(T'_m, T^j_n) \leq m+n-5$. By Lemma 4.2, we have $r(T'_m, T^j_n) \geq m-2+n-3$. Thus, $r(T'_m, T^j_n) = m+n-5$. This proves (ii). The proof is complete.

THEOREM 6.4. *Let $j \in \{1,2\}$, $m, n \in \mathbb{N}$, $m \geq 7$ and $m-1 \nmid n-4$. Suppose that $n = m+1 \geq 12$ or $n \geq \max\{m+2, 19-m\}$. Assume that $G_m \in \{P_m, T^*_m, T^1_m, T^2_m\}$ or G_m is a connected graph of order m such that $\mathrm{ex}(m+n-5; G_m) \leq (m-2)(m+n-6)/2$. If $n \geq (m-3)^2+3$ or $m+n-6 = (m-1)x+(m-2)y$ for some nonnegative integers x and y, then $r(G_m, T^j_n) = m+n-5$.*

Proof. If $n \geq (m-3)^2+3$, then $m+n-6 \geq (m-2)(m-3)$ and so $m+n-6 = (m-1)x+(m-2)y$ for some $x, y \in \{0,1,2,\dots\}$ by Lemma 5.1. Now suppose $m+n-6 = (m-1)x+(m-2)y$, where $x, y \in \{0,1,2,\dots\}$. Set $G = xK_{m-1} \cup yK_{m-2}$. Then $\Delta(\overline{G}) \leq n-4$. Thus, G does not contain any copies of G_m, and $\overline{G}$ does not contain any copies of T^j_n. Hence $r(G_m, T^j_n) \geq 1+|V(G)| = m+n-5$. On the other hand, by Lemma 6.1 we have $r(G_m, T^j_n) \leq m+n-5$. Thus $r(G_m, T^j_n) = m+n-5$. This proves the theorem.

COROLLARY 6.1. *Let $m, n \in \mathbb{N}$, $m \geq 7$, $m-1\,|\,n-b$, $b \in \{2,3,5\}$, $n \geq \max\{m+2, 19-m\}$ and $j \in \{1,2\}$. Assume that $G_m \in \{P_m, T^*_m, T^1_m, T^2_m\}$ or G_m is a connected graph of order m with*

$$\mathrm{ex}(m+n-5; G_m) \leq \frac{(m-2)(m+n-6)}{2}.$$

Then $r(G_m, T^j_n) = m+n-5$.

Proof. Set $k = (n-b)/(m-1)$. Then $k \in \mathbb{N}$. For $b = 2$ we have $k \geq 2$. Since

$$m+n-6 = \begin{cases} (k-2)(m-1)+3(m-2) & \text{if } b=2, \\ (k-1)(m-1)+2(m-2) & \text{if } b=3, \\ (k+1)(m-1) & \text{if } b=5, \end{cases}$$

the result follows from Theorem 6.4.

THEOREM 6.5. *Let $m \in \mathbb{N}$, $m \geq 12$ and $i, j \in \{1, 2\}$. Then*

$$r(T_m^i, T_{m+1}^j) = r(T_m^*, T_{m+1}^j) = 2m-5.$$

Proof. Let $T_m \in \{T_m^i, T_m^*\}$. By Theorems 2.1, 3.1 and [10, Theorem 4.1],

$$\mathrm{ex}(2m-5; T_m) = \frac{(m-2)(2m-5)-3(m-4)}{2},$$

$$\mathrm{ex}(2m-5; T_{m+1}^j) = \frac{(m-1)(2m-5)-5(m-5)}{2} \quad \text{or} \quad \left[\frac{(m-3)(2m-5)}{2}\right].$$

Since

$$\frac{(m-2)(2m-5)-3(m-4)}{2} + \frac{(m-3)(2m-5)}{2} = \frac{(2m-5)(2m-6)+7-m}{2} < \binom{2m-5}{2}$$

and

$$\frac{(m-2)(2m-5)-3(m-4)}{2} + \frac{(m-1)(2m-5)-5(m-5)}{2} = 2m^2-12m+26 < 2m^2-11m+15 = \binom{2m-5}{2},$$

we see that $\mathrm{ex}(2m-5; T_m) + \mathrm{ex}(2m-5; T_{m+1}^j) < \binom{2m-5}{2}$. Hence, applying Lemma 4.1 we deduce that $r(T_m, T_{m+1}^j) \leq 2m-5$. Since $\Delta(T_m) = m-3$ and $\Delta(T_{m+1}^j) = m-2$, by Lemma 4.2(i) we have $r(T_m, T_{m+1}^j) \geq m-3+m-2 = 2m-5$. Hence $r(T_m, T_{m+1}^j) = 2m-5$. This proves the theorem.

Acknowledgements. The first author is supported by the National Natural Science Foundation of China (grant No. 11371163), and the second author is supported by the Fundamental Research Funds for the Central Universities (grant No. 2014QNA58).

REFERENCES

[1] S. A. Burr, *Generalized Ramsey theory for graphs—a survey*, in: Graphs and Combinatorics, R. A. Bari and F. Harary (eds.), Lecture Notes in Math. 406, Springer, Berlin, 1974, 52–75.

[2] S. A. Burr and P. Erdős, *Extremal Ramsey theory for graphs*, Util. Math. 9 (1976), 247–258.
[3] S. A. Burr and J. A. Roberts, *On Ramsey numbers for stars*, Util. Math. 4 (1973), 217–220.
[4] R. J. Faudree and R. H. Schelp, *Path Ramsey numbers in multicolorings*, J. Combin. Theory Ser. B 19 (1975), 150–160.
[5] Y. Guo and L. Volkmann, *Tree-Ramsey numbers*, Australas. J. Combin. 11 (1995), 169–175.
[6] L. K. Hua, *Introduction to Number Theory*, Springer, Berlin, 1982 (translated from the Chinese by P. Shiu).
[7] S. P. Radziszowski, *Small Ramsey numbers*, Revision #14, Electron. J. Combin. 2014, Dynamic Survey DS1, 94 pp.
[8] A. F. Sidorenko, *Asymptotic solution for a new class of forbidden r-graphs*, Combinatorica 9 (1989), 207–215.
[9] Z. H. Sun, *Ramsey numbers for trees*, Bull. Austral. Math. Soc. 86 (2012), 164–176.
[10] Z. H. Sun and L. L. Wang, *Turán's problem for trees*, J. Combin. Number Theory 3 (2011), 51–69.
[11] M. Woźniak, *On the Erdős–Sós conjecture*, J. Graph Theory 21 (1996), 229–234.

Zhi-Hong Sun
School of Mathematical Sciences
Huaiyin Normal University
Huaian, Jiangsu 223001, P.R. China
E-mail: zhihongsun@yahoo.com
http://www.hytc.edu.cn/xsjl/szh

Lin-Lin Wang
School of Science
China University of Mining and Technology
Xuzhou, Jiangsu 221116, P.R. China
E-mail: wanglinlin_1986@aliyun.com

Yi-Li Wu
School of Mathematical Sciences
Huaiyin Normal University
Huaian, Jiangsu 223001, P.R. China
E-mail: yiliwu@126.com

Received 27 July 2014;
revised 27 January 2015 (6330)