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SELF-SIMILARITY IN CHEMOTAXIS SYSTEMS

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Abstract. We consider a system which describes the scaling limit of several chemotaxis systems. We focus on self-similarity, and review some recent results on forward and backward self-similar solutions to the system.

1. Introduction. Self-similarity is one of the fundamental properties of chemotaxis systems, particularly in the study of blowup solutions. For example, $n = 2$ is selected for the formation of collapse by the dimensional analysis [14], and the quantized blowup mechanism is obtained by the self-similarity of the limit equation derived from the backward self-similar transformation [42]. The present paper is devoted to the study of

$$(1)\qquad \begin{aligned} u_t &= \nabla\cdot(\nabla u - u\nabla v),\\ \tau v_t &= \Delta v + u, \qquad x\in\mathbb{R}^n,\ t>0, \end{aligned}$$

where $\tau \geq 0$ is a constant. It describes the scaling limit of several chemotaxis systems, or one of their simplified forms, or motion of the mean field of many self-gravitational particles in astrophysics, and so forth.

System (1) is invariant under the transformation

$$(u,v)\mapsto(u_\mu,v_\mu)=(\mu^2u(\mu x,\mu^2t),v(\mu x,\mu^2t)),$$

and a solution (u,v) invariant under this transformation is called a *self-similar solution*:

$$(u,v)=(u_\mu,v_\mu),\qquad \mu>0.$$

Consequently, we obtain the *forward self-similar solution* of the form

$$(2)\qquad \begin{aligned} u(x,t) &= \frac{1}{t}\,\phi\biggl(\frac{x}{\sqrt{t}}\biggr),\\ v(x,t) &= \psi\biggl(\frac{x}{\sqrt{t}}\biggr), \qquad x\in\mathbb{R}^n,\ t>0, \end{aligned}$$

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a solution to (1) global in time, and the *backward self-similar solution*,

$$u(x,t) = \frac{1}{T-t}\,\phi\left(\frac{x}{\sqrt{T-t}}\right), \tag{3}$$

$$v(x,t) = \psi\left(\frac{x}{\sqrt{T-t}}\right), \tag{4}$$

a blowup solution to (1) with blowup time $t = T > 0$, where ϕ and ψ are some functions on $\mathbb{R}^n$. Studies on these solutions and their applications are described in §2 and §3, respectively.

2. Forward self-similar solution

General existence. If the initial values

$$u_0(x) = \lim_{t\downarrow 0} \frac{1}{t}\,\phi\left(\frac{x}{\sqrt{t}}\right), \qquad v_0(x) = \lim_{t\downarrow 0} \psi\left(\frac{x}{\sqrt{t}}\right)$$

are taken in (2), then

$$u_0(\mu x) = \mu^{-2} u_0(x), \qquad v_0(\mu x) = v_0(x), \qquad \mu > 0, \tag{5}$$

and conversely, the forward self-similar solution is obtained by showing unique existence of a solution to (1) satisfying

$$u(\,\cdot\,,0) = u_0 \quad \text{and} \quad v(\,\cdot\,,0) = v_0 \quad \text{if } \tau > 0$$

for such u_0, v_0. This approach is taken in [2, 3, 4, 32, 33, 6, 19], and henceforth, the solution to (1) is said to be *strong* (resp. *weak*) if it is strongly (resp. weakly) continuous in time in the specified function space.

First, the problem (1) with $\tau = 0$ is formulated as

$$u(t) = e^{t\Delta} u_0 - \int_0^t \nabla e^{(t-s)\Delta} \cdot u(s)(\Gamma * u)(s)\, ds \tag{6}$$

for

$$(e^{t\Delta} f)(x) = \int_{\mathbb{R}^2} G(x-y,t) f(y)\, dy,$$

$$G(x,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-|x|^2/4t},$$

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \log \dfrac{1}{|x|} & (n = 2), \\ \dfrac{1}{(n-2)\omega_n}\, |x|^{2-n} & (n = 3), \end{cases}$$

where ω_n is the $(n-1)$-dimensional volume of the boundary of the unit ball in $\mathbb{R}^n$. This problem has a unique strong solution in $M^p_q(\mathbb{R}^n)$ with $n/2 < p \le n$, $2 - p/n \le q \le p$, locally in time for large initial data, and also globally in time for small initial data, where

$$M_q^p(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) \mid \|f; M_q^p\| < \infty\},$$
$$\|f; M_q^p\|^q = \sup_{x\in\mathbb{R}^n,\, 0<R\le 1} R^{n(q/p-1)} \int_{B(x,R)} |f|^q,$$

denotes the Morrey space. In the limiting case $p = n/2$ with $n \ge 3$, we obtain a unique strong solution local in time in the space

$$X = \{u \in C([0,T]; M_p^{n/2}) \mid \sup_{0<t\le T} t^{1/4}\|u(t); M_{4q/3}^{2n/3}\| < \infty\},$$

provided that

$$\ell(u_0) = \limsup_{t\downarrow 0} \|e^{t\Delta}u_0; M_{4q/3}^{2n/3}\| \ll 1,$$

where $n \ge 3$, $3/2 \le q \le n/2$ ([2]). If $n \ge 4$, there is a unique weak solution global in time for small initial data in $\mathcal{PM}^{n-2}(\mathbb{R}^n)$, where

$$\mathcal{PM}^a(\mathbb{R}^n) = \{v \in \mathcal{S}'(\mathbb{R}^n) \mid \widehat{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \|v\|_{\mathcal{PM}^a} < \infty\},$$
$$\|v\|_{\mathcal{PM}^a} = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^n} |\xi|^a|\widehat{v}(\xi)|,$$

with $\mathcal{S}'(\mathbb{R}^n)$ and $\widehat{v}$ standing for the space of tempered distributions and the Fourier transformation of $v \in \mathcal{S}'(\mathbb{R}^n)$, respectively [6, 32]. This is also the case of $L^{n/2,\infty}(\mathbb{R}^n)$, the Marcinkiewicz space defined by

$$L^{p,\infty} = \{v \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \mid \|v\|_{p,\infty} < \infty\},$$
$$\|v\|_{p,\infty} = \sup\Big\{|E|^{-1+1/p}\int_E |v| \,\Big|\, E \subset \mathbb{R}^n \text{ a Borel set with } 0 < |E| < \infty\Big\},$$

and if $n \ge 3$, there is a unique weak solution global in time for small initial data [33]. These results guarantee the existence of the forward self-similar solution to (1) with $\tau = 0$, $n \ge 3$.

In the other case of $\tau > 0$, $n \ge 3$, similarly, there is a unique weak solution global in time for small initial data in $u_0 \in L^{n/2,\infty}(\mathbb{R}^n)$ and $v_0 \in \mathrm{BMO}(\mathbb{R}^n)$, which ensures the existence of the forward self-similar solution [4, 19].

In the case $n = 2$, the problem (1) has the stability property with the parameter $\tau \ge 0$, i.e., solutions of the parabolic-parabolic system (with $\tau > 0$) converge to the solution of the parabolic-elliptic system (with $\tau = 0$) as $\tau \to 0$ in a subset of the space of pseudomeasures [34].

L^1-*Solution.* For $u(x,t)$ given by (2), we have

$$\int_{\mathbb{R}^n} u(x,t)\,dx = t^{(n-2)/2}\int_{\mathbb{R}^n} \phi(y)\,dy,$$

and therefore, $\|u(\,\cdot\,,t)\|_1 =$ constant if and only if $n = 2$. Since the L^1-norm of $u = u(\,\cdot\,,t)$ is expected to be preserved in (1), only in this case can the forward self-similar solution describe the ultimate profile of the L^1-solution to (1) globally in time.

The profile functions ϕ, ψ, on the other hand, solve

$$\begin{aligned} &\nabla \cdot (\nabla \phi - \phi \nabla \psi) + \frac{1}{2} x \cdot \nabla \phi + \phi = 0, \\ &\Delta \psi + \frac{\tau}{2} x \cdot \nabla \psi + \phi = 0 \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{7}$$

and therefore, either $\phi > 0$ or $\phi \equiv 0$ by the maximum principle. In the case of $n = 2$ and $\phi > 0$, furthermore, the first equation is equivalent to

$$\nabla \cdot \phi \nabla \left(\log \phi - \psi + \frac{|x|^2}{4} \right) = 0. \tag{8}$$

This relation holds if

$$\phi(x) = \sigma e^{-|x|^2/4} e^{\psi(x)} \tag{9}$$

with a constant $\sigma > 0$, and then (7) is reduced to

$$\Delta \psi + \frac{\tau}{2} x \cdot \nabla \psi + \sigma e^{-|x|^2/4} e^{\psi} = 0, \qquad x \in \mathbb{R}^2. \tag{10}$$

Radially symmetric self-similar solution. Let $n = 2$. Assuming radial symmetry, $\phi = \phi(r) > 0$, $\psi = \psi(r)$, $r = |x|$ in (7), we obtain

$$(\log \phi - \psi)' + \frac{r}{2} = 0$$

by (8). Thus, it follows that

$$\log \phi - \psi = -\frac{r^2}{4} + c$$

with a constant c, and therefore, (9) holds with $\sigma = e^c > 0$:

$$\begin{aligned} &\psi'' + \left(\frac{1}{r} + \frac{\tau}{2} r \right) \psi' + \sigma e^{-r^2/4} e^{\psi} = 0, \qquad r > 0, \\ &\psi'(0) = 0. \end{aligned} \tag{11}$$

Equation (7) with $\tau = 0$ is invariant under adding constants to ψ, and this induces the same property for (11). In view this, we adopt the normalization $\psi(0) = 0$ in this case, and then $\sigma = \phi(0)$ follows. The solution $\psi(r)$ exists for all $r > 0$, and satisfies $\lim_{r \uparrow \infty} \psi(r) = -\infty$. The structure of the solution set is clear in this case, and will be described in Theorems 3 and 5.

If $\tau > 0$, on the contrary, $\int_0^\infty r\psi(r)\, dr < \infty$ follows from

$$\lim_{r \uparrow \infty} \psi(r) = 0. \tag{12}$$

Existence of a positive solution to (11) with (12) is discussed in this case [24, 23], and in particular, the following result is obtained in [25].

THEOREM 1. *For* (11)–(12), *given* $0 < \tau < 2$ *there is* $\sigma^* > 0$ *such that if* $\sigma > \sigma^*$ *then there is no positive solution*, *if* $0 < \sigma < \sigma^*$ *then there are at least two positive solutions*, *and if* $\sigma = \sigma^*$, *then there is a unique positive solution.*

Two problems arise here; radial symmetry and L^1-behavior of the solution.

Radial symmetry of self-similar solution. The relation (9) and radial symmetry of the solution are proven by a Liouville type theorem [22] and the moving plane method [28], respectively. We obtain the following theorem [30, 29].

THEOREM 2. *Let* $(\phi, \psi) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ *be a solution to* (7), $n = 2$.

(i) *If* $\tau = 0$, $\phi \geq 0$, $\phi \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, *and* $\psi_+ = \max\{\psi, 0\} \in L^\infty(\mathbb{R}^2)$, *then* (9) *holds with a constant* $\sigma > 0$, *and* $\phi = \phi(r)$, $\psi = \psi(r)$ *are decreasing functions of* $r = |x| > 0$.

(ii) *The same conclusion holds if* $\tau > 0$, $\varphi, \psi \geq 0$, *and* $\phi(x), \psi(x) \to 0$ *as* $|x| \to \infty$. *In this case*, $\phi, \psi \in L^1(\mathbb{R}^2)$.

L^1-norms of radially symmetric self-similar solutions. Given profile functions $\phi = \phi(r), \psi = \psi(r)$ satisfying (7), we define the volume functions [1, 2]

$$\Phi(s) = \frac{1}{2}\int_0^s \phi(\sqrt{t})\, dt = \int_0^{\sqrt{s}} r\phi(r)\, dr,$$

$$\Psi(s) = \frac{1}{2}\int_0^s \psi(\sqrt{t})\, dt = \int_0^{\sqrt{s}} r\phi(r)\, dr,$$

and obtain

$$\int_0^r \phi(y)\, dy = 2\pi \int_0^\infty r\phi(r)\, dr = 2\pi\Phi(\infty),$$

$$\int_0^r su(s,t)\, ds = \int_0^r \frac{s}{t}\,\phi\left(\frac{s}{\sqrt{t}}\right) ds = \frac{1}{2}\int_0^{r^2/\tau} \phi(\sqrt{s})\, ds = \Phi\left(\frac{r^2}{t}\right),$$

$$\int_0^r sv(s,t)\, ds = t\Psi\left(\frac{r^2}{t}\right),$$

where

$$u(r,t) = \frac{1}{t}\,\phi\left(\frac{r}{\sqrt{t}}\right), \qquad v(r,t) = \psi\left(\frac{r}{\sqrt{t}}\right).$$

Since (u, v) solves (1), we see that

$$ru_t = (ru_r)_r - ru_r v_r - u(rv_r)_r, \qquad \tau r v_t = (rv_r)_r + ru,$$

and hence

$$\int_0^r su_t(s,t)\,ds = ru_r - ruv_r, \qquad \tau\int_0^r sv_t(s,t)\,ds = rv_r + \int_0^r su(s,t)\,ds.$$

This means

$$U_t = r(r^{-1}U_r)_r - U_r(r^{-1}V_r)_r, \qquad \tau V_t = r(r^{-1}V_r)_r + U$$

for

$$U(r,t) = \int_0^r su(s,t)\,ds = \Phi\left(\frac{r^2}{r}\right), \qquad V(r,t) = \int_0^r sv(s,t)\,ds = t\Psi\left(\frac{r^2}{t}\right),$$

and then it follows that

$$\Phi'' + \frac{1}{4}\,\Phi' - 2\Phi'\Psi'' = 0, \qquad 4s\Psi'' + \tau s\Psi' - \tau\Psi + \Phi = 0, \qquad s > 0, \tag{13}$$

where $\,' = d/ds$.

In the case of $\tau = 0$, this relation is reduced to

$$\Phi'' + \frac{1}{4}\,\Phi' + \frac{\Phi\Phi'}{2s} = 0, \qquad s > 0, \tag{14}$$

as $4s\Psi'' = -\Phi$. Since $\phi \geq 0$, we have

$$\Phi(0) = 0, \qquad \Phi'(s) \geq 0, \quad s > 0, \qquad \Phi(\infty) = \lim_{s\uparrow\infty}\Phi(s) = \frac{\lambda}{2\pi} \tag{15}$$

for $\lambda = \|\phi\|_1$. Studies on (14) with (15) are summarized as follows [1, 2, 8], where $\lambda = \|\phi\|_1$ acts as a control parameter instead of $\sigma = \phi(0)$.

THEOREM 3. *In the case of* $\tau = 0$, $n = 2$, *the problem* (7) *has a nontrivial radially symmetric solution* (ϕ, ψ) *satisfying* $\phi \geq 0$, $\|\phi\|_1 = \lambda$ *if and only if* $0 < \lambda < 8\pi$, *and it is unique for each* $\lambda \in (0, 8\pi)$.

To treat the case $\tau > 0$, we put $W(s) = -4s\Phi''(s)$. Then, by the second equation of (13),

$$\Phi' = (-4s\Psi'')' - \tau s\Psi'' = W' + \frac{\tau}{4}\,W.$$

Since $s\Psi''(s) = \sqrt{s}\psi'(\sqrt{s})/4$, we have $W(0) = \lim_{s\downarrow 0} W(s) = 0$, and

$$W(s) = e^{-\tau s/4}\int_0^s e^{\tau t/4}\Phi'(t)\,dt.$$

This means

$$-s\Psi''(s) = \frac{1}{4}\,e^{-\tau s/4}\int_0^s e^{\tau t/4}\Phi'(t)\,dt,$$

and putting this into the first equation of (13), we obtain

$$\Phi'' + \frac{1}{4}\,\Phi' + \frac{1}{2s}\,e^{-\tau s/4}\Phi'\int_0^s e^{\tau t/4}\Phi'(t)\,dt = 0.$$

From this relation, we obtain an a priori L^1-estimate of the solution [30].

THEOREM 4. *If* (ϕ, ψ) *is a radially symmetric solution to* (7) *with* $\tau > 0$, $n = 2$, *satisfying* $\phi, \psi \in L^1(\mathbb{R}^2)$, $\phi, \psi \geq 0$, *then*

$$\|\phi\|_1 \leq \max\left\{\frac{4}{3}\pi^3, \frac{4}{3}\pi^3\tau^2\right\}.$$

If $0 < \tau \leq 1/2$, *we obtain* $\|\phi\|_1 < 8\pi$.

The above estimate can be improved as follows (cf. [5]):

$$\|\phi\|_1 \leq \min\left\{\frac{4}{3}\pi^3 \max\{\tau, 1\}, 8\pi(\tau+1)\right\}.$$

Structure of the solution set. Let S be the set of all solutions $(\phi, \psi) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ to (7) satisfying

$$\begin{aligned} &\phi \geq 0,\ \phi \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2),\ \psi_+ \in L^\infty(\mathbb{R}^2),\ \psi(0) = 0 \quad && \text{if } \tau = 0, \\ &\phi, \psi \geq 0, \quad \phi(x), \psi(x) \to 0 \quad \text{as } |x| \to \infty && \text{if } \tau > 0. \end{aligned} \tag{16}$$

By Theorem 2, this set is composed of radially symmetric solutions to (10) satisfying (9), and furthermore, the problem is formulated as

$$\Delta\psi + \frac{\tau}{2} x \cdot \nabla\psi + \frac{\lambda e^{-|x|^2/4} e^{\psi}}{\int_{\mathbb{R}^2} e^{-|x|^2/4} e^{\psi}\, dx} = 0, \qquad x \in \mathbb{R}^2, \tag{17}$$

using $\lambda = \|\phi\|_1$. Then, the following results [30, 29] are obtained by the blowup analysis [11, 21].

THEOREM 5. *If* $n = 2$, *the above set* S *of solutions to* (7) *forms a one-parameter family denoted by* $\{(\phi(s), \psi(s)) \mid s \in \mathbb{R}\}$ *with*

$$\begin{aligned} &s \mapsto (\phi(s), \psi(s)) \in C_{\mathrm{loc}}(\mathbb{R}^2) \times C_{\mathrm{loc}}(\mathbb{R}^2), \\ &s \mapsto \lambda(s) = \|\phi(s)\|_1 \end{aligned}$$

continuous. Moreover, $\lambda(s) \to 0$ *as* $s \downarrow -\infty$ *and* $\lambda(s) \to 8\pi$, $\phi(s) \rightharpoonup 8\pi\delta_0$ *as* $s \uparrow \infty$ *in the sense of measures.*

If $\tau = 0$, then $s \mapsto \lambda(s)$ is strictly increasing by Theorem 3, while Theorem 4 guarantees the upper bound of $\lambda(s)$, e.g., $0 < \lambda(s) < 8\pi$ in the case of $0 < \tau \leq 1/2$.

Convergence to the self-similar solution. If (u, v) is a radially symmetric solution to (1) with $\tau = 0$,

$$\begin{aligned} &u_t = \nabla \cdot (\nabla u - u\nabla v), \\ &0 = \Delta v + u, \qquad x \in \mathbb{R}^2,\ t > 0, \\ &u(\,\cdot\,, 0) = u_0, \end{aligned} \tag{18}$$

and if we define

$$M(s,t) = Q(\sqrt{s}, t), \qquad Q(r,t) = \int_{B(0,r)} u(x,t)\, dx,\ r > 0,$$

then

$$M_t = 4sM_{ss} + \frac{1}{\pi} MM_s, \quad s \in (0,\infty),\ t>0,$$
$$M(0,t)=0, \quad M(\infty,t) = \lim_{s\uparrow\infty} M(s,t) = \lambda,$$
$$M(s,0) = M_0(s),$$

where $\lambda = \|u_0\|_1$ and

$$M_0(s) = Q_0(\sqrt{s}), \quad Q_0(r) = \int\limits_{B(0,r)} u_0(x)\,dx.$$

By Theorem 3, for each $\lambda \in (0,8\pi)$ there exists a unique radially symmetric self-similar solution $(\phi,\psi) = (\phi_\lambda,\psi_\lambda)$ to (7) satisfying $\|\phi_\lambda\|_1 = \lambda$. Then, defining $M_\lambda = M_\lambda(s,t)$ by

$$M_\lambda(s,t) = Q_\lambda(\sqrt{s},t), \quad Q_\lambda(r,t) = \int\limits_{B(0,r)} u_\lambda(x,t)\,dx,$$
$$u_\lambda(x,t) = \frac{1}{t}\,\phi_\lambda\left(\frac{|x|}{\sqrt{t}}\right),$$

we obtain the following theorem [9].

Theorem 6. *For each* $\lambda \in (0,8\pi)$,

$$\lim_{t\uparrow\infty} \|M(\,\cdot\,,t) - M_\lambda(\,\cdot\,,t)\|_\infty = 0.$$

Similar problems to (18) with $\lambda = 8\pi$ for $\Omega = B(0,R)$ or $\Omega = \mathbb{R}^2$ without radial symmetry are also studied [9, 8, 10].

Convergence to the self-similar solution (continued). To describe the case $\tau > 0$ of this convergence problem, we assume $\tau = 1$ for simplicity:

$$\begin{aligned} &u_t = \nabla\cdot(\nabla u - u\nabla v),\\ &v_t = \Delta v + u, \quad x\in\mathbb{R}^2,\ t>0,\\ &u(\,\cdot\,,0) = u_0, \quad v(\,\cdot\,,0) = v_0. \end{aligned} \tag{19}$$

This is transformed into

$$\begin{aligned} u(t) &= e^{t\Delta}u_0 - \int_0^t (\nabla e^{(t-s)\Delta})\cdot(u(s)\nabla v(s))\,ds,\\ v(t) &= e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}u(s)\,ds. \end{aligned} \tag{20}$$

First, applying the contraction mapping principle in the space

$$X_p = \{u : (0,\infty)\to L^p(\mathbb{R}^2) \text{ Bochner measurable} \mid \sup_{t>0} t^{1-1/p}\|u(\,\cdot\,,t)\|_p < \infty\},$$

$$\|u\|_{X_p} = \sup_{t>0} t^{1-1/p}\|u(\,\cdot\,,t)\|_p$$

for $4/3 < p < 2$, we obtain the following fact [4, 27].

PROPOSITION 2.1. *There are $M > 0$, $\lambda_0 > 0$, $\beta_0 \ge 0$ sufficiently small such that $\|u_0\|_1 < \lambda_0$, $\|\nabla v_0\|_2 \le \beta_0$ implies the existence of a unique solution (u, v) to (20) satisfying $u \in X_p$ and $\|u\|_{X_p} \le M$.*

The expected self-similar solution describing the asymptotic behavior of the above solution is defined by $u_0 = \lambda\delta_0$ and $v_0 = 0$ in (5):

$$\begin{aligned} u(t) &= \lambda G(\,\cdot\,,t) - \int_0^t (\nabla e^{(t-s)\Delta})\cdot(u(s)\nabla v(s))\,ds, \\ v(t) &= \int_0^t e^{(t-s)\Delta}u(s)\,ds. \end{aligned} \tag{21}$$

Similarly, we obtain the following fact:

PROPOSITION 2.2. *Given $M > 0$, $\lambda_0 > 0$, β_0 prescribed in the previous proposition, for each $\lambda \in (0, \lambda_0)$ there exists a self-similar solution (u, v) to (21) satisfying $\|u\|_{X_p} \le M$, $\|u(\,\cdot\,,t)\|_1 = \lambda$.*

The above self-similar solution has the profile functions (ϕ, ψ) satisfying (2), and hence

$$t^{1-1/p}\|u(\,\cdot\,,t)\|_p = \|\phi\|_p = \text{constant}, \qquad t > 0.$$

Next, we obtain convergence to the self-similar solution for small initial data in the following sense [27]. It is an open question whether $\lambda_0 = 8\pi$ is valid or not.

THEOREM 7. *Let (u_0, v_0) satisfy the assumptions of Proposition 2.1 and*

$$(1 + |x|^2)u_0 \in L^1(\mathbb{R}^2), \qquad \nabla v_0 \in L^1(\mathbb{R}^2),$$

and define (u_λ, v_λ) of Proposition 2.2 for $\lambda = \|u_0\|_1 < \lambda_0$. Then

$$t^{1-1/p}\|u(\,\cdot\,,t) - u_\lambda(\,\cdot\,,t)\|_p = O(t^{-\sigma}) \qquad \text{as } t \to \infty,$$

where $\sigma \in (0, 1/2)$.

3. Backward self-similar solution. The profile functions (ϕ, ψ) of the backward self-similar solution defined by (4) satisfy

$$\begin{aligned} &\nabla\cdot(\nabla\phi - \phi\nabla\psi) - \frac{1}{2}\,x\cdot\nabla\phi - \phi = 0, \\ &\Delta\psi - \frac{\tau}{2}\,x\cdot\nabla\psi + \phi = 0, \qquad x \in \mathbb{R}^n. \end{aligned} \tag{22}$$

Such a solution induces self-similar blowup to (1), while its actual existence depends on the dimension n. In more detail, the cases $n = 2$, $3 \le n \le 9$, and $n \ge 10$ are distinguished.

Non-existence. This subsection is mostly devoted to the case of $n = 2$. First, a radially symmetric solution to (22) satisfies

$$\begin{aligned} &(\phi' - \phi\psi')' + \frac{1}{r}(\phi' - \phi\psi') - \frac{r}{2}\phi' - \phi = 0, \\ &\psi'' + \left(\frac{1}{r} - \frac{\tau}{2}r\right)\psi' + \phi = 0, \qquad r > 0, \end{aligned} \tag{23}$$

and similarly to the forward self-similar case, the first equation reduces to

$$\phi = \sigma e^{r^2/4} e^{\psi}, \tag{24}$$

where $\sigma > 0$ is a constant. It follows that

$$\begin{aligned} &\psi'' + \left(\frac{1}{4} - \frac{\tau}{2}r\right)\psi' + \sigma e^{r^2/4} e^{\psi} = 0, \qquad r > 0, \\ &\psi'(0) = 0, \end{aligned} \tag{25}$$

while (25) admits no positive solution in the case of $\tau > 0$ ([24]).

To describe the case $\tau = 0$, we recall that a solution (u, v) to (1) is reasonable if the L^1-norm of $u(\,\cdot\,, t)$ is preserved. If (u, v) is defined by (4), then

$$Q(r,t) = \int_{B(0,r)} u(x,t)\,dx = \int_{|y| \le r/\sqrt{T-t}} \phi(y)\,dy,$$

which implies

$$\lim_{t \uparrow T} \int_{B(0,r)} u(x,t)\,dx \begin{cases} < \infty & \text{if } \phi \in L^1(\mathbb{R}^2), \\ = \infty & \text{if } \phi \notin L^1(\mathbb{R}^2). \end{cases}$$

Putting

$$Q(r,t) = \Phi(s), \qquad s = \frac{r}{\sqrt{T-t}},$$

we obtain

$$\Phi(\infty) = \lim_{s \uparrow \infty} \Phi(s) = \frac{\|\phi\|_1}{2\pi} < \infty$$

in the case $\phi \in L^1(\mathbb{R}^2)$. On the other hand,

$$\Phi_{ss} - \frac{s}{2}\Phi_s + \left(\frac{\Phi}{2\pi} - 1\right)\frac{\Phi_s}{s} = 0, \qquad s > 0, \tag{26}$$

by (23) similarly to (14). Since (26) admits no bounded non-trivial solution [15], we obtain the following theorem.

THEOREM 8. *If $n = 2$, $\tau > 0$, there is no non-trivial radially symmetric solution to (22) satisfying $\psi > 0$. If $n = 2$, $\tau = 0$, there is no non-trivial radially symmetric solution to (22) satisfying $\phi \in L^1$.*

The reduction to (24) is valid similarly to the forward self-similar solution [30]. First, writing

$$\nabla \cdot [e^{|x|^2/4} e^{\psi} \nabla (\phi e^{-|x|^2/4} e^{-\psi})] = 0$$

for the first equation of (22), we obtain

$$\Delta \zeta + \nabla b \cdot \nabla \zeta = 0, \qquad x \in \mathbb{R}^2,$$

where

$$\zeta = -\phi e^{-|x|^2/4} e^{-\psi} \leq 0, \qquad b = -|x|^2/4 - \psi.$$

It follows that

$$x \cdot \nabla b \leq 0, \qquad |x| \gg 1,$$

if

$$\nabla \psi(x) = o(|x|), \qquad |x| \to \infty, \tag{27}$$

and then Lemma 2.1 of [30] is applicable. Thus, ζ is a constant denoted by $-\sigma$, and it follows that

$$\phi = \sigma e^{|x|^2/4} e^{\psi}.$$

First, we consider the case $\tau = 0$. If $\lambda = \|\phi\|_1 < \infty$, it follows that

$$\phi = \frac{\lambda e^{\psi + |x|^2/4}}{\int_{\mathbb{R}^2} e^{\psi + |x|^2/4}}$$

and then

$$-\Delta \psi = \frac{\lambda e^{\psi + |x|^2/4}}{\int_{\mathbb{R}^2} e^{\psi + |x|^2/4}}, \qquad x \in \mathbb{R}^2,$$

by the second equation of (22). Putting

$$w = \psi + |x|^2/4 + \log \lambda - \log\Big(\int_{\mathbb{R}^2} e^{\psi + |x|^2/4}\Big),$$

we obtain

$$-\Delta w = e^w - 1, \quad x \in \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w = \lambda, \tag{28}$$

while we have the following fact.

PROPOSITION 3.1. *The problem* (28) *admits no solution.*

Proof. Suppose $w = w(x)$ is such a solution, and put

$$\overline{w}(r) = \frac{1}{2\pi r} \int_{|x|=r} w(x)\, ds.$$

Then

$$\int_{\mathbb{R}^2} e^w = 2\pi \int_0^\infty r\, dr \cdot \frac{1}{2\pi r} \int_{|x|=r} e^w\, ds \geq 2\pi \int_0^\infty e^{\overline{w}(r)} r\, dr \tag{29}$$

by Jensen's inequality, while

$$(r\overline{w}_r)_r = r - \frac{1}{2\pi}\int\limits_{|x|=r} e^w\,ds, \qquad r > 0,$$

from (28). This implies

$$r\overline{w}_r \geq \frac{r^2}{2} - \frac{\lambda}{2\pi},$$

and hence $\overline{w}_r(r) \geq r/4$ for r sufficiently large. We obtain $\overline{w}(r) \geq r^2/8 - C$ and hence $\int_\Omega e^w = \infty$ by (29). The proof is complete.

In the case of $\tau > 0$, we set

$$\overline{\psi}(r) = \frac{1}{2\pi r}\int\limits_{|x|=r} \psi(x)\,ds.$$

Then the equation

$$\Delta\psi - \frac{\tau}{2}\,x\cdot\nabla\psi + \sigma e^{|x|^2/4}e^{\psi} = 0, \qquad x \in \mathbb{R}^2,$$

implies

$$r^{-1}(r\overline{\psi}_r)_r - \frac{\tau}{2}\,r\overline{\psi}_r + \frac{\sigma e^{r^2/4}}{2\pi r}\int\limits_{|x|=r} e^{\psi}\,ds = 0, \qquad r > 0, \tag{30}$$

and therefore,

$$(re^{-(\tau/4)r^2}\overline{\psi}_r)_r + re^{-(\tau/4)r^2}\cdot\frac{\sigma e^{r^2/4}}{2\pi r}\int\limits_{|x|=r} e^{\psi}\,ds = 0.$$

We obtain

$$re^{-(\tau/4)r^2}\overline{\psi}_r(r) = -\int_0^r \xi e^{-(\tau/4)\xi^2}\,d\xi\cdot\frac{\sigma e^{\xi^2/4}}{2\pi\xi}\int\limits_{|x|=\xi} e^{\psi}\,ds < 0.$$

In the case of $\psi > 0$, this implies

$$(r\overline{\psi}_r)_r = \frac{\tau}{2}\,r^2\overline{\psi}_r - \frac{\sigma r e^{r^2/4}}{2\pi r}\int\limits_{|x|=r} e^{\psi}\,ds < -\sigma r e^{r^2/4}, \qquad r > 0,$$

and therefore,

$$r\overline{\psi}_r(r) < 2\sigma(1 - e^{r^2/4}) < -2\sigma\cdot\frac{r^2}{4} = -\frac{\sigma}{2}\,r^2.$$

We obtain

$$\overline{\psi}_r(r) < -\frac{\sigma}{2}\,r, \qquad r > 0,$$

and hence

$$\overline{\psi}(r) < \overline{\psi}(0) - \frac{\sigma}{4}\,r^2 \to -\infty, \qquad r \uparrow \infty,$$

a contradiction. Thus, the following theorem is proven.

THEOREM 9. *If* $n = 2$, $\tau = 0$, *there is no non-trivial solution* (ϕ, ψ) *to* (22) *satisfying* (27) *and* $\phi \in L^1(\mathbb{R}^2)$. *If* $n = 2$, $\tau > 0$, *there is no non-trivial solution* (ϕ, ψ) *to* (22) *satisfying* (27) *and* $\psi > 0$.

Aggregation rate. Backward self-similar transformation describes local behavior of the blowup solution, in particular, the quantized blowup mechanism in the simplified system of chemotaxis [18, 31, 17, 26, 42],

$$
\begin{aligned}
&u_t = \nabla \cdot (\nabla u - u \nabla v), \\
&-\Delta v = u - \frac{1}{|\Omega|} \int_\Omega u, \qquad x \in \Omega,\ t > 0, \\
&\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > 0,
\end{aligned} \tag{31}
$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. First, if the solution $u = u(x,t)$ has a blowup time $T < \infty$, then there is a collapse [38],

$$
u(x,t) \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0} + f(x) \tag{32}
$$

as $t \uparrow T$ in the sense of measures on $\overline{\Omega}$, where $\mathcal{S}$ is the blowup set,

$$
m(x_0) \geq m_*(x_0) = \begin{cases} 8\pi & (x_0 \in \Omega), \\ 4\pi & (x_0 \in \partial\Omega), \end{cases}
$$

and $0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$. Next, we obtain mass quantization $m(x_0) = m_*(x_0)$ ([42]), using the backward self-similar transformation,

$$
\begin{aligned}
&z(y,s) = (T-t)u(x,t), \qquad w(y,s) = v(x,t), \\
&y = (x - x_0)/(T-t)^{1/2}, \qquad s = -\log(T-t).
\end{aligned} \tag{33}
$$

More precisely, first, we have

$$
\lim_{b \uparrow \infty} \limsup_{t \uparrow T} \left| \int_{B(x_0, b(T-t)^{1/2}) \cap \Omega} u(x,t)\, dx - m(x_0) \right| = 0, \tag{34}
$$

which is referred to as the effect of "parabolic envelope". Then any $s_k \uparrow \infty$ admits $\{s'_k\} \subset \{s_k\}$ such that $z(y, s'_k + s)$ converges in $C_{\text{weak}}(-\infty, \infty; \mathcal{M}(\mathbb{R}^2))$ as $k \to \infty$, where the 0-extension of $z(y, s'_k + s)$ is taken where the latter is not defined. If we take even reflection in case $x_0 \in \partial\Omega$, furthermore, the limit measure $\zeta = \zeta(dy, s)$ is a weak solution to

$$
\begin{aligned}
&z_s = \nabla \cdot (\nabla z - z \nabla(w + |y|^2/4)), \\
&\nabla w = \nabla \Gamma * z \qquad \text{in } \mathbb{R}^2 \times (-\infty, \infty).
\end{aligned} \tag{35}
$$

Then, similarly to the pre-scaled case [20], we obtain $\mu \leq 8\pi$ as a necessary condition for the existence of a solution global in time, where

$$\mu = \zeta(\mathbb{R}^2, s) = \begin{cases} m(x_0) & (x_0 \in \Omega), \\ 2m(x_0) & (x_0 \in \partial\Omega). \end{cases} \tag{36}$$

Since $\mu \geq 8\pi$ is obtained by (34), this implies $m(x_0) = m_*(x_0)$. In more detail, first, we show that the assumption $\zeta(\mathbb{R}^2, 0) > 8\pi$ with sufficient concentration of $\zeta(dy, 0)$ at the origin implies the blowup of $\zeta(dy, s)$ in finite time, a contradiction, and then we remove this condition using self-similarity of (35). Actually, this concentration condition is described by the local second moment, which results in $\mu \leq 8\pi$ in the limit of the scaling parameter.

There is, on the other hand, convergence of the global second moment, and this, combined with mass quantization, implies that a type (I) blowup point is impossible [37]. To prove this, we use the key inequality to derive (34),

$$\left| \frac{d}{dt} \int_\Omega u(x,t)\varphi(x)\, dx \right| \leq C\|\varphi\|_{C^2(\overline{\Omega})}$$

with a constant $C > 0$ independent of $\varphi \in C^2(\overline{\Omega})$. More precisely, in (34) we applied this to $\varphi = \varphi_{x_0,R}$, a smooth cut-off function supported by $\overline{B(x_0, R)}$, while now we take the second moment, $\varphi = |x - x_0|^2\varphi_{x_0,R}$. This results in

$$\left| \frac{d}{dt} \int_\Omega |x - x_0|^2 \varphi_{x_0,R}(x) u(x,t)\, dx \right| \leq C.$$

Operating with $\int_t^T \cdot\, dt$, we obtain

$$\int_\Omega |x - x_0|^2 \varphi_{x_0,R}(x) u(x,t)\, dx \leq C(T-t) + \int_\Omega |x - x_0|^2 \varphi_{x_0,R}(x) f(x)\, dx$$

by (32), and therefore, for $R(t) = (T-t)^{1/2}$,

$$\begin{aligned} &\frac{1}{R(t)^2} \int_\Omega |x - x_0|^2 \varphi_{x_0,bR(t)}(x) u(x,t)\, dx \\ &\qquad \leq C + \frac{1}{R(t)^2} \int_\Omega |x - x_0|^2 \varphi_{x_0,bR(t)}(x) f(x)\, dx \leq C + b^2 \langle \varphi_{x_0,bR(t)}, f \rangle. \end{aligned}$$

Given $t_k \uparrow T$, now we take $\{s_k'\} \subset \{s_k\}$ with $s_k' \uparrow \infty$, where $s_k = -\log(T - t_k)$ and then let $b \uparrow \infty$. This implies

$$\langle |y|^2, \zeta(dy, s) \rangle \leq C, \qquad -\infty < s < \infty. \tag{37}$$

Putting

$$I(s) = \langle |y|^2, \zeta(dy, s) \rangle,$$

we obtain

$$\frac{dI}{ds} = 4\mu - \frac{\mu^2}{2\pi} + I$$

by (35), and then from (37) it follows that

$$I(s) = \langle |y|^2, \zeta(dy,s)\rangle = \frac{\mu^2}{2\pi} - 4\mu, \qquad -\infty < s < \infty. \tag{38}$$

This $\zeta(dy,s)$, derived from $s_k = -\log(T-t_k)$, takes the form

$$\zeta(dy,s) = \sum_{y_0 \in \mathcal{B}_s} 8\pi\delta_{y_0}(dy) + g(y,s)\,dy \tag{39}$$

for each $s \in \mathbb{R}$, similarly to the quantized blowup mechanism arising in the blowup solution in infinite time [39, 40]. Here, $\mathcal{B}_s$ is a finite set and $0 \le g = g(\,\cdot\,,s) \in L^1(\mathbb{R}^2) \cap C(\mathbb{R}^2 \setminus \mathcal{B}_s)$. Since $\mu = 8\pi$, (38) implies $I(s) \equiv 0$, and therefore, we obtain $g(y,s)\,dy = 0$, $\sharp(\mathcal{B}_s) = 1$, and $y_0 = 0$. Thus,

$$z(y, s' + s) \rightharpoonup m_*(x_0)\delta_0 \tag{40}$$

in $C_{\mathrm{weak}}(-\infty,\infty;\mathcal{M}(\mathbb{R}^2))$ as $s' \uparrow \infty$, where $z(y,s)$ is the backward self-similar transformation of $u(x,t)$ defined by (33) with 0-extension taken where u is not defined. (It is not the scaling limit $\zeta(dy,s)$, where the even extension is taken with respect to the line parallel to the tangent in case $x_0 \in \partial\Omega$.)

We say that $x_0 \in \mathcal{S}$ is of type (II) if

$$\limsup_{t\uparrow T} R(t)^2 \|u(\,\cdot\,,t)\|_{L^\infty(B(x_0,bR(t))\cap\Omega)} = \infty \tag{41}$$

for some $b > 0$, and of type (I) in the other case, i.e.,

$$\limsup_{t\uparrow T} R(t)^2 \|u(\,\cdot\,,t)\|_{L^\infty(B(x_0,bR(T))\cap\Omega)} < \infty \tag{42}$$

for any $b > 0$. The relation (40) says that the total blowup mechanism is enclosed by parabolic shape hypersurfaces in arbitrarily small parabolic regions. Thus, we obtain

$$\lim_{t\uparrow T} R(t)^2 \|u(\,\cdot\,,t)\|_{L^\infty(B(x_0,bR(t))\cap\Omega)} = \infty$$

for any $b > 0$.

THEOREM 10. *Every $x_0 \in \mathcal{S}$ is of type* (II), *and what is more*, *the relation* (40) *holds in $C_{\mathrm{weak}}(-\infty,\infty;\mathcal{M}(\mathbb{R}^2))$ as $s' \uparrow \infty$.*

The above asymptotic profile has been first observed in the family of blowup solutions constructed by matched asymptotic expansion [16]. These solutions are provided with a super-self-similar blowup rate $0 < r(t) \ll R(t)$, and obey the profile of emergence [42],

$$\lim_{t\uparrow T} \mathcal{F}_{x_0,br(t)}(u(t)) = \infty$$

for any $b > 0$, where

$$\mathcal{F}_{x_0,R}(u) = \int_{\Omega\cap B(x_0,R)} u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\cap B(x_0,R)\times\Omega\cap B(x_0,R)} G(x,x')u(x)u(x')\,dx\,dx'$$

is the local free energy defined by the Green function $G = G(x,x')$ for

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega, \qquad \int_\Omega v = 0, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Thus, emergence is a consequence of mass quantization and blowup envelope.

There is an alternative proof of the collapse mass quantization, assuming type (II) blowup with residual vanishing [41]. For the reader's convenience, we describe the argument to conclude this subsection. First, we use the reverse second moment and show that

$$\frac{d}{ds}\langle(4-|y|^2)_+, \zeta(dy,s)\rangle \geq \langle(4-|y|^2)_+, \zeta(dy,s)\rangle - 8\zeta(\overline{B}_2,s) + \frac{1}{2\pi}\,\zeta(B_2,s)^2,$$

where $B_2 = B(0,2)$. This implies

$$\frac{d}{ds}\langle(4-|y|^2)_+, \nu(dy,s)\rangle \geq \langle(4-|y|^2)_+, \nu(dy,s)\rangle + I(s)$$

for

$$\nu(dy,s) = \zeta(dy,s) - 8\pi\delta_0(dy), \qquad I(s) = 32\pi - 8\zeta(\overline{B}_2,s) + \frac{1}{2\pi}\,\zeta(B_2,s)^2.$$

Next, we note that

$$\langle(4-|y|^2)_+, \nu(dy,0)\rangle > 0 \tag{43}$$

implies $\zeta(B_2,0) > 8\pi$, and consequently, $I(0) > 0$. Then, using a continuation argument, we infer that (43) implies

$$\lim_{s\uparrow\infty}\langle(4-|y|^2)_+, \nu(dy,s)\rangle = \infty,$$

a contradiction. If $0 \in \mathcal{B}_0$, this implies $\nu(dy,0) = 0$, and hence $F(y,0) = 0$ on B_2. Since $F(y,s)$ satisfies a parabolic equation in the residual open set $\bigcup_s(\mathbb{R}^2 \setminus \mathcal{B}_s) \times \{s\}$, we obtain $F \equiv 0$ by the unique continuation theorem.

To treat the other case of $0 \neq y_0 \in \mathcal{B}_0$, we take the moving (reverse second) moment $(4 - |y - y_0(s)|^2)_+$, where $y_0(s) = y_0 e^s$. Using $\zeta'(dy,s) = \zeta(dy + \{y_0(s)\}, s)$, we can argue similarly, and obtain $F(y,s) \equiv 0$ with $\mathcal{B}_0 \cap B(y_0,2) = \emptyset$. Since $\mathcal{B}_0 \neq \emptyset$ if $x_0 \in \mathcal{S}$ is of type (II), this results in

$$\zeta(dy,s) = \sum_{i=1}^n 8\pi\delta_{y_i(s)}(dy),$$

with

$$|y_i(s) - y_j(s)| \geq 2 \quad (i \neq j,\ -\infty < s < \infty). \tag{44}$$

The movement of $y_i(s)$ is detected again by the reverse second moment (cf. [42, p. 318]), and it follows that

$$y_i(s) = y_i(0)e^s, \quad -\infty < s < \infty,$$

which, however, contradicts (44) as $s \downarrow -\infty$.

Aggregation rate (continued). Proposition 3.1 indicates the non-existence of a classical L^1-stationary solution to (35), and this induces an alternative proof of the non-existence of a type (I) blowup point. Since the argument uses the Lyapunov function, we have to put an additional condition

$$\lim_{b\uparrow\infty} \limsup_{t\uparrow T} R(t)^2 \|u(\,\cdot\,,t)\|_{L^\infty(B(x_0,bR(t))\cap\Omega)} < \infty. \tag{45}$$

We say that $x_0 \in \mathcal{S}$ is *uniformly of type (II)* in this case, and then any $s_k \uparrow \infty$ admits $\{s'_k\} \subset \{s_k\}$ generating a classical solution $z = z(y,s) \geq 0$ to (35) satisfying

$$\|z(\,\cdot\,,s)\|_1 = 8\pi, \quad \|z(\,\cdot\,,s)\|_\infty \leq C. \tag{46}$$

We now show this is impossible.

In fact, first, the maximum principle guarantees $z > 0$, and therefore,

$$\liminf_{t\uparrow T} \inf\{R(t)^2 u(x,t) \mid x \in B(x_0, bR(t)) \cap \Omega\} > 0 \tag{47}$$

for any $b > 0$. Next, we apply the transformation [42]

$$\begin{aligned} &z(y,s) = e^{-s}A(y',s'), \quad w(y,s) = B(y',s'),\\ &y' = e^{-s/2}y, \quad s' = -e^{-s}, \end{aligned} \tag{48}$$

and obtain

$$A_{s'} = \nabla' \cdot (\nabla' A - A\nabla' B), \quad \nabla' B = \nabla' \Gamma * A, \quad y' \in \mathbb{R}^2,\ s' < 0. \tag{49}$$

This is nothing but (1) with $\tau = 0$, $n = 2$. The transformation (48), incidentally, preserves the homogeneous Morrey space $\dot{M}_q^{n/2}(\mathbb{R}^n)$, where

$$\begin{aligned} \dot{M}^p_q(\mathbb{R}^n) &= \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) \mid \|f; \dot{M}^p_q\| < \infty\},\\ \|f; \dot{M}^p_q\|^q &= \sup_{x\in\mathbb{R}^n,\, 0<R<\infty} R^{n(q/p-1)} \int_{B(x,R)} |f|^q, \end{aligned}$$

and this suggests the threshold phenomenon concerning the existence of self-similar solutions.

Here, $A = A(dy', s')$ is regarded as an element in $C_*(-\infty, 0; \mathcal{M}(\mathbb{R}^2))$. We have $A(\mathbb{R}^2, s') = 8\pi$ by $\zeta(\mathbb{R}^2, s) = 8\pi$ and also $A(\,\cdot\,, s') \in X_0$, where

$$X_0 = \left\{ f \in X \;\middle|\; f \geq 0,\ \int_{\mathbb{R}^2} f(y)(1+|y|^2)\,dy < \infty \right\}$$

for $X = L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. For the moment, we simplify the notation and take

$$u_t = \nabla\cdot(\nabla u - u\nabla v), \quad \nabla v = \nabla\Gamma * u \quad \text{in } \mathbb{R}^2\times(0,T) \tag{50}$$

in X, i.e., we study the solution $u = u(\,\cdot\,,t) > 0$ to this problem satisfying

$$\sup_{t\in(0,T)} \|u(t)\|_{X_0} < \infty. \tag{51}$$

First, (50) is interpreted via the Duhamel formula as in (6), using

$$\|\nabla\Gamma * u\|_\infty \le C(\|u\|_\infty + \|u\|_1),$$

and then it follows that

$$\|\Delta^{1/4}u(t)\|_2 \le Ct^{-1/4}, \quad 0<t<T. \tag{52}$$

Since Calderón–Zygmund's estimate guarantees [44]

$$\|D^\alpha\Gamma * u\|_p \le C_p\|u\|_p \quad (|\alpha| = 2,\ 1<p<\infty),$$

it follows that

$$\|\Delta^{1/2}(u_1\cdot\nabla\Gamma * u_2)\|_2 \le C\|(1-\Delta)^{1/2}u_1\|_2\|u_2\|_X$$

and then the interpolation theorem implies

$$\|\Delta^{1/4}(u\cdot\nabla\Gamma * u)\|_2 \le C\|(1-\Delta)^{1/4}u\|_2\|u\|_X \tag{53}$$

since

$$\|u_1\cdot\nabla\Gamma * u_2\|_2 \le C\|u_1\|_2\|u_2\|_X.$$

From (6), (51), (52), (53), we obtain

$$\|\nabla u(t)\|_2 \le Ct^{-1/2}, \quad 0<t<T,$$

and hence

$$\frac{d}{dt}(u,\varphi) = -(\nabla u - u\nabla v, \nabla\varphi)$$

for $\varphi \in H^1(\mathbb{R}^2)$ and $t>0$. Then $\|u(t)\|_1 = \|u_0\|_1 = \lambda$ is obtained by approximating $\varphi \equiv 1$. Similarly, we obtain

$$\|\Delta^{3/4}u(t)\|_2 \le Ct^{-3/4}, \quad \|\Delta u(t)\|_2 \le Ct^{-1}, \quad \|u_t(t)\|_2 \le Ct^{-1}, \quad 0<t<T,$$

which provides sufficient regularity to $z = z(y,s)$.

The logarithmic HLS inequality [12], on the other hand, guarantees

$$-\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\log|x-y|f(y)\,dx\,dy \le \frac{\lambda}{2}\int_{\mathbb{R}^2} f\log f\,dx + C_\lambda,$$

$$C_\lambda = -\frac{\lambda^2}{2}\,(1+\log\pi+\log\lambda)$$

for $f\in X_0$ with $\|f\|_1 = \lambda$. Since

$$\frac{1}{2}\,\frac{d}{dt}\langle\Gamma_\varepsilon * z, z\rangle = (z_t, \Gamma_\varepsilon * z)$$

for $\Gamma_\varepsilon \in C_0^\infty(\mathbb{R}^2)$, we can justify

$$\frac{1}{2}\frac{d}{dt}\langle \Gamma * z, z\rangle = (z_t, w).$$

Next, we use

$$(z_t, H(z) - |y|^2/4) = -(\nabla z - z\nabla(|y|^2/4), \nabla(H(z) + |y|^2/4))$$

for $H \in C_0^\infty[0,\infty)$ and approximate $\log u$ by this $H(z)$. Thus, we obtain $\nabla\sqrt{z} \in L^2(\mathbb{R}^2 \times (-s_0, s_0))$ for any $s_0 > 0$, and also

$$\int_{-s}^{s} ds \int_{\mathbb{R}^2} |2\nabla\sqrt{z} - \sqrt{z}\nabla(w + |y|^2/4)|^2\, dy \le \mathcal{F}(z(-s)) - \mathcal{F}(z(s))$$

for $s > 0$, where

$$\mathcal{F}(z) = \int_{\mathbb{R}^2} z(\log z - 1) - \frac{1}{2}\langle \Gamma * z, z\rangle + \frac{1}{4}\int_{\mathbb{R}^2} |y|^2 z.$$

Then the uniform estimate (46) guarantees

$$\int_{-\infty}^{\infty} ds \int_{\mathbb{R}^2} |2\nabla\sqrt{z} - \sqrt{z}\nabla(w + |y|^2/4)|^2\, dy < \infty$$

and in particular, there is $s_k \to \infty$ satisfying

$$\int_{\mathbb{R}^2} |2\nabla\sqrt{z} - \sqrt{z}\nabla(w + |y|^2/4)|^2(\,\cdot\,, s_k)\, dy \to 0. \tag{54}$$

Finally, parabolic regularity is applicable to (35) by (46), and there are $\{s_k'\} \subset \{s_k\}$ and z_∞, w_∞ such that

$$z(\,\cdot\,, s_k') \to z_\infty, \qquad \nabla w(\,\cdot\,, s_k') \to \nabla w_\infty$$

locally uniformly in $\mathbb{R}^2$ such that

$$\|z_\infty\|_1 \le 8\pi, \qquad z_\infty \in L^\infty(\mathbb{R}^2), \qquad \nabla w_\infty = \nabla\Gamma * z_\infty.$$

Furthermore, we obtain $z_\infty > 0$ by (47), and therefore (54) yields

$$\log z_\infty - w_\infty - |y|^2/4 = \text{constant} \quad \text{in } \mathbb{R}^2.$$

This implies

$$-\Delta w_\infty = \frac{\lambda e^{|y|^2/4} e^{w_\infty}}{\int_{\mathbb{R}^2} e^{|y|^2/4} e^{w_\infty}}, \qquad y \in \mathbb{R}^2,$$

$$\lambda = \|z_\infty\|_1 \le 8\pi, \qquad \int_{\mathbb{R}^2} e^{|y|^2/4} e^{w_\infty} < \infty,$$

which is impossible by Proposition 3.1.

The assumption (45) may be weakened in this argument, but at least $z \log z(\,\cdot\,, s) \in L^1(\mathbb{R}^2)$ is necessary.

Existence. The case $\tau = 0$ is mostly studied for (22) with $n \geq 3$. If $n = 3$, there is a family of radially symmetric backward self-similar solutions $\{(u_k, v_k)\}$ to (1) satisfying $u_k(r, T) \sim c_k r^{-2}$ as $r \downarrow 0$ for $c_k \downarrow 2$. It is obtained formally by the method of matched asymptotic expansion [15]. Here, we describe the method of phase plane, first adopted in the study of stationary solutions [7].

In fact, a radially symmetric stationary solution to (1) satisfies

$$\begin{aligned} 0 &= \nabla \cdot (\nabla u - u \nabla v), \\ 0 &= \Delta v + u, \qquad x \in \mathbb{R}^n. \end{aligned}$$

If $u = u(r)$ is such a solution, then

$$\Phi(r) = \frac{1}{r^{n-2}} \int_0^r s^{n-1} u(s)\, ds$$

solves

$$\begin{aligned} &\Phi'' + \frac{n-3}{r}\Phi' - \frac{2(n-2)}{r^2}\Phi + \frac{\Phi}{r^2}\{(n-2)\Phi + r\Phi'\} = 0, \qquad r > 0, \\ &\Phi(0) = \Phi'(0) = 0. \end{aligned} \tag{55}$$

Using $V(s) = \Phi(r) - 2$, $s = \log r$, we obtain

$$\begin{aligned} &V'' + (n-2)V' + (n-2)(V+2)V + VV' = 0, \qquad s \in \mathbb{R}, \\ &V(-\infty) = 2, \qquad V'(-\infty) = 0. \end{aligned}$$

This is written as

$$V' = -(n-2)V + W, \qquad W' = -2(n-2)V - WV, \tag{56}$$

where $W = (n-2)V + V'$. Equilibrium points of this system are

$$(V, W) = (0, 0), \qquad (V, W) = (-2, -2(n-2)),$$

and

$$L(V, W) = \frac{1}{2}V^2 + W + \log \frac{W + 2(n-2)}{2(n-2)}$$

is a Lyapunov function:

$$\frac{d}{ds} L(V, W) = -(n-2)V^2 \leq 0.$$

Then (56) with $V(-\infty) = -2$, $V'(-\infty) = 0$ generates a heteroclinic orbit $\mathcal{O}$, and the linearized eigenvalues around $(0, 0)$ are

$$\mu_\pm = -\frac{n-2}{2} \pm \frac{\sqrt{(n-2)(n-10)}}{2}.$$

Thus, $\mathcal{O}$ spirals to $(0, 0)$ if and only if $2 < n < 10$.

We apply an analogous argument for the backward self-similar solution (u,v) defined by (4) using the profile functions (ϕ,ψ). More precisely, we introduce

$$\begin{aligned}\Phi(r,t) &= \frac{1}{r^{n-2}}\int_0^r s^{n-1}u(s,t)\,ds = \frac{1}{r^{n-2}(T-t)}\int_0^r s^{n-1}\phi\left(\frac{s}{\sqrt{T-t}}\right)ds\\ &= \left(\frac{r}{\sqrt{T-t}}\right)^{2-n}\int_0^{r/\sqrt{T-t}} \xi^{n-1}\phi(\xi)\,d\xi\end{aligned}$$

and define

$$\Psi(s) = \Phi(r,t), \qquad s = \frac{r}{\sqrt{T-t}}.$$

This means

$$\Psi(s) = \frac{1}{s^{n-2}}\int_0^s t^{n-2}\phi(t)\,dt$$

and then it follows that

$$\begin{aligned}&\Psi'' + \left(\frac{n-3}{2} - \frac{s}{2}\right)\Psi' - \frac{2(n-2)}{s^2}\,\Psi + \frac{\Psi}{s^2}\,\{(n-2)\Psi + s\Psi'\} = 0, \quad s>0,\\ &\Psi(0) = \Psi'(0) = 0.\end{aligned} \tag{57}$$

These relations are summarized by

$$\begin{aligned}&r^2u(r,t) = \frac{r^2}{\sqrt{T-t}}\,\phi\left(\frac{r}{\sqrt{T-t}}\right) = s^2\phi(s), \qquad s = \frac{r}{\sqrt{T-t}},\\ &s^2\phi(s) = s\Phi'(s) + (n-2)\Phi(s),\end{aligned} \tag{58}$$

and $s\uparrow\infty$ if and only if $t\uparrow T$ for fixed $r>0$.

Equation (57) has the exact solution [35]

$$\Psi(s) = \frac{4s^2}{2(n-2)+s^2}$$

and from this we obtain the following fact.

PROPOSITION 3.2. *If $n\ge 3$, $\tau = 0$, then* (1) *admits a backward self-similar solution*

$$u(r,t) = \frac{1}{T-t}\,\phi\left(\frac{r}{\sqrt{T-t}}\right), \qquad \phi(r) = \frac{16(n-2)}{(2(n-2)+r^2)^2} + \frac{4(n-2)}{2(n-2)+r^2}.$$

Here,

$$r^2u(r,t)\to 4(n-2) \qquad \text{as } t\uparrow T.$$

To detect other solutions, we put $\Psi(s) = W(t)$ for $s = \alpha t$, where $\alpha>0$.

Then (57) reads

$$\begin{aligned} W'' + \left(\frac{n-3}{t} - \frac{\alpha^2 t}{2}\right) W' - \frac{2(n-2)}{t^2} W \\ + \frac{W}{t^2}\{(n-2)W + tW'\} = 0, \quad t > 0, \end{aligned}$$

$$W(0) = W'(0) = 0,$$

and (55) is regarded as the limiting equation as $\alpha \downarrow 0$. If $3 \le n \le 9$, we obtain $\alpha_k \downarrow 0$ and the corresponding $W_k = W_k(t)$ such that

$$\begin{aligned} &W_k(t) > 0, \quad t > 0, \\ &W_k(t) - 2 \text{ has } 2k \text{ zeros in } t \in (0, \infty), \\ &\lim_{t \uparrow \infty} W_k(t) = c_k \in (0, 2). \end{aligned}$$

Then the following theorem is obtained [35].

THEOREM 11. *In the case of* $3 \le n \le 9$, *there is a family* $\{(u_k, v_k)\}$ *of radially symmetric backward self-similar solutions to* (1) *such that*

$$r^2 u(r,t) \to (n-2)c_k \quad \text{as } t \uparrow T,$$

where $0 < c_k < 2$.

If $n \ge 10$, there is a blowup rate higher than the backward self-similar solution, i.e., type (II) blowup rate [36].

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