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ON SPACES WITH THE IDEAL CONVERGENCE PROPERTY

BY

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Abstract. Let $I \subseteq P(\omega)$ be an ideal. We continue our investigation of the class of spaces with the I-ideal convergence property, denoted $\mathcal{IC}(I)$. We show that if I is an analytic, non-countably generated P-ideal then $\mathcal{IC}(I) \subseteq s_0$. If in addition I is non-pathological and not isomorphic to I_b, then $\mathcal{IC}(I)$ spaces have measure zero. We also present a characterization of the $\mathcal{IC}(I)$ spaces using clopen covers.

1. Introduction. Throughout this paper X is a separable metric space and $I \subseteq \mathcal{P}(\omega)$ is an ideal on ω containing all finite subsets of ω. The power set $\mathcal{P}(\omega)$ is considered to be a topological space with the product topology induced from 2^ω by identifying subsets of ω with their characteristic functions. We assume that $\mathcal{P}(\omega)$ is a closed subset of the interval $[0,1]$. Recall that an ideal $I \subseteq \mathcal{P}(\omega)$ is called a *P-ideal* if whenever $A_0, A_1, A_2, \ldots \in I$ is a sequence of sets then there exists a set $A_\infty \in I$ such that $A_n \subseteq^* A_\infty$ for all $n < \omega$, i.e., $|A_n \setminus A_\infty| < \omega$. We are especially interested in the analytic P-ideals $I \subseteq \mathcal{P}(\omega)$ because of Solecki's theorem stating in particular that for any such ideal there exists a finite lower semicontinuous submeasure $\varphi : \mathcal{P}(\omega) \to [0,\infty]$ such that $I = \mathrm{Exh}(\varphi) := \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\}$ (see [6, Theorem 3.1]).

We say that a sequence of functions $f_n : X \to \mathbb{R}$ is *I-convergent* to a function $g : X \to \mathbb{R}$, denoted $I\text{-}\lim f_n = g$, if for every $\varepsilon > 0$ and every $x \in X$ the set $\{n \in \omega : |f_n(x) - g(x)| \ge \varepsilon\} \in I$. In [3] the authors studied spaces X where I-convergence of sequences of continuous functions implies pointwise convergence of a subsequence indexed by elements of a set from the dual filter $\mathcal{F}(I) := \{B \subseteq \omega : B^c \in I\}$. More specifically, recall

Definition 1. Let I be an ideal on ω and let X be a separable metric space. We say that X has the *I-ideal convergence property* if whenever $f_n : X \to \mathbb{R}$ is a sequence of continuous functions I-convergent to the zero function then there exists a set $M = \{m_0 < m_1 < m_2 < \cdots\} \in \mathcal{F}(I)$ such that for all $x \in X$, $\lim_{i\to\infty} f_{m_i}(x) = 0$. The class of all spaces with the I-ideal

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convergence property is denoted by $\mathcal{IC}(I)$ and the set of all subspaces of X with the I-ideal convergence property is denoted by $\mathcal{IC}_X(I)$.

In [3] we said that an ideal $I \subseteq \mathcal{P}(\omega)$ is *generated by a set* $C \subseteq \omega$ if $I = I_C := \{A \subseteq \omega : A \subseteq^* C\}$. It is easy to show (see part $\Leftarrow$ of the proof of Proposition 2 of [3]) that for any space X, if I is generated by a single set then $\mathcal{IC}_X(I) = \mathcal{P}(X)$. Clearly, if I is a P-ideal which is *countably generated*, i.e., there exists a sequence $A_0, A_1, A_2, \ldots \subseteq \omega$ such that $I = \{A \subseteq \omega : A \subseteq^* \bigcup_{n<\omega} A_n\}$, then I is also generated by a single set. For a characterization of countably generated ideals see Proposition 1.2.8 of [2].

In this note we show that it is consistent that for all analytic, non-countably generated, non-pathological P-ideals I the class $\mathcal{IC}(I)$ contains countable spaces only. This result should be viewed in the context of Corollary 5 of [3] stating that under CH, for any analytic P-ideal I there exists an uncountable space X in $\mathcal{IC}(I)$. Note that the class of analytic, non-countably generated, non-pathological P-ideals is very broad and includes most ideals discussed in the literature including *summable* and *Erdős–Ulam ideals* (see comments following Corollary 1.9.4, p. 31 of [2]). In Theorem 3 we give a characterization of spaces with I-ideal convergence property using clopen covers.

2. Main results. For the readers' convenience we recall a few more definitions and simple facts. A set $X \subseteq \mathbb{R}$ is called a *σ-set* (or $X \in \sigma$) if every relative G_δ subset of X is also a relative F_σ set in X (see [5, p. 210]). A mapping $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is called a *submeasure* if $\varphi(\emptyset) = 0$ and $\varphi(A) \le \varphi(A \cup B) \le \varphi(A) + \varphi(B)$ (see [2, p. 20]). A submeasure φ is called *lower semicontinuous* if for all $A \subseteq \omega$, $\lim_{n\to\infty} \varphi(A \cap n) = \varphi(A)$. Since $\mathbb{R}$ contains a closed subset homeomorphic to $\mathcal{P}(\omega)$, to simplify the notation we assume that $\mathcal{P}(\omega) \subseteq [0,1] \subseteq \mathbb{R}$. For any two integers $n_1 \le n_2 < \omega$, $\langle n_1, n_2\rangle := \{k < \omega : n_1 \le k \le n_2\} = [n_1, n_2] \cap \omega$, and for $m < \omega$ we identify m with $\langle 0, m-1\rangle$. Following the idea in the proof of Proposition 2 of [3] we prove the following lemma.

LEMMA 1. *Let $I \subseteq \mathcal{P}(\omega)$ be an ideal and let $X \subseteq I$ be such that $X \in \mathcal{IC}(I)$. Then there exists a set $C \in I$ such that $X \subseteq I_C \subseteq I$. In particular, if $X = I$ then $X = I_C$.*

Proof. Let $g_n : X \to \mathbb{R}$ be defined as follows:

$$g_n(A) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A. \end{cases}$$

It is easy to verify that each g_n is continuous as both inverse images, $g_n^{-1}[\{1\}]$ and $g_n^{-1}[\{0\}]$, are open in X. For $A \in X$ we have $\{n : g_n(A) > 0\} = \{n : g_n(A) = 1\} = \{n : n \in A\} = A \in I$, so $I\text{-}\lim g_n = 0$. Since $X \in \mathcal{IC}(I)$ there exists a set $M=\{m_0 < m_1 < m_2 < \cdots\} \in \mathcal{F}(I)$ such that $\lim_{i\to\infty} g_{m_i}(A)=0$ for all $A \in X$. So for all $A \in X$, $g_{m_i}(A) = 0$ for sufficiently large i, hence for those i, $m_i \notin A$. It follows that $M \subseteq^* A^c$ and consequently $A \subseteq^* M^c$. Setting $C = M^c$ we obtain $X \subseteq I_C$. ∎

LEMMA 2. *If I is an analytic, non-countably generated P-ideal on ω then $\mathcal{IC}_{\mathbb{R}}(I)$ does not contain intervals.*

Proof. Let $\varphi : \mathcal{P}(\omega) \to [0,\infty]$ be a finite lower semicontinuous submeasure on $\mathcal{P}(\omega)$ such that

$$I = \operatorname{Exh}(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\} \tag{2.1}$$

(see [6, Theorem 3.1]). Define a descending sequence of sets $\{A_k : k < \omega\}$ as follows: $A_0 = \omega$, $A_k = \{n < \omega : \varphi(\{n\}) \le 1/k\}$. We shall consider the limit $\lim_{k\to\infty} \lim_{n\to\infty} \varphi(A_k \setminus n)$.

CASE 1. Assume

$$\lim_{k\to\infty} \lim_{n\to\infty} \varphi(A_k \setminus n) = 0 \tag{2.2}$$

but

$$\text{for every } k, \qquad \lim_{n\to\infty} \varphi(A_k \setminus n) > 0. \tag{2.3}$$

For every $k < \omega$ there exists a $t > k$ such that $|A_k \setminus A_t| = \omega$. Otherwise there would be a k_0 such that $\lim_{n\to\infty} \varphi(A_k \setminus n) = \lim_{n\to\infty} \varphi(A_{k_0} \setminus n)$ for all $k \ge k_0$, contradicting (2.2). Define an increasing sequence k_l, $l < \omega$, as follows: $k_0 = 0$, $k_{l+1} = \min\{k : |A_{k_l} \setminus A_k| = \omega\}$. Set $B_l = A_{k_l} \setminus A_{k_{l+1}}$.

CLAIM. $I = \{A : \forall l\ |B_l \cap A| < \omega\}$.

If $A \in I$ then $\lim_{n\to\infty} \varphi(A\setminus n) = 0$ so $|B_l \cap A| < \omega$ because otherwise we would have $\lim_{n\to\infty} \varphi(A\setminus n) \ge 1/k_{l+1}$. On the other hand, if all sets $B_l \cap A$ are finite then for every k there exists an n such that $A \setminus n \subseteq A_k$. It follows that $\lim_{n\to\infty} \varphi(A \setminus n) \le \lim_{k\to\infty} \lim_{n\to\infty} \varphi(A_k \setminus n) = 0$, which proves the Claim.

This shows that there exists a bijection $\alpha : \omega \to \omega^2$ such that $\{\alpha[A] : A \in I\} = I_b$ where

$$I_b := \{A \subseteq \omega^2 : \forall n < \omega\ \exists m < \omega\ \forall k < \omega\ ((n,k) \in A \Rightarrow k \le m)\}. \tag{2.4}$$

By Theorem 4 of [3] it follows that $\mathcal{IC}_{\mathbb{R}}(I) = \mathcal{IC}_{\mathbb{R}}(I_b) \subseteq \sigma$. Intervals are not σ-sets [5], hence the assertion of Lemma 2 in Case 1 is proved.

CASE 2. Assume that

$$\lim_{k\to\infty} \lim_{n\to\infty} \varphi(A_k \setminus n) = 0 \tag{2.5}$$

and

$$\text{there exists a } k_0 \text{ such that } \lim_{n\to\infty} \varphi(A_{k_0} \setminus n) = 0. \tag{2.6}$$

Then $A_{k_0} \in I$ and I is countably generated because $I = \{A : A \subseteq^* A_{k_0}\}$. The inclusion $I \supseteq \{A : A \subseteq^* A_{k_0}\}$ is clear while if $|A \setminus A_{k_0}| = \omega$ then $\liminf_{n\to\infty} \varphi(A \setminus n) \geq 1/k_0$ so $A \notin I$.

CASE 3. Finally assume that

$$\lim_{k\to\infty} \lim_{n\to\infty} \varphi(A_k \setminus n) = \varepsilon > 0. \tag{2.7}$$

We recursively define a sequence $\{n_k : k < \omega\} \subseteq \omega$ as follows. Since $\lim_{n\to\infty} \varphi(A_0 \setminus n) \geq \varepsilon$ there exists an integer n_0 such that for all $n \geq n_0$, $\varphi(A_0 \setminus n) > \varepsilon/2$. Now suppose we have defined the terms $n_0, n_1, \ldots, n_k$ such that for all $j = 1, \ldots, k$, if $n > n_j$ then $\varphi(A_j \setminus \langle n_{j-1}, n\rangle) > \varepsilon/2$. Since $\lim_{n\to\infty} \varphi(A_{k+1} \setminus n) \geq \varepsilon$ and $\varphi(A_{k+1} \setminus n) \geq \varphi(A_{k+1} \setminus (n+1))$ for all $n \in \omega$, we have $\varphi(A_{k+1} \setminus n_k) > \varepsilon/2$. By the lower semicontinuity of φ, $\lim_{n\to\infty} \varphi((A_{k+1} \setminus n_k) \cap n) = \varphi(A_{k+1} \setminus n_k)$ so there exists an integer $n_{k+1} \geq k+1$ such that for all $n \geq n_{k+1}$, $\varphi(A_{k+1} \cap \langle n_k, n\rangle) > \varepsilon/2$. Also by lower semicontinuity the set $Z = \{A : \forall k > 0\ (\varphi(A \setminus n_k) \leq 2/k)\}$ is a closed subset of $\mathcal{P}(\omega)$ and by (2.1), $Z \subseteq I$.

Now towards a contradiction, based on Proposition 3 of [3], without loss of generality we may assume that $[0,1] \in \mathcal{IC}_{\mathbb{R}}(I)$. By the same Proposition 3(2), $\mathcal{IC}_{\mathbb{R}}(I)$ also contains Z. By Lemma 1 there exists a set $C \in I$ such that $Z \subseteq \{A : A \subseteq^* C\}$. As $C \in I$, by (2.1) let $l_0 \in \omega$ be such that $\varphi(C \setminus m) < \varepsilon/2$ for all $m \geq 2^{l_0}$. Since for $l < \omega$, $\varphi(A_{2^l+1} \cap \langle n_{2^l}, n_{2^l+1}\rangle) > \varepsilon/2$, we may select numbers $d_l \in (A_{2^l+1} \cap \langle n_{2^l}, n_{2^l+1}\rangle) \setminus C$ and define the set $D = \{d_l : l \geq l_0\}$. As D is infinite and disjoint from C, clearly $D \not\subseteq^* C$. In particular

$$D \notin Z. \tag{2.8}$$

Now for $k > 0$ let $l_k = \max\{l_0, \lceil \log_2 n_k \rceil\}$. We have

$$\varphi(D \setminus n_k) \leq \sum_{l \geq l_k} \varphi(\{d_l\}) \leq \sum_{l \geq l_k} \frac{1}{2^l+1} \leq \frac{2}{2^{\lceil \log_2 n_k \rceil}} \leq \frac{2}{n_k} \leq \frac{2}{k}.$$

It follows that $D \in Z$, contradicting (2.8). ■

The family $\mathcal{IC}_{\mathbb{R}}(I)$ is closed under continuous images (Proposition 3(1) of [3]). Propositions 2.6.1 and 2.6.13 of [7] imply the following:

REMARK 1. If I is as in Lemma 2 and $X \in \mathcal{IC}_{\mathbb{R}}(I)$ then X is totally imperfect (i.e., does not contain any perfect sets).

A subset $X \subseteq \mathbb{R}$ is called an *s_0-set* (or $X \in s_0$) if for every perfect subset $P \subseteq \mathbb{R}$ there exists another perfect subset $Q \subseteq P \setminus X$ (see [5, p. 217]).

THEOREM 1. *If I is an analytic, non-countably generated P-ideal on ω then $\mathcal{IC}_{\mathbb{R}}(I) \subseteq s_0$.*

Proof. Suppose $X \in \mathcal{IC}_{\mathbb{R}}(I)$ and let $P \subseteq \mathbb{R}$ be a perfect set. Let $P_1 \subseteq P$ be a perfect set homeomorphic to the Cantor set (see [7, Theorem 2.6.3]).

Let $h : P_1 \to P_1 \times P_1$ be a homeomorphism and let $p : P_1 \times P_1 \to P_1$ be the projection, $p(x, y) = x$. It is well known that there is a continuous surjection $g : P_1 \to [0, 1]$ (see [7, Theorem 2.6.13]). The composition $q = g \circ p \circ h : P_1 \to [0, 1]$ is a continuous surjection so $q[P_1 \cap X] \in \mathcal{IC}_{\mathbb{R}}(I)$ by Proposition 3 of [3]. By Lemma 2 there exists $y \in [0, 1] \setminus q[P_1 \cap X]$. It is easy to see that $Q = q^{-1}[\{y\}]$ is a perfect subset of P disjoint from X. ∎

Under the Continuum Hypothesis we have an example of a non-countably generated, non-analytic P-ideal J such that $\mathcal{IC}(J)$ contains large spaces.

EXAMPLE 1. (*CH*) *There exists a maximal P-ideal J such that $\mathbb{R} \in \mathcal{IC}(J)$.*

Proof. We will construct a sequence of subsets $X_\alpha \in [\omega]^\omega$, $\alpha < \mathfrak{c}$, such that $X_\alpha \subseteq^* X_\beta$ whenever $\alpha < \beta < \mathfrak{c}$. The sequence $\{X_\alpha : \alpha < \mathfrak{c}\}$ will be such that the ideal J dual to the filter $\mathcal{F} = \{B \subseteq \omega : \exists\alpha < \mathfrak{c}\ (X_\alpha \subseteq^* B)\}$ will have the desired properties.

Let $\{A_{\alpha+1} : \alpha<\mathfrak{c}\}$ be a sequence of all subsets of ω and $\{\langle f_n^{\alpha+1} : n<\omega\rangle : \alpha < \mathfrak{c}\}$ be an indexed family of all sequences of continuous functions $f_n^{\alpha+1} : \mathbb{R} \to \mathbb{R}$. Set $X_0 = \omega$. Suppose that for some $\alpha < \mathfrak{c}$ the sets X_β, $\beta \le \alpha$, are defined. Define $X'_{\alpha+1}$ as follows. If the sequence $\langle f_n^\alpha : n \in X_\alpha\rangle$ is pointwise convergent to the zero function then we set $X'_{\alpha+1} = X_\alpha$. Otherwise there exists an $\varepsilon > 0$ and $x \in \mathbb{R}$ such that $E = \{n : |f_n^\alpha(x)| \ge \varepsilon\}$ is an infinite subset of X_α. In that case we set $X'_{\alpha+1} = E$. Now to obtain $X_{\alpha+1}$ we consider the intersection $X'_{\alpha+1} \cap A_\alpha$. If it is infinite then let $X_{\alpha+1} = X'_{\alpha+1} \cap A_\alpha$. Otherwise set $X_{\alpha+1} = X'_{\alpha+1} \cap (\omega \setminus A_\alpha)$.

To finish the proof we need to define the X_λ for limit ordinals $0 < \lambda < \mathfrak{c}$. Let $\{\gamma_n : n < \omega\} \subseteq \lambda$ be a sequence of ordinals cofinal in λ. For each $n < \omega$ let $Y_n = \bigcap_{k\le n} X_{\gamma_k}$. We have $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$ and each Y_n is infinite. Recursively pick points $x_n \in Y_n \setminus \{x_m : m < n\}$ and let $X_\lambda = \{x_n : n < \omega\}$. It is easy to verify that $X_\lambda \subseteq^* X_\alpha$ for all $\alpha < \lambda$.

Now having defined the sequence $\{X_\alpha : \alpha < \mathfrak{c}\}$, it is easy to show that the ideal J dual to the filter $\mathcal{F} = \mathcal{F}(J) = \{B \subseteq \omega : \exists\alpha < \mathfrak{c}\ (X_\alpha \subseteq^* B)\}$ is a maximal P-ideal such that $\mathbb{R} \in \mathcal{IC}(J)$. ∎

Recall that a measure on $\mathcal{P}(\omega)$ is a submeasure which is additive, i.e., $\mu : \mathcal{P}(\omega) \to [0, \infty]$ is a *measure* if $\mu(\emptyset) = 0$, $\mu(A) \le \mu(A\cup B)$, and $\mu(A\cup B) = \mu(A) + \mu(B)$ for any two disjoint sets $A, B \in \mathcal{P}(\omega)$. Before stating our next theorem we recall a definition from [2, p. 21]. We say that a submeasure $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is *non-pathological* if

$$\varphi(A) = \sup\{\mu(A) : \mu \text{ is a measure on } \mathcal{P}(\omega) \text{ and } \forall B \subseteq \omega\ (\mu(B) \le \varphi(B))\}. \tag{2.9}$$

An ideal $I \subseteq \mathcal{P}(\omega)$ is *non-pathological* if $I = \mathrm{Exh}(\varphi)$ for some non-pathological submeasure φ. Let λ and λ^* be the Lebesgue measure and the outer Lebesgue measure respectively. Set $\mathcal{N} = \{Y \subseteq \mathbb{R} : \lambda(Y) = 0\}$ and recall I_b is as in 2.4.

THEOREM 2. *If I is an analytic, non-pathological, non-countably generated P-ideal not isomorphic to I_b, then $\mathcal{IC}_{\mathbb{R}}(I) \subseteq \mathcal{N}$.*

Proof. Suppose $I = \mathrm{Exh}(\varphi)$ for some non-pathological submeasure φ. Let $X \in \mathcal{IC}_{\mathbb{R}}(I)$. By Lemma 2, X is zero-dimensional and we may assume that $X \subseteq [0,1]$. Similarly to (but not exactly as in) the proof of Lemma 2 we define a sequence of sets $A_0, A_1, A_2, \ldots$ as $A_k = \{n < \omega : \varphi(\{n\}) \leq 1/2^k\}$ and again consider the limit (2.2). By the proof of Lemma 2 our assumptions on I imply that

$$\lim_{k\to\infty} \lim_{n\to\infty} \varphi(A_k \setminus n) = \varepsilon \quad \text{ for some } \varepsilon > 0.$$

Let $\{n_k : k < \omega\}$ be defined exactly as in Case 3 of Lemma 2. Define $I_k = A_k \cap [n_k, n_{k+1} - 1]$ and let μ_k be a measure on $\mathcal{P}(\omega)$ such that $\mu_k \leq \varphi$ and $\mu_k(I_k) > \frac{1}{2}\varphi(I_k)$ (see (2.9)). For each k we form a partition of X into open (in X) sets $\{U_{ki} : i \in I_k\}$ such that $\lambda^*(U_{ki}) < 2\mu_k(\{i\})/\mu_k(I_k)$. We define $F : X \to \mathcal{P}(\omega)$ by $F(x) = \{i : \exists k\ (x \in U_{ki})\}$. Since the sets I_k are finite and pairwise disjoint the function F is continuous and we have $F[X] \in \mathcal{IC}_{\mathbb{R}}(I)$ by Proposition 3(1) of [3]. Also for each $x \in X$, $F(x) \in I$ because $\lim_{n\to\infty} \varphi(F(x) \setminus n) = 0$. This is due to the fact that we are using $1/2^k$ in the definition of A_k and $|F(x) \cap (A_k \setminus A_{k+1})| \leq 1$. So with $F[X] \subseteq I$ and $F[X] \in \mathcal{IC}_{\mathbb{R}}(I)$ by Lemma 1 there exists a set $C \in I$ such that $F(x) \subseteq^* C$ for all $x \in X$. It follows that

$$X \subseteq \bigcup_{n<\omega} \bigcap_{k>n} \bigcup_{i\in C\cap I_k} U_{ki}.$$

Notice that for any $k < \omega$,

$$\begin{aligned}\lambda^*\Big(\bigcup_{i\in C\cap I_k} U_{ki}\Big) &\leq \sum \lambda^*_{i\in C\cap I_k}(U_{ki}) \leq \sum_{i\in C\cap I_k} 2\mu_k(\{i\})/\mu_k(I_k) \\ &= 2\mu_k(C\cap I_k)/\mu_k(I_k) \leq 2\mu_k(C\cap I_k)/(\varepsilon/2) \\ &\leq 4\varphi(C\setminus n_k)/\varepsilon.\end{aligned}$$

Since $C \in I$ the last quantity converges to zero as $k \to \infty$. It follows that

$$\lambda^*\Big(\bigcap_{k>n} \bigcup_{i\in C\cap I_k} U_{ki}\Big) = 0$$

and $\lambda^*(X) = 0$ as well. ∎

COROLLARY 1. *It is consistent with ZFC that if I is an analytic, non-pathological, non-countably generated P-ideal, then $\mathcal{IC}_{\mathbb{R}}(I)$ contains countable sets only.*

Proof. Suppose $X \in \mathcal{IC}_{\mathbb{R}}(I)$. If I is not isomorphic to I_b then by Theorem 2 and Proposition 3(1) of [3] all continuous images of X are also in $\mathcal{IC}_{\mathbb{R}}(I)$ and they have Lebesgue measure zero. Bartoszyński and Shelah [1] showed that consistently such spaces may be countable only. If I is isomorphic to I_b then see Corollary 7 of [3]. ∎

Recall $\mathrm{add}^*(I) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq I \text{ and } \nexists B \in I\ \forall A \in \mathcal{A}\ (A \subseteq^* B)\}$ and $\mathrm{non}(\mathcal{N}) := \min\{|X| : X \subseteq \mathbb{R},\ \lambda^*(X) > 0\}$. By Proposition 4 of [3] for P-ideals I if $X \subseteq \mathbb{R}$ with $|X| < \mathrm{add}^*(I)$ then $X \in \mathcal{IC}_{\mathbb{R}}(I)$. By Theorem 2 above we obtain

COROLLARY 2. *If I is an analytic, non-pathological, non-countably generated P-ideal non-isomorphic to I_b, then* $\mathrm{add}^*(I) \leq \mathrm{non}(\mathcal{N})$.

Our last theorem gives a characterization of spaces with the I-ideal convergence property using clopen covers.

THEOREM 3. *Suppose I is a P-ideal on ω and let X be a separable, zero-dimensional metric space. Then $X \in \mathcal{IC}(I)$ if and only if for every sequence of clopen sets $U_0, U_1, U_2, \ldots \subseteq X$ with $\{n : x \notin U_n\} \in I$ for any $x \in X$, there exists a set $A \in \mathcal{F}(I)$ such that*

$$X \subseteq \bigcup_{m<\omega} \bigcap_{a \in A \setminus m} U_a.$$

Proof. Assume that $X \in \mathcal{IC}(I)$. Let $U_0, U_1, U_2, \ldots \subseteq X$ be a sequence of clopen sets such that for any $x \in X$,

$$\{n : x \notin U_n\} \in I. \tag{2.10}$$

Define a sequence of functions $h_n : X \to \mathbb{R}$ by

$$h_n(x) = \begin{cases} 0 & \text{if } x \in U_n, \\ 1 & \text{if } x \notin U_n. \end{cases}$$

Since U_n is clopen, h_n is continuous. By (2.10), $I\text{-}\lim h_n(x) = 0$. As $X \in \mathcal{IC}(I)$, there exists a set $A = \{a_0 < a_1 < a_2 < \cdots\} \in \mathcal{F}(I)$ such that $\lim_{n\to\infty} h_{a_n}(x) = 0$ for all $x \in X$. It follows that for every $x \in X$ there exists $m < \omega$ such that for any $a_n > m$ we have $h_{a_n}(x) = 0$, so $x \in U_{a_n}$. Hence

$$X \subseteq \bigcup_{m<\omega} \bigcap_{a \in A \setminus m} U_a.$$

To prove the other implication suppose that $f_n : X \to \mathbb{R}$ is a sequence of continuous functions with $I\text{-}\lim f_n = 0$. Since X is a separable, zero-dimensional metric space, any two disjoint closed subsets of X may be separated by a clopen set (see [4, pp. 35 and 357]). It follows that there exist clopen sets $U_n^k \subseteq X$ such $\{x : |f_n(x)| \leq 1/k\} \subseteq U_n^k \subseteq \{x : |f_n(x)| < 2/k\}$. Since $I\text{-}\lim f_n(x) = 0$ for every $x \in X$, we have $\{n : x \notin U_n^k\} \in I$ for every

$k < \omega$. By our assumption for each $k < \omega$ there exists a set $A^k \in \mathcal{F}(I)$ such that $X \subseteq \bigcup_{m<\omega} \bigcap_{n\in A^k\setminus m} U_n^k$. Since I is a P-ideal there exists a set $M = \{m_0 < m_1 < m_2 < \cdots\} \in \mathcal{F}(I)$ with $M \subseteq^* A^k$ for all $k < \omega$. It follows that $\lim_{i\to\infty} f_{m_i}(x) = 0$ for all $x \in X$. Hence $X \in \mathcal{IC}(I)$. ∎

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