COLLOQUIUM MATHEMATICUM

VOL. 111 2008 NO. 1

LIFTING VECTOR FIELDS TO THE rTH ORDER FRAME BUNDLE

BY

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Dedicated to Professor Witold Roter on the occasion of his 75th birthday with respect and gratitude

Abstract. We describe all natural operators $\mathcal{A}$ lifting nowhere vanishing vector fields X on m-dimensional manifolds M to vector fields $\mathcal{A}(X)$ on the rth order frame bundle $L^rM = \operatorname{inv} J^r_0(\mathbb{R}^m, M)$ over M. Next, we describe all natural operators $\mathcal{A}$ lifting vector fields X on m-manifolds M to vector fields on L^rM. In both cases we deduce that the spaces of all operators $\mathcal{A}$ in question form free $(m(C^{m+r}_r - 1) + 1)$-dimensional modules over algebras of all smooth maps $J^{r-1}_0\widetilde{T}\mathbb{R}^m \to \mathbb{R}$ and $J^{r-1}_0 T\mathbb{R}^m \to \mathbb{R}$ respectively, where $C^n_k = n!/(n-k)!k!$. We explicitly construct bases of these modules. In particular, we find that the vector space over $\mathbb{R}$ of all natural linear operators lifting vector fields X on m-manifolds M to vector fields on L^rM is $(m^2C^{m+r-1}_{r-1}(C^{m+r}_r - 1) + 1)$-dimensional.

0. Introduction. Let $\mathcal{M}f_m$ denote the category of m-dimensional manifolds and their embeddings (i.e. diffeomorphisms onto open subsets), and $\mathcal{FM}$ denote the category of fibered manifolds and their fibered map.

In this note we describe how a nowhere vanishing vector field X on an m-dimensional manifold M can induce a vector field $\mathcal{A}(X)$ on the rth order frame bundle $L^rM = \operatorname{inv} J^r_0(\mathbb{R}^m, M) = \{j^r_0\psi \mid \psi : \mathbb{R}^m \to M \text{ is an } \mathcal{M}f_m\text{-map}\}$ over M. This problem is reflected in the concept of $\mathcal{M}f_m$-natural operators $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ in the sense of [4], where $T : \mathcal{M}f_{\dim(L\mathbb{R}^m)} \to \mathcal{FM}$ is the natural bundle of tangent vectors (the tangent functor) and $\widetilde{T} : \mathcal{M}f_m \to \mathcal{FM}$ is the natural bundle of non-zero tangent vectors.

We recall that an $\mathcal{M}f_m$-*natural operator* $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ is a family of $\mathcal{M}f_m$-invariant regular operators (functions)

$$\mathcal{A} = \mathcal{A}_M : \Gamma\widetilde{T}M \to \Gamma T(L^rM)$$

from the set $\Gamma\widetilde{T}M$ of all nowhere vanishing vector fields on M (sections of the bundle $\widetilde{T}M$) into the set $\Gamma T(L^rM)$ of all vector fields on L^rM (sections

2000 *Mathematics Subject Classification*: Primary 58A20.

Key words and phrases: natural bundle, natural operator, jet.

of the tangent bundle $TL^rM \to L^rM$ of L^rM) for any m-manifold M. (Of course, for some m-manifolds M one can have $\Gamma\widetilde{T}M = \emptyset$; then $\mathcal{A}_M = \emptyset$.) The invariance means that if $X_1 \in \Gamma\widetilde{T}M$ and $X_2 \in \Gamma\widetilde{T}N$ are two related nowhere vanishing vector fields on m-manifolds M and N (respectively) by a $\mathcal{M}f_m$-map $\varphi : M \to N$ then $\mathcal{A}_M(X_1)$ and $\mathcal{A}_N(X_2)$ are related by $L^r\varphi$, where $L^r\varphi : L^rM \to L^rN$ is the induced map (defined by the composition of r-jets, $L^r\varphi(j^r_0\psi) = j^r_0(\varphi \circ \psi)$, $j^r_0\psi \in L^rM$). The regularity means that $\mathcal{A}$ transforms smoothly parametrized families of nowhere vanishing vector fields into smoothly parametrized families of vector fields. Replacing $\widetilde{T} : \mathcal{M}f_m \to \mathcal{FM}$ by $T : \mathcal{M}f_m \to \mathcal{FM}$ we obtain the concept of $\mathcal{M}f_m$-natural operators $\mathcal{A} : T \rightsquigarrow TL^r$.

An $\mathcal{M}f_m$-natural operator $\mathcal{A} : T \rightsquigarrow TL^r$ is said to be *linear* if $\mathcal{A}_M : \Gamma TM \to \Gamma T(L^rM)$ is $\mathbb{R}$-linear for any m-manifold M.

An $\mathcal{M}f_m$-natural operator $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ is said to be of *vertical type* if $\mathcal{A}_M(X)$ is a vertical vector field on $L^rM \to M$ for any nowhere vanishing vector field X on an arbitrary m-manifold M.

Let k be a non-negative integer. An $\mathcal{M}f_m$-natural operator $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ is said to be *of order* $\leq k$ if for any nowhere vanishing vector fields X_1 and X_2 on M and $x \in M$ the equality of k-jets $j^k_x(X_1) = j^k_x(X_2)$ implies $\mathcal{A}_M(X_1) = \mathcal{A}_M(X_2)$ on the fiber $(L^rM)_x$ of L^rM over x.

An example of an $\mathcal{M}f_m$-natural operator $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ of order $\leq r$ is the flow operator $\mathcal{L}^r$ sending a (nowhere vanishing) vector field X on an m-manifold M into the complete lift $\mathcal{L}^rX$ of X to L^rM. We recall that $\mathcal{L}^rX$ is the vector field on L^rM such that if $\{\varphi_t\}$ is the flow of X then $\{L^r\varphi_t\}$ is the flow of $\mathcal{L}^rX$.

Because of the $\mathcal{M}f_m$-invariance of $\mathcal{M}f_m$-operators with respect to (inverse) manifold charts, any $\mathcal{M}f_m$-natural operator $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ is fully determined by its "restriction" $\mathcal{A}_{\mathbb{R}^m} : \Gamma\widetilde{T}\mathbb{R}^m \rightsquigarrow \Gamma T(L^r\mathbb{R}^m)$. Conversely, by a chart argument, any $\mathcal{M}f_m$-invariant regular operator (function) $A : \Gamma\widetilde{T}\mathbb{R}^m \to \Gamma T(L^r\mathbb{R}^m)$ can be extended uniquely to an $\mathcal{M}f_m$-natural operator $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ with $\mathcal{A}_{\mathbb{R}^m} = A$. That is why all $\mathcal{M}f_m$-natural operators $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ form a set.

In this note we classify all $\mathcal{M}f_m$-natural operators $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$. Next we classify all $\mathcal{M}f_m$-natural operators $\mathcal{A} : T \rightsquigarrow TL^r$. In both cases we deduce (see Theorems 1 and 2 for detailed formulation)

THEOREM A. *The set of all $\mathcal{M}f_m$-natural operators $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ (resp. $\mathcal{A} : T \rightsquigarrow TL^r$) is a free $(m(C^{m+r}_r - 1) + 1)$-dimensional module over the algebra of smooth maps $J^{r-1}_0\widetilde{T}\mathbb{R}^m \to \mathbb{R}$ (resp. $J^{r-1}_0T\mathbb{R}^m \to \mathbb{R}$), where $C^n_k = n!/k!(n-k)!$. In particular, the vector space over $\mathbb{R}$ of all linear $\mathcal{M}f_m$-natural operators $T \rightsquigarrow TL^r$ is $(m^2C^{m+r-1}_{r-1}(C^{m+r}_r - 1) + 1)$-dimensional.*

In this paper we introduce the module structures and construct explicitly the bases of the modules.

We shall use the following notations: $G^r_m = \operatorname{inv} J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ is the differential group of order r, $T : \mathcal{M}f_m \to \mathcal{FM}$ is the tangent bundle, $\widetilde{T} : \mathcal{M}f_m \to \mathcal{FM}$ is the natural bundle over m-manifolds of non-zero tangent vectors, $L^r : \mathcal{M}f_m \to \mathcal{FM}$ is the natural bundle of frames of order r, J^r is the functor of r-jet prolongation of fibered manifolds.

It is well-known (see [4]) that G^r_m is a Lie group, and the Lie algebra $\mathcal{L}ie(G^r_m)$ of G^r_m is the Lie algebra $(J^r_0 T\mathbb{R}^m)_0$ of r-jets at $0 \in \mathbb{R}^m$ of vector fields on $\mathbb{R}^m$ vanishing at $0 \in \mathbb{R}^m$.

Some natural operators transforming vector fields to natural bundles were used in many papers where the problem of prolongation of geometric structures was studied (see e.g. [6], [8]). That is why natural operators $\mathcal{A} : T \rightsquigarrow TF$ transforming vector fields to some natural bundles $F : \mathcal{M}f_m \to \mathcal{FM}$ where studied by many authors ([1]–[5], [7]). For example, I. Kolář [3] classified all $\mathcal{M}f_m$-natural operators $\mathcal{A} : T \rightsquigarrow TT^A$, where T^A is the Weil functor corresponding to a Weil algebra A. In [2], J. Gancarzewicz studied natural linear operators $\mathcal{A} : T \rightsquigarrow TF$ for many natural bundles $F : \mathcal{M}f_m \to \mathcal{FM}$.

In what follows, all manifolds and maps are assumed to be smooth (of class $\mathcal{C}^\infty$).

1. The $\mathcal{M}f_m$-natural operators $\mathcal{B} : \widetilde{T} \rightsquigarrow T^{(0,0)}L^r$. If (in the definition of natural operators $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$) we replace $T : \mathcal{M}f_{\dim(L^r\mathbb{R}^m)} \to \mathcal{FM}$ by the natural bundle $T^{(0,0)} : \mathcal{M}f_{\dim(L^r\mathbb{R}^m)} \to \mathcal{FM}$ of tensor fields of type $(0,0)$ we obtain the concept of $\mathcal{M}f_m$-natural operators $B : \widetilde{T} \rightsquigarrow T^{(0,0)}L^r$ lifting nowhere vanishing vector fields on M into maps $L^rM \to \mathbb{R}$.

We have the following general example of $\mathcal{M}f_m$-natural operators $\widetilde{T} \rightsquigarrow T^{(0,0)}L^r$. Suppose we have a map $\lambda : J^{r-1}_0\widetilde{T}\mathbb{R}^m \to \mathbb{R}$, where $J^{r-1}_0\widetilde{T}\mathbb{R}^m$ is the manifold of all $(r-1)$-jets at 0 of nowhere vanishing vector fields on $\mathbb{R}^m$ (the fiber at $0 \in \mathbb{R}^m$ of the $(r-1)$-jets prolongation of $\widetilde{T}\mathbb{R}^m$). Then given a nowhere vanishing vector field X on M we have $\mathcal{B}^{\langle\lambda\rangle}(X) : L^rM \to \mathbb{R}$ given by

$$\mathcal{B}^{\langle\lambda\rangle}(X)(j^r_0\varphi) = \lambda(j^{r-1}_0(\varphi^{-1}_* X))$$

for all $j^r_0\varphi \in (L^rM)_x$, $x \in M$, where $\varphi : \mathbb{R}^m \to M$ is an $\mathcal{M}f_m$-map with $\varphi(0) = x$. The correspondence $\mathcal{B}^{\langle\lambda\rangle} : \widetilde{T} \rightsquigarrow T^{(0,0)}L^r$ is an $\mathcal{M}f_m$-natural operator of order $\leq r-1$ transforming nowhere vanishing vector fields on M into maps $L^rM \to \mathbb{R}$.

The set of all $\mathcal{M}f_m$-natural operators $B : \widetilde{T} \rightsquigarrow T^{(0,0)}L^r$ is (in an obvious way) an algebra. Actually, given $\mathcal{M}f_m$-natural operators $\mathcal{B}_1, \mathcal{B}_2 : \widetilde{T} \rightsquigarrow$

$T^{(0,0)}L^r$ we have the $\mathcal{M}f_m$-natural operator $\mathcal{B}_1\mathcal{B}_2 : \widetilde{T} \rightsquigarrow T^{(0,0)}L^r$ given by

$$(\mathcal{B}_1\mathcal{B}_2)_M(X) = (\mathcal{B}_1)_M(X)(\mathcal{B}_2)_M(X)$$

for any nowhere vanishing vector field X on an m-manifold M, where on the right of the above formula we have the multiplication of real-valued functions. (If $\Gamma\widetilde{T}M = \emptyset$ then of course $(\mathcal{B}_1)_M = \emptyset$, $(\mathcal{B}_2)_M = \emptyset$ and $(\mathcal{B}_1\mathcal{B}_2)_M = \emptyset$.) Similarly we define the sum $\mathcal{B}_1 + \mathcal{B}_2 : \widetilde{T} \rightsquigarrow T^{(0,0)}L^r$.

PROPOSITION 1. *The map $\lambda \mapsto \mathcal{B}^{\langle\lambda\rangle}$ is an algebra isomorphism from the algebra of smooth maps $J_0^{r-1}\widetilde{T} \to \mathbb{R}$ onto the algebra of all $\mathcal{M}f_m$-natural operators $\widetilde{T} \rightsquigarrow T^{(0,0)}L^r$.*

Proof. Clearly, the map $\lambda \mapsto \mathcal{B}^{\langle\lambda\rangle}$ is an algebra monomorphism.

Any $\mathcal{B} : \widetilde{T} \rightsquigarrow T^{(0,0)}L^r$ of order $\le r-1$ defines $\lambda : J_0^{r-1}\widetilde{T}\mathbb{R}^m \to \mathbb{R}$ by

$$\lambda(j_0^{r-1}X) = \mathcal{B}(X)_{j_0^r(\mathrm{id}_{\mathbb{R}^m})}.$$

By an order argument λ is well-defined. It is smooth because of the regularity of $\mathcal{B}$ (a standard argument using the Boman theorem, [4]).

Then by the invariance with respect to (inverse) manifold charts one can easily see that $\mathcal{B} = \mathcal{B}^{\langle\lambda\rangle}$.

By the same method as in [4] one can show that any $\mathcal{B}$ in question is of order $\le r-1$.

Thus the map $\lambda \mapsto \mathcal{B}^{\langle\lambda\rangle}$ is epimorphic. ∎

2. The $\mathcal{M}f_m$-natural operators $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ of vertical type. Let us denote by

$$E_\alpha^i = j_0^r\left(x^\alpha \frac{\partial}{\partial x^i}\right),$$

where $i = 1, \ldots, m$ and $\alpha \in (\mathbb{N} \cup \{0\})^m$ with $1 \le |\alpha| \le r$, the usual basis in $(J_0^r T\mathbb{R}^m)_0 = \mathcal{L}ie(G_m^r)$.

We denote by E^* the fundamental vector field corresponding to $E \in \mathcal{L}ie(G_m^r)$ on any principal G_m^r-bundle L^rM. Then all $(E_\alpha^i)^*$ for i and α as above form a basis over $\mathcal{C}^\infty(L^rM)$ of the vertical vector fields on L^rM for any M. Thus we have the corresponding (constant) $\mathcal{M}f_m$-natural operators $(E_\alpha^i)^* : \widetilde{T} \rightsquigarrow TL^r$ defined by $(E_\alpha^i)^*_M(X) = (E_\alpha^i)^*$ for any nowhere vanishing vector field X on an m-manifold M. Clearly, all $\mathcal{M}f_m$-natural operators E_α^i are of vertical type.

The space of all $\mathcal{M}f_m$-natural operators $\widetilde{T} \rightsquigarrow TL^r$ transforming nowhere vanishing vector fields on m-manifolds M into vector fields on L^rM is (in an obvious way) a module over the algebra of $\mathcal{M}f_m$-natural operators $\widetilde{T} \rightsquigarrow T^{(0,0)}L^r$. (Actually, given $\mathcal{M}f_m$-natural operators $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ and $\mathcal{B} : \widetilde{T} \rightsquigarrow T^{(0,0)}L^r$ we have the $\mathcal{M}f_m$-natural operator $\mathcal{B}\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ given by

$$(\mathcal{B}\mathcal{A})_M(X) = \mathcal{B}_M(X)\mathcal{A}_M(X)$$

for any nowhere vanishing vector field X on an m-manifold M, where on the right of the above formula we have the multiplication of vector fields by real-valued functions.) Then by Proposition 1 it is a module over the algebra of all maps $J^{r-1}_0\widetilde{T}\mathbb{R}^m \to \mathbb{R}$.

PROPOSITION 2. *The (sub)module of all vertical type $\mathcal{M}f_m$-natural operators $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ is free. The corresponding $\mathcal{M}f_m$-natural operators $(E^i_\alpha)^*$ form a basis over $\mathcal{C}^\infty(J^{r-1}_0\widetilde{T}\mathbb{R}^m)$ of this module.*

Proof. Since the fundamental vector fields $(E^i_\alpha)^*$ on L^rM form a basis of the module of vertical vector fields on L^rM, we see that any $\mathcal{M}f_m$-natural operator $\mathcal{A}$ (of vertical type) in question is of the form

$$\mathcal{A}(X) = \sum \lambda^\alpha_i(X)(E^i_\alpha)^*$$

for some uniquely determined maps $\lambda^\alpha_i(X) : L^rM \to \mathbb{R}$, where X is a nowhere vanishing vector field on an m-manifold M. Because of the invariance of $\mathcal{A}$ with respect to $\mathcal{M}f_m$-maps, $\lambda^\alpha_i : \widetilde{T} \rightsquigarrow T^{(0,0)}L^r$ are $\mathcal{M}f_m$-natural operators. ∎

3. The decomposition

PROPOSITION 3. *Let $\mathcal{A} : \widetilde{T} \rightsquigarrow TL^r$ be an $\mathcal{M}f_m$-natural operator of order $\le r$. There is a unique smooth map $\lambda : J^{r-1}_0\widetilde{T}\mathbb{R}^m \to \mathbb{R}$ such that $\mathcal{A} - \mathcal{B}^{\langle\lambda\rangle}\mathcal{L}^r$ is of vertical type, where $\mathcal{L}^r : \widetilde{T} \rightsquigarrow TL^r$ is the flow operator.*

Proof. Suppose that $\mathcal{A}(X)_{j^r_0(\mathrm{id}_{\mathbb{R}^m})} = \mathcal{L}^r\widetilde{X}_{j^r_0(\mathrm{id}_{\mathbb{R}^m})}$ and $X(0) \ne \mu\widetilde{X}(0)$ for all $\mu \in \mathbb{R}$. Then there is an $\mathcal{M}f_m$-map $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ preserving $j^r_0(\mathrm{id}_{\mathbb{R}^m})$ such that $J^rT\varphi(j^r_0X) = j^r_0X$ and $J^rT\varphi(j^r_0\widetilde{X}) \ne j^r_0\widetilde{X}$. Then

$$\mathcal{A}(X)_{j^r_0(\mathrm{id}_{\mathbb{R}^m})} = \mathcal{L}^r(\varphi_*\widetilde{X})_{j^r_0(\mathrm{id}_{\mathbb{R}^m})} \ne \mathcal{L}^r(\widetilde{X})_{j^r_0(\mathrm{id}_{\mathbb{R}^m})} = \mathcal{A}(X)_{j^r_0(\mathrm{id}_{\mathbb{R}^m})}.$$

This is a contradiction.

Then

$$T\pi^r \circ \mathcal{A}(X)_{j^r_0(\mathrm{id}_{\mathbb{R}^m})} = \lambda(j^{r-1}_0X)X_0$$

for some uniquely determined smooth map $\lambda : J^{r-1}_0\widetilde{T}\mathbb{R}^m \to \mathbb{R}$ and all nowhere vanishing vector fields on $\mathbb{R}^m$ with coefficients being polynomials of degree $\le r-1$, where $\pi^r : L^r\mathbb{R}^m \to \mathbb{R}^m$ is the usual projection.

Then $(\mathcal{A}(X) - \mathcal{B}^{\langle\lambda\rangle}(X)\mathcal{L}^rX)_{j^r_0(\mathrm{id}_{\mathbb{R}^m})}$ is vertical for all nowhere vanishing vector fields on $\mathbb{R}^m$ with coefficients being polynomials of degree $\le r-1$. Since the orbit with respect to the $\mathcal{M}f_m$-maps preserving $j^r_0(\mathrm{id}_{\mathbb{R}^m})$ of the space of all j^r_0X for nowhere vanishing X with coefficients being polynomials of degree $\le r-1$ is dense in $J^r_0\widetilde{T}\mathbb{R}^m$ (see [4, Lemma 42.4]), $(\mathcal{A}(X) - \mathcal{B}^{\langle\lambda\rangle}(X)\mathcal{L}^rX)_{j^r_0(\mathrm{id}_{\mathbb{R}^m})}$ is vertical for all nowhere vanishing vector fields X on $\mathbb{R}^m$ with coefficients being polynomials of degree $\le r$. Then $(\mathcal{A}(X) - \mathcal{B}^{\langle\lambda\rangle}(X)\mathcal{L}^r(X))_{j^r_0(\mathrm{id}_{\mathbb{R}^m})}$ is vertical for all nowhere vanishing vector fields on $\mathbb{R}^m$ by an order argument. So $\mathcal{A}-\mathcal{B}^{\langle\lambda\rangle}\mathcal{L}^r$ is of vertical type because of the $\mathcal{M}f_m$-invariance and the fact that L^r is a transitive natural bundle (i.e. L^rM is the $\mathcal{M}f_m$-orbit of $j^r_0(\mathrm{id}_{\mathbb{R}^m})$). ∎

4. The classification theorem. From [4] it follows that any $\mathcal{M}f_m$-natural operator $\mathcal{A}:\widetilde{T}\rightsquigarrow TL^r$ is of order $\le r$. Then summing up Propositions 2 and 3 we get

THEOREM 1. *The space of all $\mathcal{M}f_m$-natural operators $\widetilde{T}\rightsquigarrow TL^r$ is a free $(m(C^{m+r}_r-1)+1)$-dimensional module over the algebra $\mathcal{C}^\infty(J^{r-1}_0\widetilde{T}\mathbb{R}^m)$ of maps $J^{r-1}_0\widetilde{T}\mathbb{R}^m\to\mathbb{R}$. The operators $\mathcal{L}^r$ and $(E^i_\alpha)^*$ for $i=1,\dots,m$ and all $\alpha\in(\mathbb{N}\cup\{0\})^m$ with $1\le|\alpha|\le r$ form a basis of this module, where given $E\in\mathcal{L}ie(G^r_m)=(J^r_0T\mathbb{R}^m)_0$ we denote by E^* the fundamental vector field corresponding to E on any principal G^r_m-bundle L^rM.*

We have the following corollary of Theorem 1.

COROLLARY 1. *The fundamental vector fields E^* for $E\in\mathcal{L}ie(G^r_m)$ are the only $\mathcal{M}f_m$-canonical vector fields on L^rM.*

5. The complete description of all $\mathcal{M}f_m$-natural operators $T\rightsquigarrow TL^r$. If we replace $\widetilde{T}$ by T in Section 1 we obtain

PROPOSITION 4. *There exists an algebra isomorphism between the algebra $\mathcal{C}^\infty(J^{r-1}_0T\mathbb{R}^m)$ of all maps $J^{r-1}_0T\mathbb{R}^m\to\mathbb{R}$ and the algebra of all $\mathcal{M}f_m$-natural operators $T\rightsquigarrow T^{(0,0)}L^r$. This isomorphism $\lambda\mapsto\mathcal{B}^{\langle\lambda\rangle}$ is defined as in Section 1 with T playing the role of $\widetilde{T}$.*

The space of all $\mathcal{M}f_m$-natural operators $T\rightsquigarrow TL^r$ is (in an obvious way) a module over the algebra of all $\mathcal{M}f_m$-natural operators $T\rightsquigarrow T^{(0,0)}L^r$. Then by Proposition 4 it is a $\mathcal{C}^\infty(J^{r-1}_0T\mathbb{R}^m)$-module.

Similarly to Section 2 we get

PROPOSITION 5. *The (sub)module of all vertical type $\mathcal{M}f_m$-natural operators $\mathcal{A}:T\rightsquigarrow TL^r$ is free. The corresponding operators $(E^i_\alpha)^*$ form a basis of this module.*

The next question is whether Proposition 3 with T instead of $\widetilde{T}$ is true. The problem consists in proving that $\lambda:J^{r-1}_0T\mathbb{R}^m\to T_0\mathbb{R}^m$ given by

$$(*)\qquad T\pi^r\circ\mathcal{A}(X)_{j^r_0(\mathrm{id}_{\mathbb{R}^m})}=\lambda(j^{r-1}_0X)X_0$$

for all vector fields X on $\mathbb{R}^m$ with coefficients being polynomials of degree $\le r-1$ can be chosen smoothly near points j^{r-1}_0X with $X_0=0$.

Of course (since the left side of $(*)$ depends smoothly on j^r_0X), the map $\Phi:J^{r-1}_0T\mathbb{R}^m\to\mathbb{R}$ given by

$$\Phi(j^{r-1}_0X)=\lambda(j^{r-1}_0X)X^1(0)$$

is smooth and $\Phi(j_0^{r-1}X) = 0$ if $X^1(0) = 0$, where

$$X_0 = \sum_{i=1}^{m} X^i(0) \frac{\partial}{\partial x^i}_0 .$$

Then (this is a well-known fact from mathematical analysis) there is a smooth map $\Psi : J_0^{r-1}T\mathbb{R}^m \to \mathbb{R}$ such that $\Phi(j_0^{r-1}X) = \Psi(j_0^{r-1}X)X^1(0)$. Then we can put $\lambda = \Psi$. Thus we have

PROPOSITION 6. *Let $\mathcal{A} : T \rightsquigarrow TL^r$ be an $\mathcal{M}f_m$-natural operator. There is a uniquely determined smooth map $\lambda : J_0^{r-1}T\mathbb{R}^m \to \mathbb{R}$ such that $\mathcal{A} - \mathcal{B}^{\langle\lambda\rangle}\mathcal{L}^r$ is of vertical type, where $\mathcal{L}^r$ is the flow operator.*

Then similarly to Theorem 1 we have

THEOREM 2. *The space of all $\mathcal{M}f_m$-natural operators $T \rightsquigarrow TL^r$ is a free $(m(C_r^{m+r} - 1) + 1))$-dimensional module over the algebra $\mathcal{C}^\infty(J_0^{r-1}T\mathbb{R}^m)$ of smooth maps $J_0^{r-1}T\mathbb{R}^m \to \mathbb{R}$. The operators $\mathcal{L}^r$ and $(E_\alpha^i)^*$ for $i = 1, \ldots, m$ and $\alpha \in (\mathbb{N} \cup \{0\})^m$ with $1 \le |\alpha| \le r$ form a basis in this module, where given $E \in \mathcal{L}ie(G_m^r) = (J_0^r T\mathbb{R}^m)_0$ we denote by E^* the fundamental vector field corresponding to E on any principal G_m^r-bundle $L^r M$.*

By the homogeneous function theorem we have the following corollary of Theorem 2:

COROLLARY 2. *The vector space over $\mathbb{R}$ of all linear $\mathcal{M}f_m$-natural operators $T \rightsquigarrow TL^r$ is $(m^2 C_{r-1}^{m+r-1}(C_r^{m+r} - 1) + 1)$-dimensional. The operators $\mathcal{L}^r$ and $\mathcal{B}^{\langle F_j^\beta \rangle}(E_\alpha^i)^*$ for $i, j = 1, \ldots, m$ and $\alpha, \beta \in (\mathbb{N} \cup \{0\})^m$ with $1 \le |\alpha| \le r$ and $0 \le |\beta| \le r - 1$ form a basis over $\mathbb{R}$ in this vector space, where (F_j^β) is the usual basis in the dual space $(J_0^{r-1}T\mathbb{R}^m)^*$.*

Acknowledgements. We would like to thank the referee for valuable remarks and corrections.

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Received 20 October 2006;
revised 10 December 2006 (4813)