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ON MINIMAL NON-TILTED ALGEBRAS

BY

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Abstract. A minimal non-tilted triangular algebra such that any proper semiconvex subcategory is tilted is called a *tilt-semicritical algebra*. We study the tilt-semicritical algebras which are quasitilted or one-point extensions of tilted algebras of tame hereditary type. We establish inductive procedures to decide whether or not a given strongly simply connected algebra is tilted.

Introduced in the early 1980's by Happel and Ringel [13], the class of tilted algebras has played a central role in the development of the representation theory of algebras. However, it is not always easy to identify a tilted algebra looking, for instance, at its ordinary quiver with relations. Indeed, the properties characterizing tilted algebras refer to the existence of special components of the corresponding Auslander–Reiten quiver. By looking at the minimal non-tilted algebras, we try to get some insight on how the tilted algebras are built up. For example, this is the case in the Liu–Skowroński criterion for a tilted algebra where existence of a connected Auslander–Reiten component with a faithful section satisfying a homological condition is required (see [1, Th. VIII.5.6], [20] and [29] for details).

We say that a subcategory B is *semiconvex* in A if there is a sequence $B = B_s, B_{s-1}, \ldots, B_1, B_0 = A$ such that $B_i = C_i[M_i']$ (resp. $[M_i']C_i$) is a one-point (co-)extension of a convex subcategory C_i of $B_{i-1} = C_i[M_i' \oplus M_i'']$ (resp. $[M_i' \oplus M_i'']C_i$) by a C_i-module M_i', possibly $M_i' = 0$, $i = 1, \ldots, s$. A *tilt-semicritical* algebra A is a triangular algebra which is *semitilted* (that is, any proper semiconvex subcategory of A is tilted) but A is non-tilted. Clearly, any triangular non-tilted algebra contains a semiconvex subcategory which is tilt-semicritical. In Section 1 we study the basic properties of tilt-semicritical algebras and give some examples.

More recently, *quasitilted algebras* were introduced [12] as algebras of the form $A = \mathrm{End}_{\mathcal{H}} T$ for a hereditary abelian k-category $\mathcal{H}$, where k is an

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algebraically closed field, and T is a tilting object in $\mathcal{H}$. If $\mathcal{H} = \operatorname{mod} H$ for a hereditary path algebra $H = k\vec{\Delta}$, where $\vec{\Delta}$ is a quiver with underlying graph Δ, then A is said to be *tilted of type* Δ. In Section 3 we characterize the tilt-semicritical algebras which are quasi-tilted and strongly simply connected as those semitilted algebras belonging to one of three different families (QT1), (QT2), (QT3) whose members are described as extensions of well-known algebras. We recall that a basic algebra $A = kQ/I$ is said to be *strongly simply connected* if for every convex subcategory B of A, the first Hochschild cohomology $H^1(B)$ vanishes [28]. In Section 4 we characterize the tilt-semicritical algebras A such that whenever $A = B[M]$ or $[M]B$, then B is a tilted algebra of Dynkin or Euclidean type, as those semitilted algebras belonging to one of three families (EA0) (= quasi-tilted), (EA1), (EA2). The algebras in these families can be effectively constructed.

Let B be a *derived-hereditary* algebra, that is, there exists an equivalence of triangulated categories $F\colon D^b(\operatorname{mod} H) \to D^b(\operatorname{mod} B)$ for H a hereditary algebra. An indecomposable B-module M is said to be *derived-directing* (resp. *derived-regular*) if $M = FX$ for an indecomposable directing (resp. regular) H-module X. We say that a derived-regular module $M = FX$ is *governing* if X governs the regular H-modules (equivalently, if $H[X]$ is quasi-tilted); see [12]. The main result of this work is the following.

Theorem. *Let A be a strongly simply connected semitilted algebra. Then A is tilted if and only if the following conditions are satisfied*:

- (T1) *A is not in one of the families* (QT1), (QT2), (QT3).
- (T2) *A is not in one of the families* (EA1), (EA2).
- (T3) *For any extension $A = B[M]$ or coextension $[M]B$ with B tilted of wild type and M an indecomposable module*, *either M is derived-directing or M is governing.*

Directing modules were studied in [14, 25]. We provide simple criteria for a module to be derived-directing. Disgracefully, there is no simple way to check for a regular module X over a hereditary algebra H if $H[X]$ is quasitilted (see the long discussion in Chapter III of [12]). This problem remains an obstacle for our Theorem to be a handy criterion to check whether or not a given algebra is tilted.

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1. Semiconvex subcategories and tilted algebras

1.1. Let k be an algebraically closed field. In this paper, an "algebra" means a basic, indecomposable, finite-dimensional, associative k-algebra with identity. By a "module" we mean a finitely generated left module. Given an algebra A, we denote by $\operatorname{mod} A$ its module category, and by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of a complete set of representatives of the isomorphism classes of indecomposable modules. We usually consider A as a k-category, defined by a quiver Q with relations I, that is, $A = kQ/I$, as in [7]. Consequently, modules are k-linear covariant functors $A \to \operatorname{mod} k$. We say that A is *triangular* if Q has no oriented cycles.

For a vertex x of Q, denote by P_x (resp. I_x) the projective cover (resp. injective envelope) of the simple module S_x associated to x. Denote by Γ_A the Auslander–Reiten quiver of A, by τ_A the Auslander–Reiten translation $D\operatorname{Tr}$ and τ_A^- its inverse. For unexplained notions of representation theory we refer the reader to [2, 7].

1.2. Given an algebra B and a B-module M, the *one-point extension* of B by M is the algebra

$$A := B[M] = \begin{pmatrix} k & 0 \\ {}_BM_k & B \end{pmatrix}$$

with the usual matrix operations. It is well-known that A-modules can be described as triples $(k^t, {}_BX, \gamma\colon k^t \to \operatorname{Hom}_B(M,X))$. When there is no danger of confusion we shall denote an A-module $(0,X,0)$ simply as X. The indecomposable projective A-modules are of the form $(0,P,0)$ for P an indecomposable projective B-module or $P_\omega = (k, M, \mathrm{id})$, which is the projective corresponding to the extension vertex ω. Then $\operatorname{rad} P_\omega = M$.

LEMMA. *Let $A = B[M]$. Then* $\operatorname{gl.dim} A = \max\{\operatorname{gl.dim} B, \operatorname{pd}_B M + 1\}$. ■

1.3. We say that B is a *semiconvex* subcategory of A if there is a sequence of algebras $B = B_s, B_{s-1}, \ldots, B_1, B_0 = A$ satisfying:

- (SC0) There is a *convex subcategory* C_i of B_{i-1} for $i = 1, \ldots, s$. That is, $C_i = kQ'_i/I'_i$ and $B_{i-1} = kQ_{i-1}/I_{i-1}$, where Q'_i is a path closed full subquiver of Q_{i-1} and $I'_i = I_{i-1} \cap kQ'_i$.
- (SC1) There is a B_{i-1}-module $M_i = M'_i \oplus M''_i$ such that $B_{i-1} = C_i[M_i]$ (resp. $[M_i]C_i$), possibly $M'_i = 0$.
- (SC2) $B_i = C_i[M'_i]$ (resp. $[M'_i]C_i$).

In case B is semiconvex in A, the inclusion is a k-linear functor $F\colon B \to A$ with a remarkable property: for any objects x, y of B, we have $A(x,y) = B(x,y) \oplus S(x,y)$ for a family of subspaces $S(x,y)$ of $A(x,y)$ satisfying $\alpha S(x,a) \subset S(x,b)$ and $S(b,y)\alpha \subset S(a,y)$ all $\alpha \in B(a,b)$. Following [4], we say that F is a *cleaving functor* and S is a *cleavage.*

1.4. PROPOSITION. *Let A be a tilted algebra and B a semiconvex subcategory of A. Then B is a tilted algebra.*

Proof. Clearly, it is enough to consider the situation where $A = C[M]$ is a one-point extension of C by $M = M' \oplus M''$ and $B = C[M']$.

Since C is convex in A, it is tilted (see for instance [10]). Moreover, the algebra

$$\widetilde{A} = \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ M' & M'' & C & 0 \\ M'' & M' & 0 & C \end{bmatrix}$$

is a Galois covering of $A = C[M' \oplus M'']$ defined by the action of $\mathbb{Z}_2$. We may assume that $\operatorname{char} k \neq 2$ (if $\operatorname{char} k = 2$, we consider a three-fold covering of A) and conclude by [6] that $\widetilde{A}$ is tilted. Since $B = C[M']$ is a convex subcategory of $\widetilde{A}$, it is tilted. ∎

1.5. Recall from the introduction that a triangular algebra A is *semitilted* if every proper semiconvex subcategory of A is tilted. We say that a semitilted algebra A is *tilt-semicritical* if A is not tilted. We shall say that a non-tilted algebra A is *tilt-critical* if every proper convex subcategory of A is tilted. Clearly, tilt-semicritical algebras are tilt-critical.

LEMMA. *Let A be a semitilted. Then*:

(a) $\operatorname{gl.dim} A \leq 3$.
(b) *Assume $A = B[M]$ and $M = \bigoplus_{i=1}^{m} M_i$ is an indecomposable decomposition. Then B is tilted and for every proper subset $S \subset \{1, \ldots, m\}$, the extension $B[\bigoplus_{i \in S} M_i]$ is tilted.*

Proof. (a) Write $A = B[M]$ with B a convex subcategory of A and M a B-module. Since B is tilted, $\operatorname{gl.dim} B \leq 2$ and $\operatorname{pd} M \leq 2$. Hence $\operatorname{gl.dim} A \leq 3$.

(b) is clear. ∎

1.6. PROPOSITION. *Let A be a triangular algebra which is not tilted. Then there is a semiconvex subcategory of A which is tilt-semicritical.*

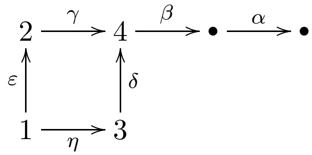
Proof. Let B be a non-tilted semiconvex subcategory of A with $\dim_k B$ minimal. We shall show that B is semitilted. Let C be a proper semiconvex subcategory of B. By definition, C is a semiconvex subcategory of A and $\dim_k C < \dim_k B$, hence C is tilted. ∎

EXAMPLES. (a) Let A be the radical square zero k-algebra given by the quiver

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \rightrightarrows \underset{4}{\bullet}$$

This algebra is not tilted, since $\operatorname{pd}_A S_1 = 3$. Let C be the convex subcategory of A in the vertices $1, 2, 3$, such that $A = [S_3 \oplus S_3]C$. Then $[S_3]C$ is a semiconvex subcategory of A given as the radical square zero algebra with quiver $\vec{\mathbb{A}}_4$. Clearly, $[S_3]C$ is tilt-semicritical.

(b) Let A be the k-algebra given by the quiver



with relations $\alpha\beta\delta = 0 = \gamma\varepsilon$.

Then A is not a tilted algebra. Let C be the convex subcategory of A obtained by "killing" the vertex 1. Then $A = C[S_2 \oplus N]$, where N is indecomposable with $\dim_k N(3) = 1 = \dim_k N(4)$. Then $B = C[S_2]$ is a semiconvex subcategory of A which is tilt-critical. Observe that $\operatorname{gl.dim} B = 2$.

2. Iterated tilted algebras of wild type

2.1. Let A be a k-algebra and $K_0(A)$ the Grothendieck group generated by the isoclasses $[S_1], \dots, [S_n]$ of simple A-modules. In case $\operatorname{gl.dim} A < \infty$, the *Euler form* $\langle -, - \rangle_A$ on $K_0(A)$ is the bilinear form such that for all A-modules X and Y we have

$$\langle [X], [Y] \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}_A^i(X, Y).$$

Defining the *Cartan matrix* $C_A = (c_{ij})$ by $c_{ij} = \langle [P_j], [P_i] \rangle_A$, we get

$$\langle v, w \rangle_A = v C_A^{-t} w^t.$$

We recall that the *Coxeter transformation* φ_A is an automorphism of $K_0(A)$ determined by

$$[P_j]\varphi_A = -[I_j] \quad \text{for } 1 \le j \le n.$$

Therefore $\varphi_A = -C_A^{-t} C_A$ and $\langle v, w\varphi_A \rangle_A = -\langle w, v \rangle_A$ for all $v, w \in K_0(A)$.

In the hereditary case $H = k\vec{\Delta}$, for any indecomposable non-projective A-module X we have

$$[X]\varphi_H = [\tau_H X].$$

Let H be a hereditary algebra of wild type. Let $\varrho = \varrho(\varphi_H)$ be the spectral radius of the Coxeter transformation φ_H. Then $1 < \varrho \in \operatorname{Spec} \varphi_H = \{\lambda \in \mathbb{C} : \lambda \text{ eigenvalue of } \varphi_H\}$ [27]. Moreover, there exist eigenvectors y^+ (resp. y^-) with all coordinates positive and $y^+\varphi_H = \varrho y^+$ (resp. $y^-\varphi_H = \varrho^{-1} y^-$). The vectors y^+, y^- play an important role in the representation theory of H. Namely (see [24]), for an indecomposable H-module X:

(a) X is preprojective or preinjective if and only if $\sigma_H(X) = \langle y^-, [X] \rangle_H \cdot \langle [X], y^+ \rangle_H < 0$.

(b) X is regular if and only if $\sigma_H(X) = \langle y^-, [X]\rangle_H \langle [X], y^+\rangle_H > 0$.

The number $\sigma_H(X)$ is called the *signature* of X.

2.2. Consider the derived category $D(A) = D^b(\operatorname{mod} A)$. If $\operatorname{gl.dim} A < \infty$, then $D(A)$ has Auslander–Reiten triangles and $[X]\varphi_A = [\tau_{D(A)} X]$ for any $X \in D(A)$.

Let A be a finite-dimensional k-algebra *derived-equivalent* to $H = k\vec{\Delta}$. By [19], A is *iterated tilted*, that is, there is a sequence of algebras $A_0 = H$, $A_1, \dots, A_m = A$ and tilting A_i-modules T_i such that $A_{i+1} = \operatorname{End}_{A_i} T_i$ for $0 \le i \le m-1$. We say that A is of *class* Δ. In this case, there is an isometry $S\colon K_0(H) \to K_0(A)$ such that $\langle v, w\rangle_H = \langle vS, wS\rangle_A$ and therefore $\varphi_A = S\varphi_H S^{-1}$. In particular, $\operatorname{Spec}\varphi_H = \operatorname{Spec}\varphi_A$ and $\varrho(\varphi_H) = \varrho(\varphi_A)$.

If A is derived-equivalent to $H = k\vec{\Delta}$ of wild type and $F\colon D(H) \to D(A)$ is an equivalence of triangulated categories inducing the isometry $S\colon K_0(H) \to K_0(A)$, we select the eigenvectors of φ_A as $y_A^+ = y_H^+ S$ and $y_A^- = y_H^- S$. We define the *signature* σ_A in the obvious way.

We recall that a module $M \in \operatorname{mod} B$ is *directing* if there are no paths of non-zero maps between indecomposable modules of the form

$$M' = N_0 \to N_1 \to \cdots \to N_i \to N_{i+1} \to N_{i+2} = \tau_B^- N_i \to \cdots \to N_s = M''$$

where M' and M'' are indecomposable direct summands of M. If M is indecomposable, this is equivalent to saying that there are no paths of non-zero non-isomorphisms between indecomposable modules of the form $M = N_0 \to N_1 \to N_s = M$ (see [14, 31]).

We say that an indecomposable A-module M is *derived-directing* (resp. *derived-regular*) if there is a directing (resp. regular) H-module X such that $FX = M$. We say that a derived-regular module M is *governing* if $M = FX$ for a module X which *governs the regular H-modules*, that is, for every non-zero indecomposable map $f\colon X^s \to Y$ such that Y has a regular direct summand, $\ker f$ is projective; equivalently, $H[X]$ is quasitilted (see [12]).

The following result follows from [17].

Proposition. *Let B be derived-equivalent to a wild hereditary algebra. Let M be an indecomposable B-module. Then*:

(a) *M is derived-directing if and only if $\sigma_B(M) < 0$. In this case $B[M]$ is derived-hereditary.*
(b) *M is derived-regular if and only if $\sigma_B(M) > 0$.*
(c) *M is governing if and only if $\sigma_B(M) > 0$ and $B[M]$ is quasitilted.*

Proof. Let $F\colon D(H) \to D(B)$ be an equivalence of triangulated categories and $S\colon K_0(H) \to K_0(B)$ the induced isometry. Assume $FX = M$ for an indecomposable H-module M. Observe that

$$\sigma_B(M) = \langle y_B^-, [M]\rangle_B \langle [M], y_B^+\rangle_B = \sigma_H(X).$$

This proves the first parts of (a) and (b). Moreover, by [3], F can be extended to an equivalence $D(H[X]) \xrightarrow{\sim} D(B[M])$. If $\sigma_B(M) < 0$, then X is directing and $H[X]$ is a tilted algebra. Therefore $B[M]$ is derived-hereditary.

(c) follows by (a), (b) and [12, III(3.9)]. ∎

3. Quasitilted tilt-critical algebras

3.1. In this section we shall characterize all the quasitilted strongly simply connected algebras which are tilt-semicritical. We start by recalling the original definition of quasitilted algebras given in [12]. Let $\mathcal{H}$ be a hereditary abelian locally finite R-category, where R is a commutative artinian ring, and let T be a tilting object in $\mathcal{H}$. The algebra $A = \operatorname{End}_{\mathcal{H}} T$ is called *quasitilted*. These algebras can be characterized by the properties: (QT1) $\operatorname{gl.dim} A \le 2$, and (QT2) for each indecomposable A-module X, $\operatorname{pd}_A X \le 1$ or $\operatorname{id}_A X \le 1$. For further details, we refer to [12].

It follows from [11] that if $A = \operatorname{End}_{\mathcal{H}} T$ is a quasitilted algebra, then $\mathcal{H}$ is either derived-equivalent to $\operatorname{mod} H$ where H is a hereditary algebra or derived-equivalent to $\operatorname{coh} \mathbb{X}$ where $\mathbb{X}$ is a weighted projective line in the sense of [8].

The following shows an important feature of quasitilted tilt-semicritical algebras.

Proposition. *Let A be a quasitilted algebra. Then A is tilt-semicritical if and only if A is tilt-critical.*

Proof. One implication is obvious. For the converse, assume that every convex subcategory of A is tilted and consider B a semiconvex subcategory of A. There is a chain $A = B_0 \supset B_1 \supset \cdots \supset B_s = B$ of subcategories such that $B_{i-1} = C_i[M_i \oplus M_i']$ and $B_i = C_i[M_i']$, where C_i is a convex subcategory of B_{i-1} and M_i is a non-trivial C_i-module, $i = 1, \dots, s$. Then C_1 is tilted. We distinguish the cases $M_1' = 0$ or not. If $M_1' = 0$, then $B_1 = C_1$ is tilted and 1.4 implies that B is tilted. If $M_1' \neq 0$, since A is quasitilted, [5, (2.2)] implies that the algebra

$$B' = \begin{pmatrix} k & 0 & 0 \\ M_1 & C_1 & 0 \\ M_1' & 0 & C_1 \end{pmatrix}$$

is tilted. Therefore the convex subcategory $B_1 = C_1[M_1]$ of B' is also tilted. Again 1.4 implies that B is tilted. In conclusion, A is tilt-semicritical. ∎

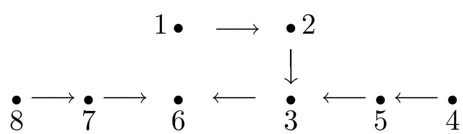
3.2. Following [16] a quasitilted algebra A is called *concealed canonical* if $A = \operatorname{End}_{\mathcal{H}} T$, with $\mathcal{H} = \operatorname{coh} \mathbb{X}$ where $\mathbb{X}$ is a weighted projective line, and T is a tilting object of infinite length, that is, $T \in \operatorname{vect} \mathbb{X}$. Equivalently, a concealed canonical algebra A can be seen as $\operatorname{End}_C T$, where C is a canonical algebra and T is a tilting C-module which is a direct sum of indecomposable modules of strictly positive rank; see [16] for details.

3.3. Example. Let $C = H[M]$ be a canonical algebra and let T' be a postprojective tilting H-module. Clearly, if P_ω is the extended projective C-module, then $T = T' \oplus P_\omega$ is a tilting C-module. Since $\operatorname{rk} T' > 0$ and $\operatorname{rk} P_\omega > 0$, we infer that $A = \operatorname{End}_C T$ is a concealed canonical algebra. Moreover, in this case, $A = B[N]$ where $B = \operatorname{End}_C T'$ is a concealed algebra and $N = \operatorname{Hom}_C(T', M)$.

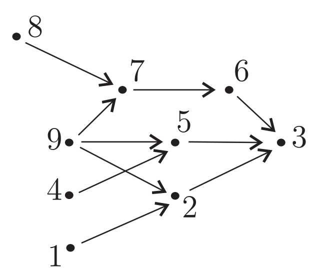
Suppose T' is the particular tilting module $T' = \bigoplus_{i=1}^{n} \tau_H^{-r} P_i$, where P_i runs over all the indecomposable projective modules in the postprojective component of C and r is a positive integer. In this case, $\operatorname{End}_C T = H[\tau_H^r M]$ is a concealed canonical algebra. Consequently, if $H[M]$ is a canonical algebra, then $H[\tau_H^r M]$ is a concealed canonical algebra for each $r \in \mathbb{N}$.

We now consider a more specific example, borrowed from [16].

Let H be the hereditary algebra given by the quiver



and consider the H-module M with vector-dimension $(0, 1, 1, 0, 1, 1, 1, 0)$. The algebra $A = H[M]$ is given by the following quiver Δ:



with a relation identifying the three paths from 9 to 3. It is not difficult to see that the extension $H[\tau_H^{-1} M]$ is a canonical algebra. By [12], A is a concealed canonical algebra and, therefore, not tilted by [11]. Denote by $A^{(l)}$ the algebra given by the full convex subquiver of Δ containing all its vertices but l. Clearly, the maximal proper convex subcategories of A are $A^{(1)}, A^{(3)}, A^{(4)}, A^{(8)}$ and $A^{(9)}$. To see that A is tilt-critical, we shall show that these subcategories are tilted. First, $A^{(3)}$ and $A^{(9)}$ are hereditary algebras.

Now, if one makes the reflection by the source 9 on either of $A^{(1)}$ or $A^{(4)}$, one gets tilted algebras of wild type (they are one-point extensions of hereditary algebras of type $\mathbb{E}_7$ by a module in a complete slice). Then $A^{(1)}$ and $A^{(4)}$ are quasitilted algebras derived-equivalent to a wild hereditary algebra, hence they are tilted algebras (see [10, 25]). Similarly, making the reflection by the source 9 in $A^{(8)}$, one gets a one-point extension of a hereditary algebra of type $\widetilde{\mathbb{E}}_6$ by a postprojective module, so a tilted algebra of wild type. Since $A^{(8)}$ is a quasitilted algebra derived-equivalent to a wild hereditary algebra, it is a tilted algebra. Hence, A is tilt-critical.

3.4. In order to prove our next result, we recall the following notation. For a module M denote by $M^{\perp}$ the *perpendicular category* given by all modules X such that $\mathrm{Hom}_A(M,X)=0$ and $\mathrm{Ext}^1_A(M,X)=0$.

PROPOSITION. *Let C be a concealed canonical algebra. If the type of C is tubular or wild, then C is tilt-critical.*

Proof. Suppose first that C is of tubular type. Then, by [21], C is in fact a tubular algebra and so, clearly, every full convex subcategory of C is tilted (see [26]) and the result is proved in this case.

Suppose now that C is of wild type, that is, $C=\mathrm{End}_{\mathrm{coh}\,\mathbb{X}}\,T$ where T is a tilting object of infinite length. Let B be a maximal proper convex subcategory of A and suppose $A=B[N]$ (the case $A=[N]B$ is similar). Then there exists a decomposition of the tilting object $T=T'\oplus T_\omega$ where ω is the extension vertex. Hence $B=\mathrm{End}_{T_\omega^{\perp}}\,T'$ and N is indecomposable. Since T_ω is an object of infinite length, by [9] we have $T_\omega^{\perp}\cong \mathrm{mod}\,H$ for some hereditary algebra H. Therefore, B is a tilted algebra and the result is shown. ∎

3.5. Let $A=kQ_A/I$ be a basic algebra. A source x of Q_A is said to be *separating* if $A=B[M]$ as a one-point extension with $M=\mathrm{rad}\,P_x$ has the property that for the indecomposable decomposition $B=\coprod_{i=1}^m B_i$, there is a corresponding decomposition $M=\bigoplus_{i=1}^m M_i$ with M_i an indecomposable B_i-module. According to [28], A is strongly simply connected if and only if for every convex subcategory B of A, every source of Q_B is separating.

Observe that a strongly simply connected tilt-semicritical algebra is tilt-critical. In particular, we get:

PROPOSITION. *Let A be a strongly simply connected algebra. Then A is not tilted if and only if it contains a convex subcategory B which is tilt-critical.* ∎

3.6. In certain situations there are handy criteria to decide the representation type of a strongly simply connected algebra. Indeed, let A be a strongly simply connected algebra. Then:

(a) A is not representation-finite if and only if A contains a convex subcategory C which is tame concealed (that is, concealed of a hereditary algebra $k\Delta$ of extended Dynkin type; see [20]).

(b) Assume A is a branch enlargement of a tame concealed algebra C, that is, $A = C[M_i, b_i]_{i=1}^{s}$ with M_i simple regular C-modules and b_i branches ($i = 1, \dots, s$) (see [26]). Then the following are equivalent:

   (i) A is tame;
   (ii) A is a domestic tubular extension or a tubular extension of C;
   (iii) A does not contain convex subcategories B which are concealed of a wild hereditary algebra $k\Delta$.

Indeed, (i)⇔(ii) is in [26]; (i)⇔(iii) follows from [22] and [23] using the Tits quadratic form.

3.7. We are now ready to show the main result of this section.

Theorem. *Let A be a quasitilted algebra. If A is tilt-critical and strongly simply connected, then it belongs to one of the following families*:

(QT1) $A = [N]C[M]$ *where C is a tame concealed algebra and N, M are simple regular modules in distinct tubes such that $C[M]$ and $[N]C$ are tilted algebras.*

(QT2) *A is a tubular algebra.*

(QT3) *A is not tilted and $A = B[M]$ (or $A = [N]B$) with B a wild algebra which is tilted of wild type and M (or N) a governing indecomposable B-module.*

Proof. Assume that A is a tilt-critical and strongly simply connected algebra. We know that $A = \operatorname{End}_{\mathcal{H}} T$ where $\mathcal{H}$ is derived-equivalent either to $\operatorname{mod} H$ for some hereditary algebra H, or to $\operatorname{coh} \mathbb{X}$ where $\mathbb{X}$ is a weighted projective line. Suppose first that $\mathcal{H}$ is derived-equivalent to $\operatorname{mod} H$ for some hereditary algebra H. If H is representation-finite or of wild type, then A is tilted by [12, 10], a contradiction. Therefore, H is of Euclidean type. By [30], A is then a semiregular branch enlargement of a tame concealed algebra, that is, $A = [K_i', b_i']_{i=1}^{t} C[K_j, b_j]_{j=1}^{s}$, where C is a tame concealed algebra with K_i', K_j simple regular C-modules in distinct tubes. If $s \geq 1$ and $t \geq 1$, since A is tilt-critical, we infer that $A = [N]C[M]$, with N, M being two simple regular modules in distinct tubes such that $C[M]$ and $[N]C$ are tilted algebras, as in (QT1). If $t = 0$, then either A is tame and therefore of type (QT2) by 3.6, or A is wild, and then it contains a convex subcategory B_0 concealed of wild type. Let B be a maximal convex subcategory of A which is tilted of wild type and $B_0 \subset B$. Then there is an indecomposable B-module N such that $[N]B$ is a convex subcategory of A (or dually, $B[N]$ is a convex subcategory of A). Then $[N]B$ is quasitilted and not tilted, so by 2.2, N is governing and $A = [N]B$ is of type (QT3).

Suppose now that $\mathcal{H}$ is derived-equivalent to $\operatorname{coh}\mathbb{X}$. Clearly, $\mathbb{X}$ is of tubular or wild type. If $\mathbb{X}$ is of tubular type, then, by [18], A is either of type (QT1) or of type (QT2). If $\mathbb{X}$ is of wild type, then A is a semiregular branch enlargement of a concealed canonical algebra C. If C is tame concealed we conclude as above. If C is not tame concealed, then C is either tubular or concealed canonical of wild type. By the minimality, A is then either of type (QT2) or of type (QT3). ∎

4. Tilt-semicritical algebras which are extensions of derived-tame algebras

4.1. We recall the following notation from [12]. Denote by $\mathcal{L}_B$ (resp. $\mathcal{R}_B$) the full subcategory of $\operatorname{mod} B$ formed by the indecomposable B-modules X such that for every *predecessor* Y of X (resp. *successor* Y of X), in the order given by non-zero maps between indecomposable modules, we have $\operatorname{pd}_B Y \le 1$ (resp. $\operatorname{id}_B Y \le 1$). If B is quasitilted, then $\operatorname{ind} B = \mathcal{L}_B \cup \mathcal{R}_B$.

Proposition ([6]). *Let $M = \bigoplus_{i=1}^m M_i$ be an indecomposable decomposition of a B-module M and $m \ge 2$. Suppose that B is quasitilted. Then*:

(a) *If $A = B[M]$ is quasitilted, then $M \in \mathcal{L}_B$ and M is directing.*
(b) *If $M \in \operatorname{add}(\mathcal{L}_B \cap \mathcal{R}_B)$ and M is directing, then $A = B[M]$ is tilted.*
(c) *If M is directing and $M_1 \in \mathcal{L}_B \setminus \mathcal{R}_B$, then $A = B[M]$ is quasitilted if and only if $B[M_1]$ is quasitilted and $\bigoplus_{i=2}^m M_i$ is a hereditary projective module with $\operatorname{Hom}_B(\bigoplus_{i=2}^m M_i, \mathcal{R}_B \setminus \mathcal{L}_B) = 0$.* ∎

4.2. Let B be a tilted algebra of type Δ, where Δ is a Dynkin or Euclidean diagram. Then the Auslander–Reiten quiver Γ_B has one of the following shapes:

(a) A single finite connected component $\mathcal{C}_B$ which is directing and contains a *slice* (= a connected full subquiver $\mathcal{S}$ such that $\bigoplus_{\mathcal{S}} X$ is a directing tilting module).

(b) $B = C[N_i, B_i]_{i=1}^s$ (or ${}_{i=1}^{s}[N_i, B_i]C$) is a *domestic branch* (*co*) *extension* of a tame concealed algebra C by simple regular modules $N_1, \dots, N_s$ and branches $B_1, \dots, B_s$ (see [26]). If $s = 0$, then $B = C$ is tame concealed. If $s \ge 1$, then $N_i \notin \mathcal{R}_B$. Indeed, there is an indecomposable projective B-module P such that N_1 is a summand of $\operatorname{rad} P$. Since N_1 is not projective, $X = \tau_B N_1$ is a module and there is a *non-sectional path* $N_1 \to \cdots \to X \to \bullet \to \tau_B^- X = N_1 \hookrightarrow P$. Hence $\operatorname{id}_B X \ge 2$ and $N_1 \notin \mathcal{R}_B$.

4.3. Proposition. *Let A be a semitilted strongly simply connected algebra such that whenever $A = B[M]$, the tilted algebra B is of Dynkin or Euclidean type. If A is tilt-semicritical then it belongs to one of the following families*:

(EA0) (QTi), $i = 1, 2, 3$.

(EA1) $A = [N]C[M]$ *with* M *and* N *indecomposable* C*-modules such that there is a non-sectional path* $N \rightsquigarrow M$. *Moreover*, C *is of finite or concealed type.*

(EA2) $A = B[M]$ *for a tilted algebra* B *of Euclidean type and* M *is indecomposable*, *neither preprojective nor preinjective*, *and not a ray-module in the sense of* [26, 4.5].

Proof. Let $A = B[M]$ with B a tilted algebra of Dynkin or Euclidean type. Since A is strongly simply connected, M is indecomposable. Suppose M is directing. By 4.1(b), either $M \notin \mathcal{L}_B$ or $M \notin \mathcal{R}_B$. Assume that $M \notin \mathcal{L}_B$ and choose a non-sectional path $I_y \to \cdots \to M$. Let N be a direct summand of I_y/S_y such that there is a non-sectional path $N \to \cdots \to M$. Since M is directing, we have $M(y) = 0$ and $A = [N]C[M]$ for a tilted algebra C of Dynkin or Euclidean type and A belongs to (EA1). In case $M \notin \mathcal{R}_B$, we choose a non-sectional path $M \to \cdots \to P_x$. By 4.2, M is preprojective and we can select x in such a way that $A = (C[M])[N]$ for $N = \operatorname{rad} P_x$ and $C = B/(x)$. In particular, N is indecomposable. Observe that M belongs to the preprojective component $\mathcal{P}_C$ of Γ_C. There are two situations to distinguish: if N does not belong to $\mathcal{P}_C$, then since $C[N] = B$ is tilted, there are no injective modules in $\mathcal{P}_C$. Then $C[M]$ is a tilted algebra of wild type, a contradiction. Therefore N belongs to $\mathcal{P}_C$, in particular N is directing. As above, either $N \notin \mathcal{L}_C$ or $N \notin \mathcal{R}_C$. If $N \notin \mathcal{L}_C$, we see that A is in (EA1). Assume that $N \notin \mathcal{R}_C$ and choose a non-sectional path $N \to \cdots \to N_1 \to P_{x_1}$. This process can be carried out only finitely many times before we stop. Then we find that A is in (EA1) as before.

Suppose now that M is indecomposable and non-directing. Let $B = \operatorname{End}_H T$ for a hereditary algebra H of affine type and a tilting module T. Then $M = \operatorname{Hom}_H(T, X)$ for a regular H-module X and $A = B[M]$ is derived-equivalent to $H[X]$. If A is not in the family (EA0), then $H[X]$ is not quasitilted. Then X is not a simple regular module, or equivalently, M is neither a ray-module (and clearly, neither preprojective nor preinjective). Hence A belongs to (EA2). ∎

5. Proof of the Main Theorem. Let A be a semitilted strongly simply connected algebra. Suppose A is tilted. Clearly, A satisfies (T1) and (T2). Assume $A = B[M]$ with B tilted of wild type and M an indecomposable module. By 2.2, A satisfies (T3).

For the converse, assume that A is not tilted. Then A is tilt-critical. If A is quasitilted, by 3.7, (T1) fails. If whenever $A = B[M]$, the tilted algebra B is of Dynkin or Euclidean type, then 4.3 implies that (T2) fails. Hence we may assume that A is not quasitilted and $A = B[M]$ for a tilted algebra B of wild type. By 2.2, (T3) should fail. ∎

REFERENCES

[1] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras I: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press, 2006.

[2] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[3] M. Barot and H. Lenzing, *One-point extensions and derived equivalence*, J. Algebra 264 (2003), 1–5.

[4] R. Bautista, P. Gabriel, A. V. Roĭter, L. Salmerón, *Representation-finite algebras and multiplicative bases*, Invent. Math. 81 (1985), 217–285.

[5] F. U. Coelho, M. I. Martins and J. A. de la Peña, *Quasitilted extension of algebras I*, Proc. Amer. Math. Soc. 129 (2001), 1289–1297.

[6] —, —, —, *Quasitilted extension of algebras II*, J. Algebra 227 (2000), 582–594.

[7] P. Gabriel, *Auslander–Reiten sequences and representation-finite algebras*, in: Representation Theory I, Lecture Notes in Math. 831, Springer, 1980, 1–71.

[8] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, in: Singularities, Representations of Algebras and Vector Bundles, Lecture Notes in Math. 1273, Springer, 1987, 265–297.

[9] —, —, *Perpendicular categories with applications to representations and sheaves*, J. Algebra 144 (1991), 273–343.

[10] D. Happel, *Quasitilted algebras*, in: Algebras and Modules I, CMS Conf. Proc. 23, Amer. Math. Soc., 1998, 55–82.

[11] —, *A characterization of hereditary categories with tilting modules*, Invent. Math. 144 (2001), 381–398.

[12] D. Happel, I. Reiten and S. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 120 (1996), no. 575.

[13] D. Happel and C. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (1982), 399–443.

[14] —, —, *Directing projective modules*, Arch. Math. (Basel) 60 (1993), 237–246.

[15] D. Happel and D. Vossieck, *Minimal algebras of infinite representation type with preprojective component*, Manuscripta Math. 42 (1983), 221–243.

[16] H. Lenzing and H. Meltzer, *Tilting sheaves and concealed-canonical algebras*, in: Representation Theory of Algebras, CMS Conf. Proc. 18, Amer. Math. Soc., 1996, 455–473.

[17] H. Lenzing and J. A. de la Peña, *On the growth of the Coxeter transformations of derived-hereditary algebras*, Internat. J. Algebra 1 (2007), no. 1-4, 95–112.

[18] H. Lenzing and A. Skowroński, *Quasi-tilted algebras of canonical type*, Colloq. Math. 71 (1996), 161–181.

[19] —, —, *Derived equivalence as iterated tilting*, Arch. Math. (Basel) 76 (2001), 20–24.

[20] S. Liu, *Tilted algebras and generalized standard Auslander–Reiten components*, ibid. 61 (1993), 12–19.

[21] H. Meltzer, *Exceptional vector bundles, tilting sheaves and tilting complexes for weighted projective lines*, Mem. Amer. Math. Soc. 171 (2004), no. 808.

[22] J. A. de la Peña, *On the representation type of one-point extensions of tame concealed algebras*, Manuscripta Math. 61 (1988), 183–194.

[23] —, *Algebras with hypercritical Tits form*, in: Topics in Algebra, Banach Center Publ. 26, Part 1, PWN, 1990, 353–369.

[24] J. A. de la Peña and M. Takane, *Spectral properties of Coxeter transformations and applications*, Arch. Math. (Basel) 55 (1990), 120–134.

[25] J. A. de la Peña and S. Trepode, *Quasitilted one point extensions of tilted algebras*, in: Representations of Algebras, Vol. II, Beijing Norm. Univ. Press, 2003, 373–386.
[26] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.
[27] —, *The spectral radius of the Coxeter transformations for a generalized Cartan matrix*, Math. Ann. 300 (1994), 331–339.
[28] A. Skowroński, *Simply connected algebras and Hochschild cohomologies*, in: Representations of Algebras, CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 431–447.
[29] —, *Generalized standard Auslander–Reiten components without oriented cycles*, Osaka J. Math. 30 (1993), 515–527.
[30] —, *Tame quasitilted algebras*, J. Algebra 203 (1998), 470–490.
[31] A. Skowroński and S. Smalø, *Directing modules*, ibid. 147 (1992), 137–146.

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