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CANTOR–SCHROEDER–BERNSTEIN QUADRUPLES FOR BANACH SPACES

BY

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Abstract. Two Banach spaces X and Y are symmetrically complemented in each other if there exists a supplement of Y in X which is isomorphic to some supplement of X in Y. In 1996, W. T. Gowers solved the Schroeder–Bernstein (or Cantor–Bernstein) Problem for Banach spaces by constructing two non-isomorphic Banach spaces which are symmetrically complemented in each other. In this paper, we show how to modify such a symmetry in order to ensure that X is isomorphic to Y. To do this, first we introduce the notion of Cantor–Schroeder–Bernstein Quadruples for Banach spaces. Then we characterize them by using some Banach spaces constructed by W. T. Gowers and B. Maurey in 1997. This new insight into the geometry of Banach spaces complemented in each other leads naturally to the Strong Square-hyperplane Problem which is closely related to the Schroeder–Bernstein Problem.

1. Introduction. Let X and Y be Banach spaces. We write $Y \overset{c}{\hookrightarrow} X$ if Y is isomorphic to a complemented subspace of X, that is, $X \sim Y \oplus A$ for some Banach space A. In this case, we say that A is a *supplement* of Y in X and we also write $Y \overset{A}{\hookrightarrow} X$. $X \sim Y$ means that X is isomorphic to Y and $X \not\sim Y$ means that X is not isomorphic to Y. If $n \in \mathbb{N} = \{1, 2, \ldots\}$, then X^n indicates the finite sum of n copies of X. It is useful to define $X^0 = \{0\}$.

Suppose that X and Y are Banach spaces *complemented in each other*, that is,

$$Y \overset{c}{\hookrightarrow} X \quad \text{and} \quad X \overset{c}{\hookrightarrow} Y. \tag{1.1}$$

In 1996, W. T. Gowers [14] solved the so-called Schroeder–Bernstein Problem for Banach spaces by showing that X is not necessarily isomorphic to Y. This answer in the negative opens two directions of research. The first is to provide new negative solutions to this problem with some particular properties (see [5]–[9], [11] and [16]). The second is to ask what additional conditions on X and Y satisfying (1.1) ensure that X is isomorphic to Y (see [10], [12] and [13]).

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Concerning this last direction, it is well known that Pełczyński's decomposition method [3, p. 63], which has played an important role in the isomorphic theory of classical Banach spaces, states that $X \sim Y$ if these spaces satisfy (1.1) and the following Decomposition Scheme:

$$\begin{cases} X \sim X^2, \\ Y \sim Y^2. \end{cases}$$

The present work is a continuation of [10], [12] and [13] in the sense that we present some alternatives to Pełczyński's decomposition method in Banach spaces. Our starting point is the fact that the first condition of (1.1), $Y \overset{c}{\hookrightarrow} X$, means that there exists a Banach space A such that $Y \overset{A}{\hookrightarrow} X$. Therefore a necessary condition for (1.1) to yield $X \sim Y$ is that

$$Y \overset{A}{\hookrightarrow} X \quad \text{and} \quad X \overset{A}{\hookrightarrow} Y. \tag{1.2}$$

So we define

Definition 1.1. Two Banach spaces X and Y are *symmetrically complemented in each other* if there exists a Banach space A satisfying (1.2).

Notice that if X and Y are Banach spaces symmetrically complemented in each other then $X^2 \sim Y^2$. Indeed, let A be a Banach space satisfying (1.2). Then

$$X \sim Y \oplus A \sim X \oplus A \oplus A = X \oplus A^2. \tag{1.3}$$

Adding X to both sides of (1.3) we deduce

$$X^2 \sim X^2 \oplus A^2 \sim Y^2.$$

We do not know examples of Banach spaces X and Y complemented in each other and satisfying $X^2 \sim Y^2$ which are not symmetrically complemented in each other.

Unfortunately, two Banach spaces which are symmetrically complemented in each other are not necessarily isomorphic. Indeed, in [14] there was constructed a Banach space Z isomorphic to Z^3 but not to Z^2. Thus

$$Z^2 \overset{Z}{\hookrightarrow} Z \quad \text{and} \quad Z \overset{Z}{\hookrightarrow} Z^2 \quad \text{but} \quad Z \not\sim Z^2.$$

Moreover, even when the space A from (1.2) is the complex scalars $\mathbb{C}$ it does not imply that $X \sim Y$. Indeed, in [16, p. 559], W. T. Gowers and B. Maurey introduced a Banach space W isomorphic to each of its subspaces of codimension 2, but not isomorphic to any of its hyperplanes. Therefore

$$W \oplus \mathbb{C} \overset{\mathbb{C}}{\hookrightarrow} W \quad \text{and} \quad W \overset{\mathbb{C}}{\hookrightarrow} W \oplus \mathbb{C} \quad \text{but} \quad W \not\sim W \oplus \mathbb{C}. \tag{1.4}$$

However, Banach spaces X and Y symmetrically complemented in each other are isomorphic whenever there exists a space A satisfying (1.2) and $A \sim A^{2m}$

for some $m \in \mathbb{N}$. Indeed, adding A^2 to both sides of (1.3) we obtain

$$X \sim X \oplus A^2 \sim X \oplus A^2 \oplus A^2 = X \oplus A^4.$$

So by induction

$$X \sim X \oplus A^{2m} \sim X \oplus A \sim Y.$$

Finally, inspired by the last remark, notice that we can strengthen (1.2) in such a way that the new conditions guarantee $X \sim Y$. For example, $X \sim Y$ whenever there exists a supplement A of Y in X satisfying

$$Y^2 \overset{A^5}{\hookrightarrow} X \quad \text{and} \quad X^2 \overset{A}{\hookrightarrow} Y.$$

Indeed, in this case

$$X \sim (Y \oplus A)^2 \oplus A^3 \sim X^2 \oplus A \oplus A^2 \sim Y \oplus A \oplus A \sim X \oplus A. \tag{1.5}$$

Now adding $X \oplus A$ to both sides of (1.5) we obtain

$$Y \sim X^2 \oplus A \sim X^2 \oplus A \oplus A \sim Y \oplus A \sim X.$$

So the natural question which originated the research of this paper is whether one can determine all quadruples (p, q, r, s) in $\mathbb{N}$ such that $X \sim Y$ whenever there exists a supplement A of Y in X satisfying

$$Y^p \overset{A^q}{\hookrightarrow} X \quad \text{and} \quad X^r \overset{A^s}{\hookrightarrow} Y.$$

To answer affirmatively this question it is convenient to introduce

Definition 1.2. A quadruple (p, q, r, s) in $\mathbb{N}$ is a *Cantor–Schroeder–Bernstein Quadruple* for Banach spaces (for short, CSBQ) if $X \sim Y$ whenever there exists a supplement A of Y in X such that

$$Y^p \overset{A^q}{\hookrightarrow} X \quad \text{and} \quad X^r \overset{A^s}{\hookrightarrow} Y. \tag{1.6}$$

We also say that $\Gamma = (q-1)(s+1) + (p-q)(s+r)$ is the *Cantor–Schroeder–Bernstein discriminant* of the quadruple (p, q, r, s).

The main aim of this paper is to present a characterization of the CSBQ's in terms of their discriminants Γ (see Theorem 1.3). This result suggests an intriguing problem concerning the geometry of the hyperplanes of Banach spaces (see Problem 1.4).

Theorem 1.3. *A quadruple (p, q, r, s) in $\mathbb{N}$ with discriminant Γ is a CSBQ if and only if one of the following conditions holds*:

(a) $\Gamma = 0$, $r = 1$ *and* $\gcd(q-1, s+1) = 1$;
(b) $\Gamma \neq 0$ *and* Γ *divides* $p-1$ *and* $r-1$.

The Banach spaces constructed by W. T. Gowers and B. Maurey in [16, p. 563] and the main result of [13] will be fundamental in the proof of Theorem 1.3 (see Remarks 2.1 and 2.2 and also the proofs of Proposition 3.1 and Lemma 4.1).

Nevertheless, we have not been able to obtain *a natural generalization of Theorem 1.3*, that is, one involving finite sums of X, X^m, $m \in \mathbb{N}$, $m \geq 2$, instead of X in the first condition of (1.6). In particular, in the simplest case of $m = 2$, we do not know whether two Banach spaces X and Y are isomorphic whenever there exists a supplement A of Y in X satisfying

$$Y^2 \overset{A}{\hookrightarrow} X^2 \quad \text{and} \quad X \overset{A}{\hookrightarrow} Y. \tag{1.7}$$

On one hand, observe that (1.7) implies that $X \sim Y$ when $A \sim X^p \oplus Y^q$ for some $p, q \in \mathbb{N} \cup \{0\}$ with $p + q \geq 1$. Indeed, in this case,

$$X \sim Y \oplus A \sim X^p \oplus Y^{q+1},$$

and

$$Y^2 \sim Y \oplus A \oplus X \sim X^2 \sim Y^2 \oplus A \sim Y \oplus Y \oplus A \sim Y \oplus X.$$

Thus, according to Remark 2.2 below, $X \sim Y$.

On the other hand, we do not know how to solve the above problem even when A is the smallest possible non-null space, that is, the field of real or complex scalars. Notice that in this case, indicating by K the field in question, we have

$$X^2 \sim Y^2 \oplus K \sim (X \oplus K)^2 \oplus K \sim X \oplus X \oplus K^2 \oplus K \sim X^2 \oplus K.$$

Thus (1.7) can be rewritten as follows:

$$X^2 \sim X^2 \oplus K \quad \text{and} \quad X \sim X \oplus K^2.$$

Hence our search for alternatives to Pełczyński's decomposition method leads naturally to:

Problem 1.4 (Strong Square-hyperplane Problem). *Let X be a Banach whose square space is isomorphic to its hyperplanes. Suppose that X is isomorphic to its subspaces of codimension* 2. *Does it follow that X is isomorphic to its hyperplanes?*

Observe that the Banach space W mentioned in (1.4) is a candidate for a negative solution to Problem 1.4. Moreover, evidently the answer to Problem 1.4 is affirmative if the following problem has a positive solution.

Problem 1.5 (Square-hyperplane Problem). *Let X be a Banach space whose square is isomorphic to its hyperplanes. Is X itself isomorphic to its hyperplanes?*

Finally, we recall that a Banach space H is *hereditarily indecomposable* (H.I.) if no closed subspace E of H contains a pair of infinite-dimensional closed subspaces M and N such that $E = M \oplus N$. In [15] W. T. Gowers and B. Maurey gave the first example of a H.I. space. We refer to [2] for a detailed survey of results about H.I. spaces. These spaces have been used to provide negative answers to several questions in Banach space theory (see for example

[1], [7], [15]–[17]). These spaces may be useful in solving Problems 1.4 and 1.5; we only remark that no H.I. space itself is a solution to Problem 1.4. Furthermore, they are not solutions to Problem 1.5 because the square of a H.I. space is not isomorphic to its hyperplanes [4, Corollary 5].

2. Preliminaries. We start by recalling some results on pairs of Banach spaces which are isomorphic to complemented subspaces of each other.

REMARK 2.1. In [16, p. 563] there were constructed Banach spaces X_t, for every $t \in \mathbb{N}$, $t \geq 2$, having the following property: $X_t^m \sim X_t^n$, with $m, n \in \mathbb{N}$, if and only if m is equal to n modulo t.

REMARK 2.2. In [13] a quintuple (p, q, r, s, t) in $\mathbb{N} \cup \{0\}$ with $p + q \geq 2$, $r + s + t \geq 3$, $(r, s) \neq (0, 0)$ and $t \geq 1$ was said to be a *Schroeder–Bernstein quintuple* (for short, SBq) if $X \sim Y$ whenever the Banach spaces X and Y satisfy (1.1) and the following Decomposition Scheme:

$$\begin{cases} X \sim X^p \oplus Y^q, \\ Y^t \sim X^r \oplus Y^s. \end{cases}$$

The number $\nabla = (p - 1)(s - t) - rq$ was called the *discriminant* of the quintuple (p, q, r, s, t).

We recall the following characterization of the SBq's obtained in [13]: Let (p, q, r, s, t) be a quintuple in $\mathbb{N}$ with $p + q \geq 2$, $r + s + t \geq 3$, $(r, s) \neq (0, 0)$ and $t \geq 1$. Then (p, q, r, s, t) is a SBq if and only if $\nabla \neq 0$ and ∇ divides $p + q - 1$ and $r + s - t$.

3. Sufficient conditions for a quadruple (p, q, r, s) in $\mathbb{N}$ to be a CSBQ. The main goal of this section is to prove the sufficiency part of Theorem 1.3, by proving Propositions 3.1 and 3.2 below.

PROPOSITION 3.1. *Let (p, q, r, s) be a quadruple in $\mathbb{N}$ with discriminant Γ. If $\Gamma \neq 0$ and Γ divides $p - 1$ and $r - 1$ then (p, q, r, s) is a CSBQ.*

Proof. Let X and Y be Banach spaces satisfying (1.6) for some supplement A of Y in X and quadruple (p, q, r, s) in $\mathbb{N}$ such that $\Gamma \neq 0$ and Γ divides $p - 1$ and $r - 1$. We will show that $X \sim Y$. It is convenient to distinguish two cases: $p \leq q$ and $p > q$.

CASE 1: $p \leq q$. There are two subcases: $\Gamma > 0$ and $\Gamma < 0$.

SUBCASE 1.1: $\Gamma > 0$. Let $m, n \in \mathbb{N} \cup \{0\}$ be such that $p - 1 = m\Gamma$ and $r - 1 = n\Gamma$. We can check that

$$\Gamma = (s + 1)(p - 1) - (q - p)(r - 1). \tag{3.1}$$

Thus

$$m(s + 1) = 1 + n(q - p). \tag{3.2}$$

By the first condition of (1.6), we have

$$X \sim Y^p \oplus A^p \oplus A^{q-p} \sim X^p \oplus A^{q-p}. \tag{3.3}$$

Adding $X^{p-1} \oplus A^{q-p}$ to both sides of (3.3) we conclude that

$$X \sim X^p \oplus A^{q-p} \sim X^{p+(p-1)} \oplus A^{2(q-p)} = X^{2(p-1)+1} \oplus A^{2(q-p)}.$$

Therefore by induction we get

$$X \sim X^{n(p-1)+1} \oplus A^{n(q-p)}. \tag{3.4}$$

Now according to the second condition of (1.6),

$$Y \sim X^r \oplus A^s. \tag{3.5}$$

Adding A to both sides of (3.5), we deduce that

$$X \sim Y \oplus A \sim X^r \oplus A^{s+1}.$$

Thus proceeding as above, we see that

$$X \sim X^{m(r-1)+1} \oplus A^{m(s+1)}. \tag{3.6}$$

By the choice of m and n, $n(p-1) = m(r-1)$. Hence bearing (3.2) in mind and using (3.4) in (3.6) we find that

$$X \sim X \oplus A. \tag{3.7}$$

Finally, adding $X^{r-1} \oplus A^s$ to both sides of (3.7), we infer that

$$\begin{aligned} Y &\sim X^r \oplus A^s \sim X \oplus X^{r-1} \oplus A^s \sim X \oplus X^{r-1} \oplus A^s \oplus A \\ &\sim X^r \oplus A^s \oplus A \sim Y \oplus A \sim X. \end{aligned}$$

Subcase 1.2: $\Gamma < 0$. Let $m, n \in \mathbb{N}$ be such that $p-1 = -m\Gamma$ and $r-1 = -n\Gamma$. Hence $n(p-1) = m(r-1)$ and according to (3.1), $n(q-p) = 1 + m(s+1)$. Thus analogously to Subcase 1.1, we use (3.6) in (3.4) to get $X \sim Y$.

Case 2: $p > q$. Since $X \sim Y \oplus A$, by the first condition of (1.6) we have

$$X \sim Y^{p-q} \oplus Y^q \oplus A^q \sim X^q \oplus Y^{p-q}. \tag{3.8}$$

Moreover, by the second condition of (1.6) we know that

$$Y \sim X^r \oplus A^s. \tag{3.9}$$

Adding Y^s to both sides of (3.9), we deduce that

$$Y^{s+1} \sim X^r \oplus A^s \oplus Y^s \sim X^r \oplus X^s = X^{r+s}.$$

Thus by (3.8) we conclude that

$$\begin{cases} X \sim X^q \oplus Y^{p-q}, \\ Y^{s+1} \sim X^{r+s}. \end{cases}$$

Since the discriminant ∇ of the quintuple $(q, p-q, r+s, 0, s+1)$ is equal to $-(q-1)(s+1) - (p-q)(r+s) = -\Gamma$, by hypothesis we have $\nabla \neq 0$, and

∇ divides $q+(p-q)-1=p$ and $r+s-(s+1)=r-1$. Furthermore $p\geq 2$, so by Remark 2.2 we conclude that $X\sim Y$. ∎

PROPOSITION 3.2. *Let (p,q,r,s) be a quadruple in $\mathbb{N}$ with discriminant $\Gamma=0$, $r=1$ and $\gcd(q-1,s+1)=1$. Then (p,q,r,s) is a CSBQ.*

Proof. By Bézout's theorem there exist $m,n\in\mathbb{N}\cup\{0\}$ such that

$$m(q-1)=n(s+1)+1 \quad \text{or} \quad n(s+1)=m(q-1)+1.$$

Since $r=1$, it follows that $\Gamma=(p-1)(s+1)=0$ and therefore $p=1$. Now, as in the proof of (3.4) and (3.5), we obtain

$$X\sim X\oplus A^{n(q-1)} \quad \text{and} \quad X\sim X\oplus A^{m(s+1)}.$$

So it suffices to proceed as in the proof of Proposition 3.1 to deduce that $X\sim Y$. ∎

4. Necessary conditions for a quadruple (p,q,r,s) in $\mathbb{N}$ to be a CSBQ. The main purpose of this section is to complete the proof of Theorem 1.3. This theorem is an immediate consequence of Propositions 4.2, 4.5 and 4.6 below. In order to prove Proposition 4.2 we need to state an auxiliary result. It is related to the Banach spaces X_t mentioned in Remark 2.1.

LEMMA 4.1. *Let $p,q,r,s\in\mathbb{N}$ and suppose that there exist $i,j,t\in\mathbb{N}$ with $t\geq 2$ satisfying*

(a) *t divides $i(q-1)+j(p-q)$;*
(b) *t divides $i(s+r)-j(s+1)$;*
(c) *t does not divide $j-i$.*

Then (p,q,r,s) is not a CSBQ.

Proof. Let $n\in\mathbb{N}$ be such that $nt-j+i>0$. Since $j+(nt-j+i)-i=nt$, by the property of X_t mentioned in Remark 2.1 we have

$$X_t^j \overset{A}{\hookrightarrow} X_t^i, \quad \text{where} \quad A=X_t^{nt-j+i}.$$

Next notice that from (a) and (b) we deduce that

$$X_t^{jp} \overset{A^q}{\hookrightarrow} X_t^i \quad \text{and} \quad X_t^{ir} \overset{A^s}{\hookrightarrow} X_t^j.$$

Furthermore, (c) implies that X_t^i is not isomorphic to X_t^j. Consequently, (p,q,r,s) is not a CSBQ. ∎

PROPOSITION 4.2. *If a quadruple (p,q,r,s) in $\mathbb{N}$ is a CSBQ with discriminant $\Gamma=0$, then $r=1$ and $\gcd(q-1,s+1)=1$.*

Proof. Suppose that the discriminant Γ of a quadruple (p,q,r,s) in $\mathbb{N}$ is equal to zero. We will show that (p,q,r,s) is not a CSBQ when $r\geq 2$ or $r=1$ and $\gcd(q-1,s+1)\geq 2$.

CASE 1: $r \geq 2$. Take $i = s+1$ and $j = s+r$. Thus $i(s+r) - j(s+1) = 0$ and since $\Gamma = 0$, it follows that $i(q-1) + j(p-q) = 0$. Moreover, $j - i = r - 1 \neq 0$. Hence it is enough to take $t \in \mathbb{N}$, $t \geq 2$, not dividing $r-1$ and apply Lemma 4.1 to see that (p, q, r, s) is not a CSBQ.

CASE 2: $r = 1$ and $\gcd(q-1, s+1) \geq 2$. Since $\Gamma = 0$, we deduce that $p = 1$. Take $i = 1$, $j = 2$ and $t = \gcd(q-1, s+1)$. Hence the conditions (a)–(c) of Lemma 4.1 are satisfied. Consequently, (p, q, r, s) is not a CSBQ. ∎

We need two lemmas.

LEMMA 4.3. *Let (p, q, r, s) be a quadruple in $\mathbb{N}$ with discriminant $\Gamma \geq 2$. Suppose that there exist integers α and β satisfying*

(a) $-\alpha(s+1) > \beta(p-q)$;
(b) $\beta(q-1) > \alpha(s+r)$;
(c) *Γ does not divide $\beta(p-1) + \alpha(r-1)$.*

Then (p, q, r, s) is not a CSBQ.

Proof. Let $t = \Gamma$ and consider the linear system

$$\left\{\begin{aligned} i(q-1) + j(p-q) &= \alpha t, \\ i(s+r) - j(s+1) &= \beta t. \end{aligned}\right. \tag{4.1}$$

The only solution of (4.1) is $i = -\alpha(s+1) - \beta(p-q)$ and $j = \beta(q-1) - \alpha(s+r)$. It follows from (a)–(c) that $i > 0$, $j > 0$ and t does not divide $j - i = \beta(p-1) - \alpha(r-1)$. Moreover, clearly t divides $i(q-1) + j(p-q)$ and $i(s+r) - j(s+1)$. Therefore Lemma 4.1 implies that (p, q, r, s) is not a CSBQ. ∎

Taking $t = -\Gamma$ and proceeding as in the proof of Lemma 4.3 we obtain:

LEMMA 4.4. *Let (p, q, r, s) be a quadruple in $\mathbb{N}$ with discriminant $\Gamma \leq -2$. Suppose that there exist integers α and β satisfying*

(a) $-\alpha(s+1) < \beta(p-q)$;
(b) $\beta(q-1) < \alpha(s+r)$;
(c) *Γ does not divide $\beta(p-1) - \alpha(r-1)$.*

Then (p, q, r, s) is not a CSBQ.

Now we are ready to complete the proof of the necessity part of Theorem 1.3, by proving Propositions 4.5 and 4.6 below.

PROPOSITION 4.5. *If a quadruple (p, q, r, s) in $\mathbb{N}$ with discriminant $\Gamma \neq 0$ is a CSBQ, then Γ divides $p-1$.*

Proof. Assume that a quadruple in $\mathbb{N}$ has discriminant $\Gamma \neq 0$ and Γ does not divide $p-1$. We will show that it is not a CSBQ. We consider two cases: $\Gamma \geq 2$ and $\Gamma \leq -2$.

CASE 1: $\Gamma \geq 2$. We distinguish three subcases: $p < q$, $p = q$ and $p > q$.

SUBCASE 1.1: $p < q$. Then $q > 1$ and according to the definition of Γ,

$$(s+r)/(q-1) < (s+1)/(q-p).$$

Take $\alpha = q - p$ and $\beta = s + 2$. Hence

$$(s+1)/(q-p) < \beta/\alpha \quad \text{and} \quad \beta(p-1) - \alpha(r-1) = \Gamma + p - 1.$$

By Lemma 4.3, we infer that (p, q, r, s) is not a CSBQ.

SUBCASE 1.2: $p = q$. Then $\Gamma = (p-1)(s+1)$ and therefore $p \geq 2$. Take $\alpha = 1 - q$ and $\beta = 1 - s - r$. Hence $\alpha < 0$, $\beta(q-1) > \alpha(s+r)$ and $\beta(p-1) - \alpha(r-1) = -\Gamma + p - 1$. Thus Lemma 4.3 implies that (p, q, r, s) is not a CSBQ.

SUBCASE 1.3: $p > q$. We consider the subcases $q = 1$ and $q > 1$.

SUBCASE 1.3.1: $q = 1$. Then $\Gamma = (p-1)(s+r) > 0$ and hence $p \geq 2$. Take $\alpha = 1 - p$ and $\beta = s$. Then $\alpha < 0$, $\beta(p-1) < -\alpha(s+1)$ and $\beta(p-1) - \alpha(r-1) = \Gamma - (p-1)$. According to Lemma 4.3, (p, q, r, s) is not a CSBQ.

SUBCASE 1.3.2: $q > 1$. Then by the definition of Γ we have

$$(s+1)/(q-p) < (s+r)/(q-1).$$

Take $\alpha = q-p$ and $\beta = s+2$. So $\beta/\alpha < (s+1)/(q-p)$ and $\beta(p-1)-\alpha(r-1) = \Gamma + p - 1$. Therefore again by Lemma 4.3 we deduce that (p, q, r, s) is not a CSBQ.

CASE 2: $\Gamma \leq -2$. Then by the definition of Γ, $p < q$ and

$$(s+1)/(q-p) < (s+r)/(q-1).$$

Take $\alpha = q-p$ and $\beta = s$. So $\beta/\alpha < (s+1)/(q-p)$ and $\beta(p-1) - \alpha(r-1) = \Gamma - (p-1)$. Thus Lemma 4.4 implies that (p, q, r, s) is not a CSBQ. ∎

PROPOSITION 4.6. *If a quadruple (p, q, r, s) in $\mathbb{N}$ with discriminant $\Gamma \neq 0$ is a CSBQ, then Γ divides $r - 1$.*

Proof. Suppose that a quadruple in $\mathbb{N}$ has discriminant $\Gamma \neq 0$ and Γ does not divide $r - 1$. We will show that it is not a CSBQ. We consider two cases: $\Gamma \geq 2$ and $\Gamma \leq -2$.

CASE 1: $\Gamma \geq 2$. Then according to the definition of Γ,

$$(q-p)/(s+1) < (q-1)/(s+r).$$

Take $\alpha = q - p - 1$ and $\beta = s + 1$. Hence $\alpha/\beta < (q-p)/(s+1)$ and $\beta(p-1) - \alpha(r-1) = \Gamma + r - 1$. By Lemma 4.3, we infer that (p, q, r, s) is not a CSBQ.

Case 2: $\Gamma \leq -2$. Then again by the definition of Γ,

$$(q-1)/(s+r) < (q-p)/(s+1).$$

Take $\alpha = q-p+1$ and $\beta = s+1$. It follows that $(q-p)/(s+r) < \alpha/\beta$ and $\beta(p-1) - \alpha(r-1) = \Gamma - (r-1)$. It suffices to apply Lemma 4.4 to conclude that (p,q,r,s) is not a CSBQ. ∎

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