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GLOBAL EXISTENCE OF SOLUTIONS TO A CHEMOTAXIS SYSTEM WITH VOLUME FILLING EFFECT

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Abstract. A system of quasilinear parabolic equations modelling chemotaxis and taking into account the volume filling effect is studied under no-flux boundary conditions. The resulting system is non-uniformly parabolic. A Lyapunov functional for the system is constructed. The proof of existence and uniqueness of regular global-in-time solutions is given in cases when either the Lyapunov functional is bounded from below or chemotactic forces are suitably weakened. In the first case solutions are uniformly bounded in time, in the second one it is shown that a uniform bound is not possible.

1. Introduction. In the present paper we study the following boundary value problem:

$$\frac{\partial u}{\partial t} = \nabla.[\alpha(u)\nabla u - u\beta(u)\nabla v], \tag{1}$$

$$\frac{\partial v}{\partial t} = D_v \Delta v - v + u \quad \text{in } U \times (0,T), \tag{2}$$

$$\nabla v \cdot \vec{n} = 0, \quad \nabla v \cdot \vec{n} = 0 \quad \text{on } \partial U \times (0,T), \tag{3}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in } U, \tag{4}$$

where U is an open bounded subset of $\mathbb{R}^n$ with boundary of class C^2 and $\vec{n}$ denotes the outer normal vector. We look for classical solutions $u, v : U \times [0,T) \to \mathbb{R}$.

The problem is studied under the following hypotheses:

T: (i) There exists $\varepsilon > 0$ such that $\beta \in C^2(-\varepsilon,\infty)$ is a positive bounded function. D_v is a positive constant.

(ii) There exists $\varepsilon > 0$ such that $\alpha \in C^2(-\varepsilon,\infty)$ is a positive bounded function.

Such systems arise in the study of chemotaxis phenomena. Chemotaxis is a chemosensitive movement of biological cells which may detect and response to some chemical secreted into environment. The system we are going to investigate describes the chemotactic movement of cells taking into account

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the volume filling effect (cf. [10]): the higher the density at x, the smaller the chance that another cell attains that position.

The previous models of chemotactic movement, the so called minimal version of the classical Keller–Segel model (cf. [15], see also the survey [12] and the bibliography therein) predict a blow-up in dimensions $n \geq 2$ (see, e.g., [14], [3], [4], [5]).

In order to avoid this property, several models have been given that were supposed to prevent blow-up which is interpreted as the overcrowding of cells. One of such attempts was presented in [10].

The idea in [10] to get rid of this phenomenon was to consider the volume filling effect which appears at high cell densities. We denote the density of cells by u and the density of the chemoattractant, a chemical which attracts cells, by v. Then the model derived in [10] is (1)–(4), where the C^2 functions α and β are given in the following way:

$$\alpha(u) = q(u) - uq'(u), \tag{5}$$

$$\beta(u) = q(u), \tag{6}$$

where $q(u)$ is interpreted as the probability that the particle attains a position (x, t) if the density of cells at this position equals u. Notice that the assumptions $q(u) \geq 0$ for $0 \leq u < \infty$ and q decreasing seem very natural. So α and β are bounded. The authors suggested considering the case when there is no value of u at which chemotaxis is switched off and

$$q(u) \xrightarrow{u\to\infty} 0. \tag{7}$$

The purpose of this paper is to find sufficient conditions on $q(u)$ for (1)–(4) to have unique global regular solutions if (7) holds. The case of $q(u) = 0$ for $u \geq M$, where M is a fixed value, was solved in [18]. Before we formulate our results in a precise way, let us state a few facts about connections between α, β and q.

If (7) holds we have two possibilities. If

$$\lim_{u\to\infty} uq'(u) > 0 \tag{8}$$

then (1)–(4) is uniformly parabolic. Otherwise α defined in (5) is not bounded away from 0, so (1)–(4) is no more uniformly parabolic. On the other hand, this case is specially interesting. Examples of q that lead to non-uniform parabolicity are $(1 + u)^{-\lambda}$, $\lambda > 0$, or $e^{-\gamma u}$, $\gamma > 0$. The latter was mentioned in [10]. The present paper is devoted to investigating non-uniformly parabolic problems.

Notice that the functions α and β satisfy

$$\frac{\beta(\eta)}{\alpha(\eta)} \leq M \tag{9}$$

for $M = 1$ and every $\eta \in \mathbb{R}$. Indeed, since q is decreasing, we have $q(u) \leq q(u) - uq'(u)$ and (9) follows. This simple observation will be of importance in the further part of the paper. We will state our global existence results under hypothesis (9).

In order to estimate the L^∞ norm of the solution to (1)–(4) we shall need the following theorem.

PROPOSITION 0. *Let u solve the problem*

$$\frac{\partial u}{\partial t} = \nabla.[\alpha(u)\nabla u - u\beta(u)\nabla v] \quad \text{in } U \times (0,T), \tag{10}$$

$$\nabla u \cdot \vec{n} = 0, \quad \nabla v \cdot \vec{n} = 0 \quad \text{on } \partial U \times (0,T), \tag{11}$$

$$u(x,0) = u_0(x) \quad \text{in } U, \tag{12}$$

$$\sup_{\tau < t < T_{\max}} \|\nabla v(\cdot,t)\|_\infty < \infty, \tag{13}$$

corresponding to $u_0 \in L^\infty(U)$. Assume that hypotheses $\mathbf{T}$ *and* (9) *are satisfied. Then*

$$\sup_{[0,T_{\max})} \|u(\cdot,t)\|_\infty < \infty,$$

where $T_{\max}$ is the maximal interval of solution's existence.

Similar theorems were proved in [8] and [6]. In the first case it was impossible to infer uniform-in-time estimates for $\|u(\cdot,t)\|_\infty$, in the second the authors did not attempt to do it. In fact following their calculations we will show the uniform in time ($T_{\max} = \infty$) estimates of $\|u(\cdot,t)\|_\infty$ provided (13) holds uniformly.

Now we are in a position to present our main results. In the theorems given below, conditions on α and β under which (1)–(4) has global unique solutions, are given. The conditions depend on the dimension. For example, in dimension one for every system (1)–(4) satisfying (9) (in particular this assumption holds if α and β satisfy (5), (6)) global bounded classical solutions exist. This means that the model presented in [10] predicts no blow-up in dimension $n = 1$.

THEOREM 1. *Assume $n = 1$, $u_0, v_0 \in W^{1,p}(U)$ for $p > 2$ are nonnegative functions and hypotheses* $\mathbf{T}$ *and* (9) *are satisfied. Then there exists a unique classical nonnegative global uniformly-in-time bounded solution to* (1)–(4).

Theorems 2 and 3 specify conditions that prevent blow-up in (1)–(4) in dimensions 2 and 3. The assumptions of Theorem 2 are excluded by those of Theorem 3 and vice versa. It seems worth underlying that according to Theorem 2 we do not need the boundedness from below of the Lyapunov functional to prevent blow-up in (1)–(4) (see Lemma 2.2). Let $B(0,r)$ be the ball centred at 0 with radius r.

THEOREM 2. *Assume $n = 2, 3$. Let $u_0, v_0 \in W^{1,p}(U)$ for $p > n$ be nonnegative and assume that hypotheses* **T** *and* (9) *are satisfied. Then there exists a unique classical nonnegative global-in-time solution to* (1)–(4) *provided there exists a positive constant M_1 such that for every $u > 0$,*

$$\beta(u) \le M_1 u^{-\gamma_1} \tag{14}$$

for some $\gamma_1 > n-1$. If $\Omega = B(0,r)$, then for every choice of β satisfying (14) *such that* (19) *holds there exists an unbounded radially symmetric solution.*

THEOREM 3. *Assume $n = 2, 3$. Let the nonnegative functions $u_0, v_0 \in W^{1,p}(U)$ for $p > n$ and assume that hypotheses* **T** *and* (9) *are satisfied. Then there exists a unique classical nonnegative global uniformly-in-time bounded solution to* (1)–(4) *provided there are positive constants M_2 and $\gamma_2 < 2/n$ such that for every $u > 0$,*

$$\frac{u\beta(u)}{\alpha(u)} \le M_2 u^{\gamma_2}. \tag{15}$$

The inspiration for this paper was [13], where it was proved that for $\alpha = 1$ and $u\beta = u^\gamma$, $\gamma < 2/n$, there exists a global bounded solution to (1)–(4). This paper extends the results of [13] to the case of a quasilinear system, including non-uniformly parabolic ones. The authors of [13] pointed out that their interests are purely mathematical. The results of our considerations, in view of [10], can be interpreted as the prevention of overcrowding in the model taking into account the volume filling effect.

The paper is organized as follows: Section 2 contains some preliminaries. In this section we also construct a Lyapunov functional for the system and present estimates of it. In Section 3 we prove the global existence in the one-dimensional case. Section 4 is devoted to proving the results in dimensions 2 and 3.

Notation. The norm in the space $L^p(U)$, $1 \le p \le \infty$, is denoted by $\|\cdot\|_p$. The same notation is used for vector-valued functions $u \in L^p(U;\mathbb{R}^n)$. The classical Sobolev spaces will be denoted by $W^{1,p}(U)$ for $1 \le p \le \infty$. Sometimes to shorten the notation we shall denote the vector-valued function (u, v) by $\overline{u}$. We shall denote the Lebesgue measure of a set A by $|A|$.

2. Preliminaries. First we consider the local existence of solutions to (1)–(4). We set

$$G = \{(u, v) : u > -\varepsilon,\ v > -\varepsilon\},$$

where ε is as in hypotheses **T**. For any $(\varphi_1, \varphi_2) \in G$, $a_{j,k}(\varphi_1, \varphi_2)$, $1 \le j, k \le n$, is a family of 2×2 matrices such that

$$a_{kk}(\varphi_1, \varphi_2) = \begin{bmatrix} \alpha(\varphi_1) & -\varphi_1\beta(\varphi_1) \\ 0 & D_v \end{bmatrix}, \qquad a_{j,k} \equiv 0 \qquad \text{for } j \ne k.$$

We define the boundary-value operator $(\mathcal{A},\mathcal{B})$ in the following way:

$$\mathcal{A}(\varphi)\overline{z} = -\sum_{j,k=1}^{n} \partial_j(a_{j,k}(\varphi)\partial_k\overline{z}), \qquad \mathcal{B}(\varphi)\overline{z} = \sum_{j,k=1}^{n} n_j a_{j,k}(\varphi)\partial_k\overline{z},$$

where $\vec{n} = (n_1, \ldots, n_n)$ and $\overline{z} = (z_1, z_2)$.

Since $a_{j,k}$ is upper triangular, the normal ellipticity of the boundary-value operator $(\mathcal{A},\mathcal{B})$, according to Amann's terminology, follows from $\mathbf{T}$(ii) and the positivity of D_v.

We are now in a position to apply Amann's existence theory for quasilinear parabolic systems. Thanks to [2, Theorem 1] and the maximum principle we obtain local existence of a regular nonnegative solution. [2, Theorem 2] yields the continuation principle and [2, Theorem 5.2] gives the Hölder regularity estimate (cf. (16)). To be more precise, we obtain the following theorem where $C^{1+\sigma}(\overline{\Omega})$ denotes the space of C^1 functions whose derivatives are Hölder continuous.

THEOREM 2.1. *Assume* $\mathbf{T}$ *and* $\overline{u}_0 \in W^{1,p}(U,\mathbb{R}^2)$, $\overline{u}_0 \geq 0$. *Then* (1)–(4) *has a unique maximal classical nonnegative solution. Let* $T_{\max}$ *be the time of the solution's existence.*

(a) *If*

$$\sup_{t\in[0,T_{\max})\cap[0,T]} \|\overline{u}(\cdot,t)\|_\infty < \infty$$

for every $T < \infty$ *and* $\overline{u}$ *is bounded away from* ∂G, *then* $T_{\max} = \infty$, *and there exists* $0 < \sigma < 1$ *such that*

$$\sup_{t<T_{\max}} \|\overline{u}(\cdot,t)\|_{C^{1+\sigma}(\overline{\Omega})} < \infty. \tag{16}$$

(b) *For every* $t \in (0, T_{\max})$,

$$\begin{aligned} \int_U u(t)\,dx &= \int_U u_0\,dx, \\ \int_U v(t)\,dx &= \int_U u_0 dx + \Big(\int_U v_0 dx - \int_U u_0 dx\Big)e^{-t}. \end{aligned} \tag{17}$$

Part (b) follows by integrating (1) and (2) and using the Stokes formula.

Now let us give a Lyapunov functional for the system (1)–(4) and prove some estimates of it. Set

$$\Phi(s) := \int_0^s\int_1^\sigma \frac{\alpha(\tau)}{\tau\beta(\tau)}\,d\tau\,d\sigma, \qquad s > 0.$$

Then it turns out that

$$L(u,v) := \int_U \Phi(u) + \frac{D_v}{2}\int_U |\nabla v|^2 + \frac{1}{2}\int_U v^2 - \int_U uv$$

is a Lyapunov functional for the system (1)–(4). Indeed, we have

LEMMA 2.1. *Let* (u,v) *be the classical solution to* (1)–(4). *Then* L *is a Lyapunov functional*, *more precisely*,

$$L(\overline{u})(t)+\int_s^t\int_U v_t^2\,dx\,d\tau+\int_s^t\int_U\frac{|\alpha(u)\nabla u-u\beta(u)\nabla v|^2}{u\beta(u)}=L(\overline{u})(s)\tag{18}$$

for every $t>s>0$.

Proof. We see that $u(x,t)>0$ for $t>0$ by the nonnegativity of u_0 and the strong maximum principle unless $u(x,t)\equiv 0$. Notice that from (1),

$$\begin{aligned}\int_U u_t(\Phi'(u)-v)\,dx&=\int_U\nabla.\{\alpha(u)\nabla u-u\beta(u)\nabla v\}(\Phi'(u)-v)\,dx\\&=-\int_U(\alpha(u)\nabla u-u\beta(u)\nabla v)(\Phi''(u)\nabla u-\nabla v)\,dx\\&=-\int_U\frac{|\alpha(u)\nabla u-u\beta(u)\nabla v|^2}{u\beta(u)}\,dx.\end{aligned}$$

Since

$$\frac{d}{dt}\int_U\Phi(u)=\int_U\Phi'(u)u_t,$$

and

$$\begin{aligned}&\frac{d}{dt}\Big(\int_U v^2\,dx+\int_U D_v|\nabla v|^2\,dx-\int_U uvdx\Big)\\&\quad=2\int_U vv_t\,dx-D_v\int_U\Delta vv_t\,dx-\int_U uv_t\,dx-\int_U u_tv\,dx\\&\quad=-\int_U v_t^2\,dx-\int_U u_tv\,dx,\end{aligned}$$

the claim follows.

Similar Lyapunov functionals for quasilinear chemotaxis systems were introduced in [17] and [13]. As the Lyapunov functional given above is of the same form as the one given in [13], the following estimates hold (for the proofs see [13, Remark after Lemma 5.1, and Lemma 5.2] and [11] for $n=2$, $\gamma=2/n$).

LEMMA 2.2. *If* (15) *holds with* $\gamma<2/n$, $n\ge 2$, *then there exists a constant* C *such that* $L\ge C$ *for every* $(u,v)\in(W^{1,\infty}(U))^2$. *Moreover*, *if* $n=2,3$ *and*

$$\frac{u\beta(u)}{\alpha(u)}\ge Mu^\gamma\tag{19}$$

for some $\gamma>2/n$ (*in the two-dimensional case for* $\gamma=2/n$ *we can choose the initial data such that the subsequent conclusion holds*), *then for any fixed*

$\lambda > 0$ there exists $\varepsilon_0 > 0$ and families $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}, (v_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)} \subset W^{1,\infty}$ such that $u_\varepsilon, v_\varepsilon > 0$ in $\overline{U}$,

$$\int_U u_\varepsilon = \lambda \quad \text{and} \quad \int_U v_\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0,$$

but

$$L(u_\varepsilon, v_\varepsilon) \to -\infty \qquad \text{as } \varepsilon \to 0.$$

We will need the following lemma and an inequality. Although this is a standard result, we include the proof for the reader's convenience. Since v is a solution to (2) we give Schauder type estimates of it.

LEMMA 2.3. *Let v be the solution to (2), $T < \infty$. Then*

(i) $\sup_{\tau<t\le T} \|\nabla v(\cdot,t)\|_q \le C \sup_{\tau<t\le T} \|u(\cdot,t)\|_\delta$,
where $q = n\delta/(n-\delta)$, $\delta \ge 1$ and C is a positive constant.

(ii) *If $\sup_{\tau<t\le T} \|u(\cdot,t)\|_p < \infty$, $p > n$, then*

$$\sup_{\tau<t\le T} \|\nabla v(\cdot,t)\|_\infty < \infty.$$

Proof. For the proof of (i) see [13, Lemma 4.1]. To prove (ii) consider (2). We shall use the concept of X^α solutions, which can be found in [16]. Let $X = L^p(U)$. Being a regular solution, v is an X^α solution after arbitrarily small time τ (after this time $v(\cdot,t) \in X^\alpha$). Thus rescaling time $t_{\text{new}} := t_{\text{old}} - \tau$ we see that the solution is given by the formula (see for example [16])

$$v(t) = P^t v_0 + \int_0^t P^{t-s} u(s)\, ds, \qquad t > 0. \tag{20}$$

Here P^t is the semigroup generated by $A_\Delta = I - D_v \Delta$. The operator is defined on X, with the domain

$$D(A_\Delta) = \left\{ u \in W^{2,p} : \frac{\partial u}{\partial \vec{n}} = 0 \right\}, \qquad p > n.$$

Since $X^\alpha \subset C^1$ for $\alpha > n/2p + 1/2$, we see that in order to prove Lemma 2.3(ii) it suffices to find estimates for $\sup_{t\in(0,T]} \|v(t)\|_{X^\alpha}$ (thanks to the condition $p > n$).

Applying A_Δ^α to both sides of (20) we see that for $t > 0$,

$$\begin{aligned} \|v(t)\|_{X^\alpha} &\le \|A_\Delta^\alpha P^t v_0\|_p + \int_0^t \|A_\Delta^\alpha P^{t-s} u(s)\|_p\, ds \\ &\le \left\| \frac{e^{-\nu t}}{t^\alpha} v_0 \right\|_p + \int_0^t \frac{e^{-\nu(t-s)}}{(t-s)^\alpha} \|u(s)\|_p\, ds \end{aligned}$$

for some $\nu > 0$ and thus $\|v(t)\|_{X^\alpha} < \infty$ and Lemma 2.3 follows.

Next notice that from Theorem 2.1(b) we know that $\|u(\cdot,t)\|_1$ is finite. Thus, with the use of the Gagliardo–Nirenberg inequality and Lemma 2.3(i) with $\delta = 1$ one obtains

$$\|\nabla v(\cdot,t)\|_{2p'}^2 \le C\|\Delta v(\cdot,t)\|_2^{2b'}, \tag{21}$$

where C is a positive constant, $p' < n/(n-2)$ and

$$b' = \frac{\frac{n}{q} - \frac{n}{2p'}}{1 - \frac{n}{2} + \frac{n}{q}}\,. \tag{22}$$

For the details see [13, (27)].

For completeness we also present the following easy proposition.

Proposition 2.1. *Assume* $\mathbf{T}$ *and hypothesis* (9) *are satisfied and* $\overline{u}_0 \in W^{1,p}(U;\mathbb{R}^2)$, $\overline{u}_0 \ge 0$. *Then to prove the global existence to* (1)–(4) *it suffices to show that*

$$\sup_{t\in[0,T_{\max})\cap[0,T]} \|u(\cdot,t)\|_\infty < \infty$$

for every $T < \infty$.

Proof. Theorem 2.1 gives us the existence of regular unique maximal solutions. In order to prove that solutions are global we only have to estimate their L^∞ norms.

Notice that on every finite interval $[0,T]$,

$$Gt + M \tag{23}$$

is a supersolution to (2), where

$$M \ge \sup_{x\in U} v_0, \qquad G \ge \sup_{t\in[0,T_{\max})\cap[0,T]} \|u(\cdot,t)\|_\infty.$$

Thus, Proposition 2.1 is proved.

Next we provide some propositions that enable us to distinguish between uniformly bounded solutions and unbounded ones.

Let the triple $(u_\infty, v_\infty, \Gamma) \in (C(\overline{\Omega}), C^2(\overline{\Omega}), \mathbb{R})$ be a solution to the stationary problem (S_λ)

$$-D_v\Delta v_\infty + v_\infty = u_\infty \qquad \text{in } U, \tag{24}$$

$$\Phi'(u_\infty) - v_\infty = \Gamma \qquad \text{in } U, \tag{25}$$

$$\nabla v_\infty \cdot \vec{n} = 0 \qquad \text{on } \partial U, \tag{26}$$

$$\int_U u_\infty(x)\,dx = \int_U v_\infty(x) = \lambda. \tag{27}$$

Since the proof of [13, Theorem 6.1] relies only on the form of the Lyapunov functional (the same as in our case) and (24), we have immediately the following result.

PROPOSITION 2.2. *Let $n \in \{2,3\}$ and $\tau\beta(\tau)/\alpha(\tau) \ge c_0\tau^\delta, \delta > 2/n$, for all $\tau > 1$. Then for every $\lambda > 0$ and for every radially symmetric solution $(u_\infty, v_\infty, \Gamma)$ to the stationary problem (S_λ) there exists a constant $c_\lambda > 0$ such that $L(u_\infty, v_\infty) \ge -c_\lambda$.*

PROPOSITION 2.3. *Suppose (u,v) is a bounded solution to (1)–(4) with the initial data (u_0, v_0) satisfying $u_0 > 0$ in $\overline{U}$ and $\lambda = \int_U u_0(x)\,dx$. Then there exist $(u_\infty, v_\infty, \Gamma)$ satisfying (S_λ) and a sequence of times t_k such that*

$$(u(t_k), v(t_k)) \to (u_\infty, v_\infty)$$

as $k \to \infty$ in $\mathcal{W} = W^{1,p}(\Omega) \times C^2(\overline{\Omega})$, $p > n$.

Proof. Let us denote the ω-limit set of $\{(u(\cdot,t), v(\cdot,t))\}$ by $\omega(u,v)$. By (16) and the parabolic regularity there exists $t_0 > 0$ such that

$$\bigcup_{t \ge t_0} \{(u(\cdot,t), v(\cdot,t))\}$$

is relatively compact in $\mathcal{W}$. Hence $\omega(u,v)$ is a nonempty compact connected set. Let $(u', v') \in \omega(u,v)$.

Since $u(\cdot,t)$ is bounded, $L(u(\cdot,t), v(\cdot,t))$ is bounded from below. We see that L is continuous as a function on $\mathcal{W}$. By La Salle's invariance principle we know that L is constant on $\omega(u,v)$. Thus, if $(U(\cdot,t), V(\cdot,t))$ is a solution to (1)–(4) emanating from (u', v'), by the entropy production term in (18) we have $V_t(x,t) = 0$ and $\nabla(\Phi'(U) - V) = 0$ for all $t > 0$. Hence V and U are constant in time, $V \equiv v' \equiv v_\infty$, $U \equiv u' \equiv u_\infty$, and Proposition 2.3 is proved.

Now let us prove Proposition 0. A generalization of the Alikakos–Moser method to the non-uniformly parabolic case was presented for the first time in [6]. However, the authors of that paper did not attempt to use it for proving a uniform-in-time estimate of the L^∞ norm of the solution (for other reasons they could not ensure a uniform estimate for $\|\nabla v(\cdot,t)\|_\infty$). We shall prove that in our case this method leads to uniform-in-time estimates. The key is to use (9) and the nonlinear test function given for any $\eta > 0$, $p > 2$ as

$$\alpha(\eta)\phi''(\eta) = p(p-1)\eta^{p-2} \tag{28}$$

and $\phi(0) = \phi'(0) = 0$.

Proof of Proposition 0. We proceed as in [6, Lemma 6.14]. Multiplying (1) by $\phi'(u)$ and slightly modifying the calculations in [7, Section 9.3] as in [8, Lemma 4.1, steps I, II, III], we arrive at the following inequality (we use (9)):

$$\frac{d}{dt}\int_U \phi(u)\,dx \le -4^k \int_U u^{2^k}\,dx + C(4^k)^{n/2+1}\Big(\int_U u^{2^{k-1}}\,dx\Big)^2. \tag{29}$$

Integrating (28) twice we see that

$$\int_U u^p\,dx \le C\int_U \phi(u),$$

hence (29) implies, for every $0 \le s < t$ (we choose $p = 2^k$),

$$\int_U u(\cdot,t)^{2^k}\,dx - \int_U u(\cdot,s)^{2^k}\,dx \le -4^k C\int_s^t\int_U u^{2^k}\,dx\,d\tau + C(4^k)^{n/2+1}\int_s^t\Big(\int_U u^{2^{k-1}}\,dx\Big)^2 d\tau.$$

Since u is regular we divide both sides by $t-s$ and letting $t-s \to 0$ we arrive at

$$\frac{d}{dt}\int_U u^{2^k}dx \le -4^k C\int_U u^{2^k}\,dx + C(4^k)^{n/2+1}\Big(\int_U u^{2^{k-1}}\,dx\Big)^2.$$

Keeping in mind the last inequality (analogous to [7, first displayed inequality, p. 215]), we finish the proof exactly as in [7].

3. Dimension $n = 1$. In this section we prove Theorem 1. The proof is split into two steps. $T_{\max}$ is the maximal time of the solution's existence. Due to Theorem 2.1 we have classical local-in-time solutions. In the first step we have to obtain a uniform-in-time bound on $\|\nabla v(\cdot,t)\|_\infty$. Then we can apply Proposition 0 to show that there is no finite time blow-up of solutions to (1)–(4) and they are uniformly bounded.

STEP 1. By Theorem 2.1(b) we see that $u \in L^\infty((0,T_{\max});L^1(U))$. We will show that this implies $\nabla v \in L^\infty((\tau,T_{\max});L^\infty(U))$ for τ arbitrarily small. To this end we need the following auxiliary result.

LEMMA 3.1. *Assume*

$$f(x,t) \in L^\infty((0,T_{\max});L^1(\mathbb{R})) \cap C((0,T_{\max})\times U) \tag{30}$$

and there exists a positive constant f_* such that $\|f(\cdot,t)\|_1 \le f_*$. If Γ is a classical solution to the Cauchy problem

$$\frac{\partial\Gamma}{\partial t} = \Delta\Gamma - \Gamma + f(t,x)$$

in the L^1 sense with the initial data $\Gamma_0(x) \in L^1(\mathbb{R})$, i.e.

$$\Gamma \in C^{2,1}(\mathbb{R}\times(0,T_{\max})) \cap C([0,T_{\max});L^1(\mathbb{R}))$$

then $\nabla\Gamma \in L^\infty((\tau,T_{\max});L^\infty(\mathbb{R}))$.

Proof. The first step of the proof is the following standard proposition.

PROPOSITION 3.1. *The fundamental solution of*

$$\frac{\partial\Gamma}{\partial t} - \Delta\Gamma + \Gamma = 0 \tag{31}$$

is

$$G_1(x,t) = e^{-t} G(x,t),$$

where G is the Gauss–Weierstrass kernel.

With the use of Proposition 3.1 we can represent $\Gamma(x,t)$ by the following formula:

$$\Gamma(x,t) = \frac{e^{-t}}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-|x-y|^2/4t} \Gamma_0(y)\,dy + \int_0^t \frac{e^{-(t-s)}}{(4\pi(t-s))^{1/2}} \int_{\mathbb{R}} e^{-|x-y|^2/4(t-s)} f(s,y)\,dy\,ds.$$

We differentiate this with respect to x to get

$$\frac{\partial}{\partial x}\Gamma(x,t) = \frac{\partial}{\partial x} \frac{e^{-t}}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-|x-y|^2/4t} \Gamma_0(y)\,dy + \int_0^t ds \int_{\mathbb{R}} \frac{e^{-(t-s)}}{4\pi^{1/2}(t-s)^{3/2}} (x-y) e^{-|x-y|^2/4(t-s)} f(s,y)\,dy. \tag{32}$$

Thus,

$$\left|\frac{\partial}{\partial x}\Gamma(x,t)\right| \le F(t) + \frac{1}{4\pi^{1/2}} \int_0^t \frac{e^{-(t-s)}}{(t-s)^{3/2}}\,ds \int_{\mathbb{R}} (t-s)\omega e^{-\omega^2} f(s,x-\omega)\,d\omega \tag{33}$$

where

$$F(t) = \frac{\partial}{\partial x} \frac{e^{-t}}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-|x-y|^2/4t} \Gamma_0(y)\,dy.$$

The second term on the right-hand side of (33) was obtained by the change of variables $(x-y)/\sqrt{t-s} \mapsto \omega$ in (32). From the general theory we know that $F(t)$ is uniformly bounded for $t > \tau > 0$. Hence,

$$\left|\frac{\partial}{\partial x}\Gamma(x,t)\right| \le F(t) + C f_* \int_0^t \frac{e^{-(t-s)}}{(t-s)^{1/2}}\,ds,$$

where the bound on xe^{-x^2} is absorbed in C.

The last formula implies that

$$\|\nabla \Gamma(\cdot,t)\|_\infty \le W, \tag{34}$$

where W is independent of time. Indeed,

$$\int_0^t \frac{e^{-(t-s)}}{(t-s)^{1/2}}\,ds = \int_0^t \frac{e^{-z}}{z^{1/2}}\,dz.$$

Since for small $\delta > 0$ and every $z > 0$, $e^{-z} < 1/z^{1/2+\delta}$, we choose $\varsigma > 0$ such that

$$\int_0^t \frac{e^{-z}}{z^{1/2}}\,dz < \int_0^{\varsigma} \frac{1}{z^{1/2}} + \int_{\varsigma}^t \frac{1}{z^{1+\delta}} \le 2\varsigma^{1/2} - \delta(t^{-\delta} - \varsigma^{-\delta}) < 2\varsigma^{1/2} + \delta\varsigma^{-\delta}.$$

In particular $\nabla v(\cdot, t)$ is uniformly bounded. This completes the proof of Lemma 3.1.

Those estimates of Green's function will be of importance in proving that

$$\nabla v(\cdot, t) \in L^\infty((\tau, T_{\max}); L^\infty(U)).$$

In view of Theorem 2.1,

$$u \in C^{2,1}(U \times (0, T_{\max})) \cap L^\infty((0, T_{\max}); L^1(U)).$$

Since v is a solution to (2), from the uniqueness and regularity of solutions to the heat equation we infer that $\frac{\partial}{\partial x} v$ satisfies

$$\frac{\partial \frac{\partial}{\partial x} v}{\partial t} - D_v \Delta \frac{\partial}{\partial x} v + \frac{\partial}{\partial x} v = \frac{\partial}{\partial x} u \tag{35}$$

in U under the zero Dirichlet boundary conditions.

Let us now define a function $\widetilde{f}$ which is a suitable continuation of u to the whole real line. First we fix a positive number δ and the domain $U = [a, b]$. From the regularity of solutions to (1)–(4) we see that there are continuous functions $A, B : [0, T_{\max}) \to \mathbb{R}_+$ such that $u(a, t) = A(t)$, $u(b, t) = B(t)$. For $(x, t) \in \mathbb{R} \times (0, T_{\max})$ we set

$$\widetilde{f}(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in [a,b] \times (0, T_{\max}), \\ M_1(t)x + M_2(t) & \text{if } (x,t) \in [a - \delta/u(a,t), a] \times (0, T_{\max}), \\ M_3(t)x + M_4(t) & \text{if } (x,t) \in [b, b + \delta/u(b,t)] \times (0, T_{\max}), \\ 0 & \text{elsewhere,} \end{cases} \tag{36}$$

where

$$M_1(t) = u^2(a,t)/\delta, \qquad M_2(t) = u(a,t) - au^2(a,t)/\delta,$$
$$M_3(t) = -u^2(b,t)/\delta, \qquad M_4(t) = u(b,t) + bu^2(b,t)/\delta.$$

From the construction we observe that $\widetilde{f}(x, t)$ differs from u only outside the set $[a, b] \times (0, T_{\max})$ and satisfies the condition (30). Moreover, $\|\widetilde{f}(\cdot, t)\|_1 = \|u(\cdot, t)\|_1 + \delta$ and $\widetilde{f} \in C([0, T_{\max}); W^{1,\infty}(\mathbb{R}))$. We denote by $\widetilde{v}$ the solution to

$$\frac{\partial \widetilde{v}}{\partial t} - D_v \Delta \widetilde{v} + \widetilde{v} = \widetilde{f} \tag{37}$$

in $\mathbb{R} \times (0, T_{\max})$. Then, in view of classical regularity results for parabolic equations, $\frac{\partial}{\partial x}\widetilde{v}$ satisfies

$$\frac{\partial \frac{\partial}{\partial x}\widetilde{v}}{\partial t} - D_v \Delta \frac{\partial}{\partial x}\widetilde{v} + \frac{\partial}{\partial x}\widetilde{v} = \frac{\partial}{\partial x}\widetilde{f} \tag{38}$$

in $\mathbb{R} \times (0, T_{\max})$. Owing to Lemma 3.1 and the fact that $\widetilde{v}$ is a solution to (37), we see that

$$\frac{\partial}{\partial x}\widetilde{v} \in L^\infty((\tau, T_{\max}); L^\infty(\mathbb{R})).$$

Next note that (35) and (38) result in

$$\frac{\partial w}{\partial t} - D_v \Delta w + w = 0$$

in $U \times (0, T_{\max})$, where $w := \frac{\partial}{\partial x}(v - \widetilde{v})$. Thus, by the maximum principle we infer

$$|w| \le |w|_{|\partial U \times (\tau, T_{\max})},$$

which implies

$$\left\| \frac{\partial}{\partial x} v(x,t) \right\|_{L^\infty(U \times (\tau, T_{\max}))} \le 2 \left\| \frac{\partial}{\partial x}\widetilde{v}(x,t) \right\|_{L^\infty(U \times (\tau, T_{\max}))} + \left\| \frac{\partial}{\partial x} w(\cdot, \tau) \right\|_{L^\infty(U)}.$$

STEP 2. Owing to Proposition 0 we have

$$\sup_{[0, T_{\max})} \|u(\cdot, t)\|_\infty < \infty.$$

In view of Theorem 2.1 and Proposition 2.1 we obtain $T_{\max} = \infty$ and the bound on the L^∞ norm is uniform. The proof is complete.

4. Dimensions $n = 2, 3$**.** In this section we give the proofs of Theorems 2 and 3. The scheme of the proof will be the same in both cases. First we assume $T_{\max}$ to be finite. Due to Theorem 2.1 we have classical local-in-time solutions. In the first step we have to obtain a bound on $\|u(\cdot, t)\|_p$, $p > n$, on finite time intervals. Then, with the use of Lemma 2.3(ii), we get a bound on $\|\nabla v(\cdot, t)\|_\infty$ on finite time intervals. This lets us apply Proposition 0 to show that there is no finite time blow-up of solutions to (1)–(4). Note that proving Theorem 3 we shall show that $\|u(\cdot, t)\|_p$, $p > n$, can be estimated independently of time, and thus the solution is uniformly bounded. By the proof of Theorem 2 we can only infer $(u, v) \in L^\infty_{\mathrm{loc}}((0, \infty); L^\infty(U))$, and it will be shown that this is essential.

As just said, Steps 2 and 3 of both proofs are the same and Step 1 differs in each case.

STEP 2. From Lemma 2.3(ii) we infer that $\|\nabla v(\cdot, t)\|_\infty < \infty$ on finite time intervals.

Step 3. We apply Proposition 0 to get

$$\sup_{\tau<t<T_{\max}} \|u(\cdot,t)\|_\infty < C(T_{\max}).$$

Theorem 2.1 and Proposition 2.1 give a contradiction to the assumption $T_{\max} < \infty$. Since $C(T_{\max})$ does not depend on $T_{\max}$ when the Lyapunov functional is bounded from below, the uniform boundedness in Theorem 3 holds.

So to complete the proofs we only have to show that there exists $p > n$ such that

$$\|u(\cdot,t)\|_p < \infty \tag{39}$$

on finite time intervals.

Step 1 of the proof of Theorem 2. We will denote by C a generic constant which may depend on $T_{\max}$ (assumed to be finite), but its value may vary from line to line.

We multiply (1) by u^{p-1} to obtain

$$\frac{1}{p}\frac{d}{dt}\int_U u^p dx + (p-1)\int_U \alpha(u)|\nabla u|^2 u^{p-2}\,dx = (p-1)\int_U u^{p-1}\beta(u)\nabla v\cdot\nabla u\,dx. \tag{40}$$

Since

$$u^{p-1}\beta(u) = u^{(p-2)/2}u^{p/2}\sqrt{\beta(u)}\sqrt{\beta(u)}$$

and (9) holds, the Cauchy–Schwarz inequality and (40) imply

$$\frac{1}{p}\frac{d}{dt}\int_U u^p\,dx + \frac{p-1}{2}\int_U \alpha(u)|\nabla u|^2 u^{p-2}\,dx \le C\int_U u^p\beta(u)|\nabla v|^2\,dx. \tag{41}$$

On the other hand, multiplying (2) by Δv and using the Cauchy–Schwarz inequality yields

$$\frac{d}{dt}\int_U |\nabla v|^2\,dx + \frac{D_v}{2}\int_U |\Delta v|^2\,dx + 2\int_U |\nabla v|^2\,dx \le C\int_U u^2\,dx. \tag{42}$$

Now adding inequalities (41) and (42) we obtain ($p > 2$)

$$\frac{d}{dt}\Big(\int_U |\nabla v|^2\,dx + \int_U u^p\,dx\Big) + \frac{D_v}{2}\int_U |\Delta v|^2\,dx \le C\Big(\int_U u^p\,dx\Big)^{2/p} + C\int_U u^p\beta(u)|\nabla v|^2\,dx. \tag{43}$$

Thanks to (14),

$$u^p\beta(u) \le M_1 u^{p-\gamma_1}.$$

Next the Hölder inequality yields

$$\int_U u^p \beta(u)|\nabla v|^2\,dx \le C\|u^{p-\gamma_1}\|_l \|\nabla v\|_{2p'}^2, \tag{44}$$

where $1/l + 1/p' = 1$.

We claim that

$$\|u^{p-\gamma_1}\|_l \|\nabla v\|_{2p'}^2 \le C\int_U u^p\,dx + C\|\Delta v\|_2^{2b'p'}.$$

Indeed, since $\|u(\cdot,t)\|_1 = \|u_0\|_1$, putting $\delta = 1$ in Lemma 2.3(i) we see that $\|\nabla v\|_q$ is finite for $q < n/(n-1)$. We choose $q = p' = n/(n-1) - \nu_1$ for small $\nu_1 > 0$. Then $l < n$. We observe that

$$(p-\gamma_1)l \le p \quad \text{provided} \quad \gamma_1 l - p(l-1) \ge 0. \tag{45}$$

Since (14) gives $\gamma_1 > n-1$ we find $p > n$ small enough to ensure that the right inequality of (45) holds. By (21) and the Young inequality applied to the right-hand side of (44) we see that the claim holds. We have used the fact that (22) ensures

$$2b'p' \le 2.$$

Owing to (44) and (14), taking $p > n$ we arrive at

$$\int_U u^p\beta(u)|\nabla v|^2\,dx \le C\int_U u^p\,dx + C_1\|\Delta v\|_2^2 + C|U|. \tag{46}$$

Obviously, using the Young inequality in (44), we can ensure $C_1 < D_v/4$.

In view of (46), by (43) we obtain

$$\frac{d}{dt}\Big(\int_U |\nabla v|^2\,dx + \int_U u^p\,dx\Big) + \frac{D_v}{4}\int_U |\Delta v|^2\,dx \le C\int_U u^p\,dx + C|U|. \tag{47}$$

With the use of the Gronwall inequality we estimate $\int_U u^p dx$. The proof of Step 1 is finished.

In order to finish the proof of Theorem 2 we still need to show that it is essential that we cannot obtain uniform-in-time boundedness of u.

Step 4 of the proof of Theorem 2. We assume (19) holds and $\Omega = B(0,R)$. We choose radially symmetric initial data $(u_0(x), v_0(x))$. At the same time we assume the solution to be uniformly bounded. We will show that this leads to a contradiction.

Indeed, by Proposition 2.3 there exists a subsequence $t_k \to \infty$ such that $(u(\cdot,t_k), v(\cdot,t_k))$ tends to a solution of (24)–(25) in $\mathcal{W}$ (for the definition see Proposition 2.3). This holds for all the choices of $(u_0(x), v_0(x))$. But then thanks to Lemma 2.2 we can define a sequence of initial data $(u_0^l(x), v_0^l(x))$, $l = 1, 2, \ldots$, in such a way that

$$L(u(\cdot,t_k)^l, v(\cdot,t_k)^l) \to -\infty \quad \text{as } t_k, l \to \infty. \tag{48}$$

The system (1)–(4) is rotationally invariant and thus $(u(\cdot,t_k)^l, v(\cdot,t_k)^l)$ is radially symmetric for all t_k. Hence $\omega(u,v)$ (for the definition see Proposition 2.3) consists of radially symmetric solutions. Indeed, since the convergence is in the space of continuous functions, one can analyze the difference between $(u(\cdot,t_k)^l, v(\cdot,t_k)^l)$ and the element of $\omega(u,v)$ on the sphere $\partial B(0,r)$, $r \le R$.

But then Proposition 2.2 contradicts (48).

STEP 1 OF THE PROOF OF THEOREM 3. In [9, Theorem 2.1] it was proved that (39) holds on finite time intervals. We shall analyze this proof in order to ensure that (39) holds uniformly-in-time.

The proof of [9, Theorem 2.1] was split into two steps. First we stated [9, Lemma 2.2] which is a generalization of the bootstrapping from [13] to the case of non-uniform parabolicity. Then we were able to repeat the argument in [13]. Notice that having [9, Lemma 2.2] (which we state below for the reader's convevience) with constants C_i, $i = 1, 2, 3$, on the right-hand side independent of time, we finish the proof exactly as in [9, Theorem 2.1] and obtain uniform estimates in (39).

LEMMA 2.2 ([9]). *Suppose that for a given v such that*

$$\sup_{[\tau,T_{\max})} \|\nabla v(\cdot,t)\|_{q_0} \le C_1, \qquad q_0 > 2, \tag{49}$$

u is the solution to (1) *under the no-flux boundary condition. Assume also*

$$\sup_{[\tau,T_{\max})} \|u(\cdot,t)\|_{\gamma_0} \le C_2. \tag{50}$$

Moreover, *suppose*

$$\left(\frac{n}{q_0} - 1\right)\gamma_0 < n(1-\gamma). \tag{51}$$

Then

$$\sup_{[\tau,T_{\max})} \|u(\cdot,t)\|_{\gamma_1} \le C_3$$

for any $\gamma_1 > \max\{\gamma_0, 2-2\gamma\}$ which fulfills

$$\left(\frac{n}{q_0} - 1\right)\gamma_1 < (n-2)(1-\gamma). \tag{52}$$

We have to check that if both C_1 and C_2 on the right-hand sides of (49) and (50) are independent of time, then so is C_3. We see that the energy estimates in [9, Lemma 2.2] are obtained with the use of the nonlinear test function (28). Then it is enough to follow the method used in the proof of Proposition 0 to ensure the uniform boundedness. We integrate [9, (19)] with respect to time from s to t, $\tau \le s < t < T_{\max}$. Then the right-hand side is estimated in the same way as in [9, Lemma 2.2], and since C_1 and C_2

are independent of time, this estimate is also independent of time. We let $t-s$ go to 0 and see that C_3 does not depend on time. The proof is finished.

Concluding remarks. At the end let us overview the possible choices of functions q generating Hillen–Painter models from the point of view of preventing blow-up.

In the one-dimensional case every nonincreasing q satisfies (9), so it prevents blow-up.

For higher dimensions let us focus on the choices of q of the form either $q(u) = (1+u)^{-\lambda}$ or $q(u) = e^{-\gamma u}$. In both 2- and 3-dimensional cases the second choice of q prevents blow-up and the solution is uniformly bounded. The first q prevents blow-up for $\lambda > n-1$, but for every such choice of q there exists an initial data such that the solution emanating from it becomes unbounded at ∞.

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