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COARSE STRUCTURES AND GROUP ACTIONS

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Abstract. The main results of the paper are:

PROPOSITION 0.1. *A group G acting coarsely on a coarse space $(X, \mathcal{C})$ induces a coarse equivalence $g \mapsto g \cdot x_0$ from G to X for any $x_0 \in X$.*

THEOREM 0.2. *Two coarse structures $\mathcal{C}_1$ and $\mathcal{C}_2$ on the same set X are equivalent if the following conditions are satisfied:*

(1) *Bounded sets in $\mathcal{C}_1$ are identical with bounded sets in $\mathcal{C}_2$.*
(2) *There is a coarse action ϕ_1 of a group G_1 on $(X, \mathcal{C}_1)$ and a coarse action ϕ_2 of a group G_2 on $(X, \mathcal{C}_2)$ such that ϕ_1 commutes with ϕ_2.*

They generalize the following two basic results of coarse geometry:

PROPOSITION 0.3 (Shvarts–Milnor lemma [5, Theorem 1.18]). *A group G acting properly and cocompactly via isometries on a length space X is finitely generated and induces a quasi-isometry equivalence $g \mapsto g \cdot x_0$ from G to X for any $x_0 \in X$.*

THEOREM 0.4 (Gromov [4, p. 6]). *Two finitely generated groups G and H are quasi-isometric if and only if there is a locally compact space X admitting proper and cocompact actions of both G and H that commute.*

1. Introduction. The proof in [2] of the Shvarts–Milnor lemma was based on the idea that isometric actions of groups ought to induce a coarse structure on the group under reasonable conditions. Since left coarse structures on countable groups are unique (in the sense of independence from the left-invariant proper metric), the Shvarts–Milnor lemma follows.

In this paper we investigate cases where group actions on sets induce a natural coarse structure on the set. As usual, the uniqueness of the coarse structure is of interest.

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We will use two approaches to coarse structures on a set X:

(1) The original one of Roe [5], based on controlled subsets of $X \times X$.
(2) The one from [3], based on uniformly bounded families in X.

The reason is that certain concepts and results have a more natural meaning in a particular approach to coarse structures. Recall that one can switch from one approach to another using the following basic facts (see [3]):

(a) If $\{B_s\}_{s\in S}$ is uniformly bounded, then $\bigcup_{s\in S} B_s \times B_s$ is controlled.
(b) If E is controlled, then there is a uniformly bounded family $\{B_s\}_{s\in S}$ such that $E \subset \bigcup_{s\in S} B_s \times B_s$.

To define a coarse structure using uniformly bounded families one needs to verify the following conditions:

(1) $\mathcal{B}_1$ is uniformly bounded implies $\mathcal{B}_2$ is uniformly bounded if each element of $\mathcal{B}_2$ consisting of more than one point is contained in some element of $\mathcal{B}_1$.
(2) $\mathcal{B}_1, \mathcal{B}_2$ uniformly bounded implies $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2)$ is uniformly bounded.

Definition 1.1. A function $f\colon (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ of coarse spaces is:

- *large scale uniform* (or *bornologous*) if $f(\mathcal{B}) \in \mathcal{C}_Y$ for every $\mathcal{B} \in \mathcal{C}_X$,
- *coarsely proper* if $f^{-1}(U)$ is bounded for every bounded subset U of Y,
- *coarse* if it is large scale uniform and coarsely proper.

Recall that two functions $f, g\colon S \to (X, \mathcal{C}_X)$ from a set S to a coarse space $(X, \mathcal{C}_X)$ are *close* if the family $\{\{f(s), g(s)\}\}_{s\in S}$ is bounded.

Definition 1.2. A coarse function $f\colon (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ of coarse spaces is a *coarse equivalence* if there is a coarse function $g\colon (Y, \mathcal{C}_Y) \to (X, \mathcal{C}_X)$ such that $f \circ g$ is close to id_Y and $g \circ f$ is close to id_X.

Here is a simple criterion for being a coarse equivalence using the approach of [3]:

Lemma 1.3. *A surjective coarse function $f\colon (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ of coarse spaces is a coarse equivalence if and only if $f^{-1}(\mathcal{B})$ is a uniformly bounded family in X for each uniformly bounded family $\mathcal{B}$ in Y.*

Proof. Let $g\colon Y \to X$ be a selection for $y \mapsto f^{-1}(y)$, and define $\mathcal{B}' = \{f^{-1}(y)\}_{y\in Y} \in \mathcal{C}_X$.

If $g\colon (Y, \mathcal{C}_Y) \to (X, \mathcal{C}_X)$ is bornologous, then $f^{-1}(\mathcal{B})$ refines $\mathrm{St}(g(\mathcal{B}), \mathcal{B}')$, resulting in $f^{-1}(\mathcal{B})$ being uniformly bounded.

Let us show that g is bornologous if f is a coarse equivalence. Choose $h\colon (Y, \mathcal{C}_Y) \to (X, \mathcal{C}_X)$ that is bornologous and $h \circ f$ is $\mathcal{B}_1$-close to id_X for some $\mathcal{B}_1 \in \mathcal{C}_X$. Therefore $h = h \circ f \circ g$ is $\mathcal{B}_1$-close to g and g is bornologous.

Assume $f^{-1}(\mathcal{B})$ is a uniformly bounded family in X for each uniformly bounded family $\mathcal{B}$ in Y. If g is bornologous, then f is a coarse equivalence as

$f \circ g = \mathrm{id}_Y$ and $g \circ f$ is $\mathcal{B}'$-close to id_X. If $\mathcal{B} \in \mathcal{C}_Y$, then $g(\mathcal{B})$ refines $f^{-1}(\mathcal{B})$, so it is uniformly bounded and g is bornologous. ∎

COROLLARY 1.4. *Suppose $f\colon X \to Y$ is a surjective function and $\mathcal{C}_1$, $\mathcal{C}_2$ are two coarse structures on Y. If $\mathcal{C}_X$ is a coarse structure on X such that both $f\colon (X, \mathcal{C}_X) \to (Y, \mathcal{C}_i)$, $i = 1, 2$, are coarse equivalences, then $\mathcal{C}_1 = \mathcal{C}_2$.*

Proof. Suppose $\mathcal{B} \in \mathcal{C}_1$ is uniformly bounded. Since $f^{-1}(\mathcal{B}) \in \mathcal{C}_X$ by 1.3 and $f\colon (X, \mathcal{C}_X) \to (Y, \mathcal{C}_2)$ is bornologous, $\mathcal{B} = f(f^{-1}(\mathcal{B})) \in \mathcal{C}_2$. Similarly, $\mathcal{C}_2 \subset \mathcal{C}_1$. ∎

REMARK 1.5. We will see in 2.2 that f being surjective is necessary.

2. Coarse structures on groups. Given a group G one can equip it with either the left coarse structure $\mathcal{C}_l(G)$ or right coarse structure $\mathcal{C}_r(G)$. For countable groups G those structures are metrizable by proper left-invariant (proper right-invariant) metrics on G.

In terms of controlled sets, $E \in \mathcal{C}_l(G)$ if and only if there is a finite subset F of G such that $x^{-1} \cdot y \in F$ for all $(x, y) \in E$. Similarly, $E \in \mathcal{C}_r(G)$ if and only if there is a finite subset F of G such that $x \cdot y^{-1} \in F$ for all $(x, y) \in E$. Notice all functions $x \mapsto g \cdot x$ ($g \in G$ being fixed) are coarse self-equivalences of $(G, \mathcal{C}_l(G))$ and all functions $x \mapsto x \cdot g$ are coarse self-equivalences of $(G, \mathcal{C}_r(G))$. We will primarily deal with the structure $\mathcal{C}_l(G)$ (notice $x \mapsto x^{-1}$ induces isomorphism of structures $\mathcal{C}_l(G)$ and $\mathcal{C}_r(G)$) but first we will characterize cases where the two structures are identical.

PROPOSITION 2.1. *The following are equivalent for any group G:*

(1) $\mathcal{C}_l(G) = \mathcal{C}_r(G)$,
(2) $\mathcal{C}_l(G) \subset \mathcal{C}_r(G)$,
(3) $\mathcal{C}_r(G) \subset \mathcal{C}_l(G)$,
(4) *G is an FC-group (conjugacy classes of all elements are finite).*

Proof. (3)⇒(4). Fix $a \in G$ and consider the family $\{\{x, a \cdot x\}\}_{x \in G}$. It is uniformly bounded in $\mathcal{C}_r(G)$, so it must be uniformly bounded in $\mathcal{C}_l(G)$, but that means the set $\{x^{-1} \cdot a \cdot x\}_{x \in G}$ is finite, i.e. the set of conjugacy classes of a is finite. The same proof shows (2)⇒(4).

(4)⇒(1). Given a uniformly bounded family $\mathcal{B}$ in $\mathcal{C}_l(G)$ there is a finite subset F of G such that $u^{-1} \cdot v \in F$ for all u, v belonging to the same element of $\mathcal{B}$. Let E be the set of conjugacy classes of all elements of F. If u, v belong to the same element of $\mathcal{B}$, then there is $f \in F$ so that $u^{-1} \cdot v = f$. Thus $v = u \cdot f$ and $v \cdot u^{-1} = u \cdot f \cdot u^{-1} \in E$. Thus $\mathcal{B}$ is uniformly bounded in $\mathcal{C}_r(G)$. The same argument shows $\mathcal{C}_r(G) \subset \mathcal{C}_l(G)$. ∎

COROLLARY 2.2. *There is a monomorphism $i\colon \mathbb{Z} \to \mathrm{Dih}_\infty$ from the integers to the infinite dihedral group Dih_∞ that induces coarse equivalences* *for both left coarse structures and right coarse structures but* $\mathcal{C}_l(\mathrm{Dih}_\infty) \neq \mathcal{C}_r(\mathrm{Dih}_\infty)$.

Proof. Consider the presentation $\{x, t \mid t^{-1}xt = x^{-1} \text{ and } t^2 = 1\}$ of Dih_∞. Identify $\mathbb{Z}$ with the subgroup of Dih_∞ generated by x. Notice $\mathbb{Z}$ is of index 2 in Dih_∞, so $\mathbb{Z} \to \mathrm{Dih}_\infty$ is a coarse equivalence for both left and right coarse structures. Since $\mathbb{Z}$ is Abelian, those coincide on that group but $\mathcal{C}_l(\mathrm{Dih}_\infty) \neq \mathcal{C}_r(\mathrm{Dih}_\infty)$ as the conjugacy class of x equals $\mathbb{Z}$. ∎

PROPOSITION 2.3. *The multiplication* $m\colon (G \times G, \mathcal{C}_l(G) \times \mathcal{C}_l(G)) \to (G, \mathcal{C}_l(G))$ *is large scale uniform if and only if* $\mathcal{C}_l(G) = \mathcal{C}_r(G)$.

Proof. Suppose F is a finite subset of G. Consider the uniformly bounded family $\{F \times \{x\}\}_{x\in G}$ in $\mathcal{C}_l(G) \times \mathcal{C}_l(G)$. Since $m(F \times \{x\}) = F \cdot x$, the family $\{F \cdot x\}_{x\in G} \in \mathcal{C}(G)_l$. Thus $\mathcal{C}_r(G) \subset \mathcal{C}_l(G)$ and $\mathcal{C}_l(G) = \mathcal{C}_r(G)$ by 2.1.

Suppose $\mathcal{C}_l(G) = \mathcal{C}_r(G)$. It suffices to show that $\{m(x\cdot F \times y\cdot E)\}_{(x,y)\in G\times G}$ is uniformly bounded for any finite subsets F and E of G. Choose a finite subset E' and a function $f\colon G \to G$ such that $x \cdot E \subset E' \cdot f(x)$ for all $x \in G$. Pick a finite subset F' of G and a function $g\colon G \to G$ such that $F \cdot E' \cdot y \subset g(y) \cdot F'$ for all $y \in G$. Now $m(x \cdot F \times y \cdot E) \subset x \cdot F \cdot E' \cdot f(y) \subset x \cdot g(f(y)) \cdot F'$ and the proof is complete. ∎

3. Inducing coarse structures by group actions. Our first task is to discuss cases of group actions of a group G on a set X inducing a coarse structure $\mathcal{C}_G$ on X such that $g \mapsto g\cdot x_0$ is a coarse equivalence from $(G, \mathcal{C}_l(G))$ to $(X, \mathcal{C}_G)$ for all $x_0 \in X$.

PROPOSITION 3.1. *Suppose a group* G *acts transitively on a set* X.

(1) *If there is a coarse structure* $\mathcal{C}_G$ *on* X *so that* $g \mapsto g \cdot x_0$ *is a coarse equivalence from* $(G, \mathcal{C}_l(G))$ *to* $(X, \mathcal{C}_G)$, *then the stabilizer of* x_0 *is finite.*
(2) *If the stabilizer of* x_0 *is finite, then there is a unique coarse structure* $\mathcal{C}_G$ *on* X *so that* $g \mapsto g \cdot x_0$ *is large scale uniform. In that case* $g \mapsto g \cdot x_0$ *is a coarse equivalence from* $(G, \mathcal{C}_l(G))$ *to* $(X, \mathcal{C}_G)$.

Proof. (1) If $\gamma\colon g \mapsto g \cdot x_0$ is a coarse equivalence, then $\gamma^{-1}(x_0)$ must be bounded in G, i.e. finite. Notice that $\gamma^{-1}(x_0)$ is precisely the stabilizer of x_0.

(2) Assume the stabilizer S of x_0 is finite. Define $\mathcal{C}_G$ as follows: $\mathcal{B} \in \mathcal{C}_G$ if $\gamma^{-1}(\mathcal{B})$ is uniformly bounded in $\mathcal{C}_l(G)$. If $\mathcal{C}_G$ is a coarse structure and $\gamma\colon (G, \mathcal{C}_l(G)) \to (X, \mathcal{C}_G)$ is bornologous, then 1.3 says γ is a coarse equivalence, and the uniqueness of $\mathcal{C}_G$ follows from 1.4.

Since $\gamma^{-1}(\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2)) = \mathrm{St}(\gamma^{-1}(\mathcal{B}_1), \gamma^{-1}(\mathcal{B}_2))$, we see that $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{C}_G$ implies $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{C}_G$. Given $\mathcal{B} \in \mathcal{C}_G$ we need to check that any family $\mathcal{B}'$, whose elements containing more than one point refine $\mathcal{B}$, also belongs to $\mathcal{C}_G$. There is a finite subset F of G such that $\gamma^{-1}(\mathcal{B})$ refines the family $\{g\cdot F\}_{g\in G}$.

Put $E = F \cup S$. If $\{x\} \in \mathcal{B}'$, then $\gamma^{-1}(x) = h \cdot S$, where $h \in G$ satisfies $x = h \cdot x_0$. Thus $\gamma^{-1}(\mathcal{B}')$ refines $\{g \cdot E\}_{g\in G}$, so $\mathcal{B}' \in \mathcal{C}_G$. ∎

PROPOSITION 3.2. *Suppose a group G acts on a set X. If there is a subset U of X such that $X = G \cdot U$ and the stabilizer*

$$S_U = \{g \in G \mid U \cap (g \cdot U) \neq \emptyset\}$$

of U is finite, then there is a coarse structure $\mathcal{C}_G$ on X so that $g \mapsto g \cdot x_0$ is a coarse equivalence from $(G, \mathcal{C}_l(G))$ to $(X, \mathcal{C}_G)$ for all $x_0 \in X$.

Proof. First define the bounded sets of $\mathcal{C}_G$. Those are subsets of sets of the form $F \cdot U$, where F is any finite subset of G. Second, define $\mathcal{C}_G$ as families $\mathcal{B}$ such that there is a bounded set V so that $\mathcal{B}$ refines $\{g \cdot V\}_{g\in G}$. Notice that, if $\mathcal{B}'$ is a family whose elements containing more than one point refine $\mathcal{B}$, then $\mathcal{B}'$ refines $\{g\cdot(V \cup U)\}_{g\in G}$ and $V \cup U$ is bounded. Thus $\mathcal{B}' \in \mathcal{C}_G$.

The important property of bounded sets V is that their stabilizers $S_V = \{g \in G \mid V \cap (g \cdot V) \neq \emptyset\}$ are finite. It suffices to prove that for $V = F \cdot U$, $F \subset G$ being finite. If $V \cap (g \cdot V) \neq \emptyset$, then there exist elements $f_i \in F$, $i = 1, 2$, such that $(f_1 \cdot U) \cap (g \cdot f_2 \cdot U) \neq \emptyset$, which implies $U \cap (f_1^{-1} g f_2 \cdot U) \neq \emptyset$. Thus $f_1^{-1} g f_2 \in S_U$ and $g \in F \cdot S_U \cdot F^{-1}$, which proves S_V is finite.

The second useful observation is that $\mathrm{St}(V, \mathcal{B})$ is bounded for any bounded set V and any $\mathcal{B} \in \mathcal{C}_G$. Indeed, if $\mathcal{B}$ refines $\{g \cdot W\}_{g\in G}$ for some bounded W, we may assume $V \subset W$, in which case V intersects only finitely many elements of $\{g \cdot W\}_{g\in G}$. Since those are all bounded and a finite union of bounded sets is bounded, we are done.

Suppose $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{C}_G$ and choose bounded sets V_i, $i = 1, 2$, such that $\mathcal{B}_i$ refines $\{g \cdot V_i\}_{g\in G}$. Put $V = \mathrm{St}(V_1, \{g \cdot V_2\}_{g\in G})$ and notice V is bounded. Our aim is to show $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2)$ refines $\{g \cdot V\}_{g\in G}$. If $(h \cdot V_1) \cap (g \cdot V_2) \neq \emptyset$, then $V_1 \cap (h^{-1} \cdot g \cdot V_2) \neq \emptyset$, so $V_1 \cup (h^{-1} \cdot g \cdot V_2) \subset V$, hence $\mathrm{St}(h \cdot V_1, \mathcal{B}_2) \subset h \cdot V$. ∎

Let us point out that, surprisingly, the structure $\mathcal{C}_G$ in 3.2 does not have to be unique, contrary to typical categorical intuition.

PROPOSITION 3.3. *There is an action of integers $\mathbb{Z}$ on the infinite dihedral group* Dih_∞ *such that $g \mapsto g \cdot x_0$ are coarse equivalences for both left and right coarse structures but $\mathcal{C}_l(\mathrm{Dih}_\infty) \neq \mathcal{C}_r(\mathrm{Dih}_\infty)$.*

Proof. Consider the presentation $\{x, t \mid t^{-1}xt = x^{-1}$ and $t^2 = 1\}$ of Dih_∞. Identify $\mathbb{Z}$ with the subgroup of Dih_∞ generated by x. Notice $\mathbb{Z}$ is of index 2 in Dih_∞, so $\mathbb{Z} \to \mathrm{Dih}_\infty$ is a coarse equivalence for both left and right coarse structures. Since $\mathbb{Z}$ is Abelian, those coincide on that group but 2.2 says that $\mathcal{C}_l(\mathrm{Dih}_\infty) \neq \mathcal{C}_r(\mathrm{Dih}_\infty)$. ∎

4. Actions by uniformly bornologous functions. We want to generalize isometric actions to the framework of coarse geometry. The appropriate concept is not only to require that each function $x \mapsto g \cdot x$ is bornologous but that those functions are uniformly bornologous.

DEFINITION 4.1. A group G acts on a coarse space $(X, \mathcal{C}_X)$ by *uniformly bornologous functions* if for any controlled set E there is a controlled set E' such that $(g \cdot x, g \cdot y) \in E'$ for all $(x, y) \in E$ and all $g \in G$.

PROPOSITION 4.2. *A group G acts on a coarse space X by uniformly bornologous functions if and only if for any uniformly bounded family $\mathcal{B} = \{B_s\}_{s\in S}$ in X the family $G \cdot \mathcal{B} = \{g \cdot B_s\}_{(g,s)\in G\times S}$ is uniformly bounded.*

Proof. Suppose the action is by uniformly bornologous functions and $\mathcal{B} = \{B_s\}_{s\in S}$ is a uniformly bounded family. Put $E = \bigcup_{s\in S} B_s \times B_s$ and notice it is a controlled set. Pick a controlled set E' such that $(g\cdot x, g\cdot y) \in E'$ for all $g \in G$ and all $(x, y) \in E$. Define $\mathcal{B}'$ as the family of all $B \subset X$ satisfying $B \times B \subset E'$. It is a uniformly bounded family containing $G \cdot \mathcal{B}$.

Suppose the family $G \cdot \mathcal{B} = \{g \cdot B_s\}_{(g,s)\in G\times S}$ is uniformly bounded for any uniformly bounded family $\mathcal{B} = \{B_s\}_{s\in S}$ in X. Assume E is a symmetric controlled set containing the diagonal. Consider the family $\mathcal{B}$ of all sets $B \subset X$ such that $B \times B \subset E \circ E \circ E \circ E$ and let

$$E' = \bigcup_{B\in\mathcal{B},\, g\in G} g \cdot B \times g \cdot B.$$

It is a controlled set and, if $(x, y) \in E$, then $\{x, y\} \times \{x, y\} \subset E \circ E \circ E \circ E$ and $(g \cdot x, g \cdot y) \in E'$. ∎

COROLLARY 4.3. *Let G be a group and $(X, \mathcal{C}_X)$ be a coarse space. If $\phi\colon (G \times X, \mathcal{C}_l(G) \times \mathcal{C}_X) \to (X, \mathcal{C}_X)$ is bornologous, then the action of G on $(X, \mathcal{C}_X)$ is by uniformly bornologous functions.*

Proof. Given a uniformly bounded family $\mathcal{B} = \{B_s\}_{s\in S}$ in X, the family $\{\{g\}\times B_s\}_{(g,s)\in G\times S}$ is uniformly bounded in $G\times X$, so $\{\phi(\{g\}\times B_s)\}_{(g,s)\in G\times S}$ is uniformly bounded, which means $G \cdot \mathcal{B}$ is uniformly bounded. ∎

REMARK 4.4. Notice that the infinite dihedral group Dih_∞ acts on itself by left multiplication so that the action is by uniformly bornologous functions but the multiplication is not bornologous (see 2.3 and 2.2).

5. Coarsely proper and cobounded actions

DEFINITION 5.1. An action ϕ of a group G on a coarse space $(X, \mathcal{C}_X)$ is *coarsely proper* if $\phi_x\colon G \to G \cdot x$ is coarsely proper for all $x \in X$.

LEMMA 5.2. *An action ϕ of a group G on a coarse space $(X, \mathcal{C}_X)$ is coarsely proper if and only if for every bounded subset U of X the family $\{g \cdot U\}_{g\in G}$ is point-finite.*

Proof. This follows from the fact that $\phi_x^{-1}(U) = \{g \in G \mid x \in g^{-1} \cdot U\}$ for all $x \in X$ and $U \subset X$. ∎

Corollary 5.3. *If an action ϕ of a group G on a coarse space $(X, \mathcal{C}_X)$ is coarsely proper and by uniformly bornologous functions, then $\phi_x \colon G \to G \cdot x$ is a coarse equivalence for all $x \in X$.*

Proof. Notice the stabilizer of x_0 is finite by 5.2 and use (2) of 3.1. ∎

Lemma 5.4. *Let ϕ be an action of a group G on a coarse space $(X, \mathcal{C}_X)$ by uniformly bornologous functions. Then it is coarsely proper if and only if the stabilizer*

$$S_U = \{g \in G \mid U \cap (g \cdot U) \neq \emptyset\}$$

of U is finite for every bounded subset U of X.

Proof. One direction is obvious in view of 5.2, so assume ϕ is an action by uniformly bornologous functions that is coarsely proper. If $S_U = \{g \in G \mid U \cap (g \cdot U) \neq \emptyset\}$ is infinite for some bounded set U, then put $V = \mathrm{St}(U, \{g \cdot U\}_{g \in G})$ and notice that $\phi_x^{-1}(V)$ contains S_U for all $x \in U$, a contradiction. ∎

Definition 5.5. An action of a group G on a coarse space $(X, \mathcal{C}_X)$ is *cobounded* if $X = G \cdot U$ for some bounded subset U of X.

Proposition 5.6. *If an action ϕ of a group G on a coarse space $(X, \mathcal{C}_X)$ is cobounded and by uniformly bornologous functions, then for every uniformly bounded family $\mathcal{B}$ there is a bounded set U such that $\mathcal{B}$ refines $\{g \cdot U\}_{g \in G}$.*

Proof. Pick a bounded set V such that $G \cdot V = X$. Given $\mathcal{B} = \{B_s\}_{s \in S} \in \mathcal{C}_X$ put $U = \mathrm{St}(V, G \cdot \mathcal{B})$. Then U is bounded and $\mathcal{B}$ refines $\{g \cdot V\}_{g \in G}$. ∎

Corollary 5.7. *If an action ϕ of a group G on a set X is cobounded and by uniformly bornologous functions under two coarse structures $\mathcal{C}_1$ and $\mathcal{C}_2$ on X, then $\mathcal{C}_1 = \mathcal{C}_2$ if and only if bounded sets in both structures are identical.*

Proof. By 5.6 both structures are generated by families $\{g \cdot U\}_{g \in G}$, where U is bounded. ∎

6. Coarse actions

Definition 6.1. An action of a group G on a coarse space $(X, \mathcal{C})$ is *coarse* if it is coarsely proper, cobounded, and by uniformly bornologous functions.

Corollary 6.2. *If an action ϕ of a group G on a coarse space $(X, \mathcal{C}_X)$ is coarse, then $\phi_x \colon (G, \mathcal{C}_l(G)) \to (X, \mathcal{C}_X)$ is a coarse equivalence for all $x \in X$.*

Proof. By 5.3 the function $g \mapsto g \cdot x_0$ is a coarse equivalence from G to $G \cdot x_0$. Notice the inclusion $G \cdot x_0 \to X$ is a coarse equivalence by the coboundedness of the action. ∎

THEOREM 6.3. *Suppose $\alpha_i \colon G_i \times X \to X$, $i = 1, 2$, are two commutative left actions of groups G_i on the same set X. If there are coarse structures $\mathcal{C}_i$, $i = 1, 2$, whose bounded sets coincide such that α_i is coarse with respect to $\mathcal{C}_i$, then*

(a) *G_1 is coarsely equivalent to G_2,*
(b) *$(X, \mathcal{C}_1)$ is coarsely equivalent to $(X, \mathcal{C}_2)$.*

Proof. Pick a bounded set (in both coarse structures) U with $G_i \cdot U = X$ for $i = 1, 2$. Pick $x_0 \in U$. Define $\psi \colon G_2 \to G_1$ so that $h^{-1} \cdot x_0 \in \psi(h) \cdot U$ for all $h \in G_2$.

To show ψ is large scale uniform consider a finite subset F of G_2 containing identity, define $V = F^{-1} \cdot U$ and define E as the set of all $g \in G_1$ so that $V \cap (g \cdot V) \neq \emptyset$. Suppose $h = h_1^{-1} h_2 \in F$ and $g_i = \psi(h_i)$ for $i = 1, 2$. Consider $y = g_1^{-1}(h_2^{-1} \cdot x_0)$ and put $g = g_1^{-1} g_2$. Our goal is to show $y \in V \cap (g \cdot V)$, resulting in $g \in E$. Since $g^{-1} \cdot y = g_2^{-1}(h_2^{-1} \cdot x_0) \in U \subset V$, we have $y \in g \cdot V$. Now, as $h_2 = h_1 \cdot h$, we see that $y = g_1^{-1}(h_2^{-1} \cdot x_0) = g_1^{-1}(h^{-1} \cdot h_1^{-1} \cdot x_0) = h^{-1}(g_1^{-1}(h_1^{-1} \cdot x_0)) \subset h^{-1} \cdot U \subset F^{-1} \cdot U = V$.

Similarly, define $\phi \colon G_1 \to G_2$ so that $g^{-1} \cdot x_0 \in \phi(g) \cdot U$ for all $g \in G_1$ and notice it is large scale uniform.

Let $\mathcal{B}$ be a uniformly bounded family in $\mathcal{C}_1$ so that all sets $g \cdot U$, $g \in G_1$, refine $\mathcal{B}$. Observe $g \mapsto g \cdot x_0$ and $g \mapsto \psi(\phi(g)) \cdot x_0$ are $\mathrm{St}(\mathcal{B}, \mathcal{B})$-close. Indeed, using the definition of ϕ and commutativity of the two actions, we get $\phi(g)^{-1} \cdot x_0 \in g \cdot U$, and by definition of ψ we have $\phi(g)^{-1} \cdot x_0 \in \psi(\phi(g)) \cdot U$. Since $g \mapsto g \cdot x_0$ is a coarse equivalence from G_1 to $(X_1, \mathcal{C}_1)$ (see 6.2), $\psi \circ \phi$ is close to the identity of G_1. Similarly, $\phi \circ \psi$ is close to the identity of G_2. ∎

7. Topological actions. Let X be a topological space and G be a group. Recall that an action of G on X is *topologically proper* if each point $x \in X$ has a neighborhood U_x such that the stabilizer $\{g \in G \mid U_x \cap (g \cdot U_x) \neq \emptyset\}$ of U_x is finite. An action of G on X is *cocompact* if there exists a compact subspace $K \subset X$ such that $G \cdot K = X$.

DEFINITION 7.1. Let X be a locally compact topological space. An action of a group G on X is *topological* if it is by homeomorphisms, it is cocompact and topologically proper.

PROPOSITION 7.2. *Suppose X is a locally compact topological space. If ϕ is a topological action of G on X, then there is a unique coarse structure $\mathcal{C}_\phi$ on X such that the action ϕ of G on $(X, \mathcal{C}_\phi)$ is coarse and the bounded sets* *in $\mathcal{C}_\phi$ are precisely the relatively compact subsets of X. The structure $\mathcal{C}_\phi$ is generated by the families $\{g \cdot K\}_{g\in G}$ where K is a compact subset of X.*

Proof. Uniqueness of ϕ follows from 5.7. Let us show that the stabilizer of each compact subset K of X is finite. If it is not, then there is an infinite subset I of G and points $x_g \in K \cap (g \cdot K)$ for each $g \in I$. The set $\{x_g\}_{g\in I}$ must be discrete (otherwise the action would not be topologically proper at its accumulation point), so infinitely many x_g's are equal, a contradiction.

Consider the structure $\mathcal{C}_\phi$ on X described in the proof of 3.2. Notice it has the required properties. ∎

COROLLARY 7.3. *Suppose $\phi\colon G \times X \to X$ and $\psi\colon H \times X \to X$ are two topological actions on a locally compact space X. If ϕ commutes with ψ, then $(X, \mathcal{C}_\phi)$ and $(X, \mathcal{C}_\psi)$ are coarsely equivalent.*

Proof. Use 6.3. ∎

REMARK 7.4. It is not true that $\mathcal{C}_\phi = \mathcal{C}_\psi$ in general. Use 3.3 and equip groups with discrete topologies.

THEOREM 7.5. *If G and H are coarsely equivalent groups, then there is a locally compact topological space X and topological actions $\phi\colon G \times X \to X$ and $\psi\colon H \times X \to X$ that commute.*

Proof. Pick a coarse equivalence $\alpha\colon G \to H$. Choose a function c assigning to each finite subset F of G a finite subset $c(F)$ of H with the property that $u^{-1} \cdot v \in F$ implies $\alpha(u)^{-1} \cdot \alpha(v) \in c(F)$.

Choose a function d assigning to each finite subset F of H a finite subset $d(F)$ of G with the property that $\alpha(u)^{-1} \cdot \alpha(v) \in F$ implies $u^{-1} \cdot v \in d(F)$.

Let E be a finite subset of H so that $H = \alpha(G) \cdot E$.

Let X be the space of all functions $\beta\colon G \to H$ satisfying the following conditions:

(1) $u^{-1} \cdot v \in F$ implies $\beta(u)^{-1} \cdot \beta(v) \in c(F)$ for all finite subsets F of G,
(2) $\beta(u)^{-1} \cdot \beta(v) \in F$ implies $u^{-1} \cdot v \in d(F)$ for all finite subsets F of H,
(3) $H = \beta(G) \cdot E$.

We consider X with the compact-open topology provided both G and H are given the discrete topologies. Notice X is closed in the space H^G of all functions from G to H equipped with the compact-open topology. Indeed, conditions (1) and (2) above hold for all $\beta \in \mathrm{cl}(X)$, so it remains to check $H = \beta(G) \cdot E$ for such β. Given $h \in H$ consider the set $F = \beta(1_G)^{-1} \cdot h \cdot E^{-1}$ and choose $\gamma \in X$ so that $\gamma(g) = \beta(g)$ for all $g \in d(F) \cup \{1_G\}$. Pick $g_1 \in G$ and $e \in E$ so that $h = \gamma(g_1) \cdot e$. Since $\gamma(1_G)^{-1} \cdot \gamma(g_1) \in F$, we see that $g_1 = 1_G^{-1} \cdot g_1 \in d(F)$ and $\gamma(g_1) = \beta(g_1)$. Thus $h \in \beta(G) \cdot E$.

Notice X is locally compact. Indeed, given $\beta \in X$ consider $U = \{\gamma \in X \mid \gamma(1_G) = \beta(1_G)\}$. It is clearly open and equals $X \cap K$, where $K \subset H^G$ is the set of all functions u satisfying $u(g) \in \beta(1_G) \cdot c(\{g\})$. Notice K is compact (it is a product of finite sets). Since X is closed in H^G, $X \cap K$ is compact as well.

The action of G on X is given by $(g \cdot \beta)(x) := \beta(g \cdot x)$. The action of H on X is given by $(h \cdot \beta)(x) := h \cdot \beta(x)$. Notice that the two actions commute. The action of H on X is cocompact: $X = H \cdot K$, where $K = \{\beta \in X \mid \beta(1_G) = 1_H\}$. The action of G on X is cocompact: $X = G \cdot L$, where L is the set of $\beta \in X$ such that $\beta(1_G) \in E^{-1}$ (which implies $\beta(g) \in E^{-1} \cdot c(\{g\})$ for all $g \in G$ so that L is compact). Indeed, for any $\gamma \in X$ there is $e \in E$ such that $1_H = \gamma(g_1) \cdot e$ for some $g_1 \in G$. Put $\beta(x) = \gamma(g_1 \cdot x)$ and notice $\beta(1_G) = e^{-1} \in E^{-1}$, so $\beta \in L$ and $\gamma = g_1 \cdot \beta$.

The action of H is proper: for $\beta \in X$ put $U = \{\gamma \in X \mid \gamma(1_G) = \beta(1_G)\}$. If $\lambda \in U \cap (h \cdot U)$, then $\lambda(1_G) = \beta(1_G)$ and $h^{-1} \cdot \lambda(1_G) = \beta(1_G)$. Thus $h = 1_H$.

The action of G is proper: for $\beta \in X$ put $U = \{\gamma \in X \mid \gamma(1_G) = \beta(1_G)\}$. If $\lambda \in U \cap (g \cdot U)$, then $\lambda(1_G) = \beta(1_G)$ and $\lambda(g^{-1}) = \beta(1_G)$. Thus $\lambda(g^{-1}) = \lambda(1_G)$, which implies $g^{-1} \in d(\{1_H\})$, so the set of such g is finite. ■

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