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ON A PARTICULAR CLASS OF WARPED PRODUCTS WITH FIBRES LOCALLY ISOMETRIC TO GENERALIZED CARTAN HYPERSURFACES

BY

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Dedicated to Professor Marek Abramowicz on his sixtieth birthday

Abstract. We prove that every generalized Cartan hypersurface satisfies the so called Roter type equation. Using this fact, we construct a particular class of generalized Robertson–Walker spacetimes.

1. Introduction. According to [8] a semi-Riemannian manifold (M, g) with $\dim M = n \geq 4$ is said to be a *Roter type manifold* if

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \eta G \tag{1}$$

holds on $U_C \cap U_S \subset M$, where ϕ, μ and η are some functions on this set, $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$ and $U_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$; here C denotes the Weyl tensor and S the Ricci tensor. For precise definitions of the symbols used here, we refer to Section 2 of this paper (and also to Sections 2 and 3 of [13]).

Obviously, we consider Roter type manifolds with $U_C \cap U_S$ non-empty. We refer to [8] and [15] for a review of results on Roter type manifolds.

In this paper we investigate warped products $\overline{M} \times_F \widetilde{N}$ with $\dim \overline{M} = 1$ and $\dim \widetilde{N} = n-1 \geq 3$ satisfying (1). We show that if they are of Roter type then the fibres $(\widetilde{N}, \widetilde{g})$ satisfy a special form of (1). We remark that manifolds $\overline{M} \times_F \widetilde{N}$ with $\dim \overline{M} = p \geq 2$ and $\dim \widetilde{N} = n - p \geq 2$ satisfying (1) were investigated in [13] and [16].

In Section 2 basic definitions are presented and we also give first results (see especially Theorem 2.1) relating to 3-dimensional manifolds or conformally flat quasi-Einstein manifolds, of dimension ≥ 4 satisfying (1). The next section contains preliminary results on warped products $\overline{M} \times_F \widetilde{N}$ with

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$\dim \overline{M} = n-1 \geq 3$ and $\dim \widetilde{N} = 1$ satisfying (1). In Section 4 we consider so called generalized Cartan hypersurfaces, i.e., tubular hypersurfaces around minimal surfaces, introduced in [3]. We prove that such hypersurfaces satisfy special relations for the Ricci tensor and the scalar curvature. Manifolds which are locally isometric to open subsets of such hypersurfaces will be used as fibres in the construction of Roter type warped products $\overline{M} \times_F \widetilde{N}$ with $\dim \overline{M} = 1$ and $\dim \widetilde{N} = n-1 \geq 3$, which are generalized Robertson–Walker spacetimes. Section 5 contains results relating to this construction. We recall that if $n \geq 4$, $p = 1$, $\overline{g}_{11} = -1$, and the fibre manifold $(\widetilde{N}, \widetilde{g})$ is a Riemannian manifold, then $\overline{M} \times_F \widetilde{N}$ is called a *generalized Robertson–Walker spacetime* (see [1] and references therein).

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2. Preliminaries. Throughout this paper all manifolds are assumed to be connected paracompact C^∞-manifolds. Let (M,g) be an n-dimensional semi-Riemannian manifold, $n \geq 3$, ∇ its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on M. We define the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$
$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

respectively, where A is a symmetric $(0,2)$-tensor on M and $X, Y, Z \in \Xi(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$ and the scalar curvature κ of (M,g) are defined by

$$S(X,Y) = \operatorname{tr}\{Z \mapsto \mathcal{R}(Z,X)Y\}, \qquad g(\mathcal{S}X, Y) = S(X,Y), \qquad \kappa = \operatorname{tr} \mathcal{S}.$$

The endomorphism $\mathcal{C}(X,Y)$ is defined by

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right)Z.$$

The $(0,4)$-tensor G, the Riemann–Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by

$$G(X_1,X_2,X_3,X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$
$$R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3, X_4),$$
$$C(X_1,X_2,X_3,X_4) = g(\mathcal{C}(X_1,X_2)X_3, X_4),$$

where $X_1, X_2, \ldots \in \Xi(M)$. Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be the $(0,4)$-tensor associated with $\mathcal{B}(X,Y)$ by

$$B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3, X_4). \tag{2}$$

The tensor B is said to be a *generalized curvature tensor* if

$$B(X_1,X_2,X_3,X_4)+B(X_2,X_3,X_1,X_4)+B(X_3,X_1,X_2,X_4)=0,$$
$$B(X_1,X_2,X_3,X_4)=B(X_3,X_4,X_1,X_2).$$

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be the tensor defined by (2). We extend $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f=0$ for every smooth function f on M. Now, for a $(0,k)$-tensor field T, $k\geq 1$, we define the $(0,k+2)$-tensor $B\cdot T$ by

$$\begin{aligned}(B\cdot T)(X_1,\ldots,X_k;X,Y) &= (\mathcal{B}(X,Y)\cdot T)(X_1,\ldots,X_k)\\ &= -T(\mathcal{B}(X,Y)X_1,X_2,\ldots,X_k)-\cdots-T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k).\end{aligned}$$

In addition, if A is a symmetric $(0,2)$-tensor, we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned}Q(A,T)(X_1,\ldots,X_k;X,Y) &= (X\wedge_A Y\cdot T)(X_1,\ldots,X_k)\\ &= -T((X\wedge_A Y)X_1,X_2,\ldots,X_k)-\cdots-T(X_1,\ldots,X_{k-1},(X\wedge_A Y)X_k).\end{aligned}$$

In this manner we obtain the $(0,6)$-tensors $B\cdot B$ and $Q(A,B)$. Setting in the above formulas $\mathcal{B}=\mathcal{R}$ or $\mathcal{B}=\mathcal{C}$, $T=R$ or $T=C$ or $T=S$, $A=g$ or $A=S$, we get the tensors $R\cdot R$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(S,G)$. For symmetric $(0,2)$-tensors E and F we define their *Kulkarni–Nomizu product* $E\wedge F$ by

$$\begin{aligned}(E\wedge F)(X_1,X_2,X_3,X_4) &= E(X_1,X_4)F(X_2,X_3)+E(X_2,X_3)F(X_1,X_4)\\ &\quad -E(X_1,X_3)F(X_2,X_4)-E(X_2,X_4)F(X_1,X_3).\end{aligned}$$

Clearly, the tensors R, C, G and $E\wedge F$ are generalized curvature tensors. For a symmetric $(0,2)$-tensor E we define the $(0,4)$-tensor $\overline{E}$ by $\overline{E}=\frac{1}{2}E\wedge E$. Thus in particular we have $\overline{g}=G=\frac{1}{2}g\wedge g$ and

$$C=R-\frac{1}{n-2}\,g\wedge S+\frac{\kappa}{(n-2)(n-1)}\,G. \tag{3}$$

We also have the following identity (see e.g. [9, Section 3]):

$$Q(E,E\wedge F)=-Q(F,\overline{E}). \tag{4}$$

Let (M,g), $n\geq 3$, be a *quasi-Einstein manifold*, that is, a semi-Riemannian manifold with the Ricci tensor S given by

$$S=\alpha g+\beta w\otimes w \tag{5}$$

for every $x\in M$, where $w\in T^*_xM$, $\alpha,\beta\in\mathbb{R}$. Quasi-Einstein manifolds arose in the study of exact solutions of the Einstein field equations as well as in considerations of quasi-umbilical hypersurfaces of conformally flat spaces. We note that for every point of $U_S\subset M$ the condition (5) is equivalent to

$$\operatorname{rank}(S-\alpha g)=1$$

and this is equivalent to

$$(S - \alpha g) \wedge (S - \alpha g) = 0. \tag{6}$$

Let now (M, g) be a quasi-Einstein manifold of dimension ≥ 4. It is easy to verify that if (1) is satisfied on $U_S \subset M$ then $C = 0$ on this set. Also a converse statement is true (see Theorem 2.1).

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* (see e.g. [2]) if the $(0, 6)$-tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point of M. This is equivalent to

$$R \cdot R = L_R Q(g, R) \tag{7}$$

on $U_R = \left\{x \in M \mid R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\right\}$, where L_R is some function on U_R. It is easy to check that L_R is uniquely determined on U_R. According to [17], if L_R is a constant on U_R, then the pseudosymmetric manifold (M, g) is called of *constant type*. It is obvious that every semisymmetric manifold $(R \cdot R = 0)$ is pseudosymmetric. The converse is not true (see [2]).

According to [8], (1) is called a *Roter type equation* and a semi-Riemannian manifold (M, g) (with $n \geq 4$) such that (1) holds on $U_C \cap U_S$ is called a *Roter type manifold*. We mention that the decomposition of R on $U_C \cap U_S$ in terms of $S \wedge S$, $g \wedge S$ and G is unique ([12, Lemma 3.2]). It is easy to check that (1) implies (7) on $U_C \cap U_S$ with

$$L_R = \phi^{-1}((n-2)(\mu^2 - \phi\eta) - \mu). \tag{8}$$

Further, we note that (1) can be presented in the form

$$R + \phi^{-1}(\mu^2 - \phi\eta)G = \frac{\phi}{2}\,(S + \phi^{-1}\mu g) \wedge (S + \phi^{-1}\mu g).$$

The equation (1) also implies (see e.g. [8])

$$R \cdot R - Q(S, R) = L Q(g, C), \tag{9}$$

with

$$L = L_R + \phi^{-1}\mu = (n-2)\phi^{-1}(\mu^2 - \phi\eta).$$

Remark 2.1

(i) In [5, Theorem 4.1] it was shown that every warped product $\overline{M} \times_F \widetilde{N}$ with $\dim \overline{M} = 1$ and $\dim \widetilde{N} = 3$ satisfies (9) with some function L. In particular, every 4-dimensional generalized Robertson–Walker spacetime has this property.

(ii) From Theorem 6.1 of [5] it follows that the warped product $\overline{M} \times_F \widetilde{N}$ of an $(n-1)$-dimensional space $(\overline{M}, \overline{g})$ of constant curvature with $n \geq 4$ and a 1-dimensional manifold $(\widetilde{N}, \widetilde{g})$ satisfies (9) with $L = -\overline{\kappa}/(n-1)$, where $\overline{\kappa}$ is the scalar curvature of $(\overline{M}, \overline{g})$.

We finish this section with

THEOREM 2.1. *Let (M, g) be a 3-dimensional semi-Riemannian manifold or a conformally flat semi-Riemannian manifold of dimension ≥ 4 and let (6) hold on $U_S \subset M$ with $\alpha \neq \kappa/n$ at every point of this set.*

(i) *If*

$$R - \beta G = \frac{\phi}{2}\,(S - (n-1)\beta g) \wedge (S - (n-1)\beta g) \tag{10}$$

on U_S for some functions β and ϕ on U_S, then

$$\beta = \frac{\kappa}{n-1} - \alpha, \tag{11}$$

$$\phi = \frac{1}{n-2}\,(\alpha - (n-1)\beta)^{-1} \tag{12}$$

on this set.

(ii) *If β and ϕ are defined by (11) and (12) then (10) holds on U_S.*

Proof. (i) The relations (3) and (6) yield

$$\left(\phi(\alpha - (n-1)\beta) - \frac{1}{n-2}\right) g \wedge S = \left(\phi((n-1)^2\beta^2 - \alpha^2) + \beta + \frac{\kappa}{(n-2)(n-1)}\right) G,$$

which implies

$$\phi(\alpha - (n-1)\beta) = \frac{1}{n-2},$$

$$\phi(\alpha^2 - (n-1)^2\beta^2) = \beta + \frac{\kappa}{(n-2)(n-1)}.$$

This immediately leads to (11) and (12).

(ii) Using (11) and (12) we obtain

$$\begin{aligned} \frac{1}{n-2} - \alpha\phi &= \frac{1}{n-2}\left(1 + \frac{\alpha}{(n-1)\beta - \alpha}\right) \\ &= \frac{n-1}{n-2}\,\frac{\beta}{(n-1)\beta - \alpha} = -(n-1)\beta\phi, \end{aligned} \tag{13}$$

$$\begin{aligned} \alpha^2\phi - \frac{\kappa}{(n-2)(n-1)} - \beta &= \alpha^2\phi - \frac{1}{n-2}\,(\beta + \alpha) - \beta \\ &= \left(\alpha\phi - \frac{1}{n-2}\right)\alpha - \frac{n-1}{n-2}\,\beta = (n-1)\alpha\beta\phi - \frac{n-1}{n-2}\,\beta \\ &= (n-1)\beta\left(\alpha\phi - \frac{1}{n-2}\right) = (n-1)^2\beta^2\phi. \end{aligned} \tag{14}$$

Furthermore, (3) and (6) give

$$R = \frac{1}{n-2}\left(g \wedge S - \frac{\kappa}{n-1}\, G\right),$$
$$R - \beta G = \frac{\phi}{2}\, S \wedge S + \left(\frac{1}{n-2} - \alpha\phi\right) g \wedge S + \left(\alpha^2\phi - \frac{\kappa}{(n-2)(n-1)} - \beta\right) G.$$

But the last equation, by making use of (13) and (14), turns into (10), which completes the proof.

EXAMPLE 2.1. Let $\overline{M} \times_F \widetilde{N}$ be a Robertson–Walker spacetime, i.e. the warped product of a line or a circle $(\overline{M}, \overline{g})$, $\overline{g}_{11} = \varepsilon = \pm 1$, and an $(n-1)$-dimensional Riemannian space $(\widetilde{N}, \widetilde{g})$ of constant curvature with the warping function F and $n-1 \geq 3$. It is known that (5) holds on $U_S \subset \overline{M} \times_F \widetilde{N}$, with $\beta = \frac{\kappa}{n-1} - \frac{\varepsilon (F')^2}{4F^2}$ (see e.g. [6, Lemma 3.1]). In view of Theorem 2.1, if $\beta \neq \kappa/n$ for every point of U_S, then (10) holds on U_S. We can easily prove that $\beta = \kappa/n$ on U_S if and only if

$$2FF'' + (n-2)(F')^2 - \frac{2\varepsilon\widetilde{\kappa}}{n-1}\, F = 0$$

on this set, where $\widetilde{\kappa}$ and κ denote the scalar curvatures of $(\widetilde{N}, \widetilde{g})$ and $\overline{M} \times_F \widetilde{N}$, respectively.

3. Warped products satisfying (1). Let $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$, with $\dim \overline{M} = p$ and $\dim \widetilde{N} = n - p$, $1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{\overline{U}; x^a\}$ and $\{\widetilde{V}; y^\alpha\}$, respectively. Further, let $F : \overline{M} \to \mathbb{R}^+$ be a positive smooth function on $\overline{M}$. The *warped product* $\overline{M} \times_F \widetilde{N}$ is the product manifold $\overline{M} \times \widetilde{N}$ with the metric $g = \overline{g} \times_F \widetilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1) \pi_2^* \widetilde{g}$, where $\pi_1 : \overline{M} \times \widetilde{N} \to \overline{M}$ and $\pi_2 : \overline{M} \times \widetilde{N} \to \widetilde{N}$ are the natural projections. Let $\{\overline{U} \times \widetilde{V}; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \widetilde{N}$. The local components of the metric g with respect to this chart read: $g_{hk} = \overline{g}_{ab}$ if $h = a$ and $k = b$, $g_{hk} = F\widetilde{g}_{\alpha\beta}$ if $h = \alpha$ and $k = \beta$, and $g_{hk} = 0$ otherwise, where $a, b, c, \ldots \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \ldots \in \{p+1, \ldots, n\}$ and $h, i, j, k, \ldots \in \{1, \ldots, n\}$. We will mark by bars (resp., tildes) tensors formed from $\overline{g}$ (resp., $\widetilde{g}$). The local components R_{hijk} of the curvature tensor R and the local components S_{hk} of the Ricci tensor S of $\overline{M} \times_F \widetilde{N}$ which generally do not vanish identically are the following (see e.g. [13], [14]):

$$\begin{aligned} R_{abcd} &= \overline{R}_{abcd}, \\ R_{\alpha bc\beta} &= -\frac{1}{2}\, T_{bc} \widetilde{g}_{\alpha\beta}, \qquad (15) \\ R_{\alpha\beta\gamma\delta} &= F\left(\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F}\, \widetilde{G}_{\alpha\beta\gamma\delta}\right), \end{aligned}$$

$$S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F}\, T_{ab},$$
(16)
$$S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \left(\frac{\operatorname{tr} T}{2} + (n-p-1)\,\frac{\Delta_1 F}{4F}\right)\widetilde{g}_{\alpha\beta},$$

(17) $$T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F}\, F_a F_b, \qquad \Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b,$$

where T denotes the $(0,2)$-tensor with local components T_{ab} and $\operatorname{tr} T = \operatorname{tr}_{\overline{g}} T = \overline{g}^{ab} T_{ab}$. The scalar curvature κ of $\overline{M} \times_F \widetilde{N}$ satisfies the relation

(18) $$\kappa = \overline{\kappa} + \frac{\widetilde{\kappa}}{F} - \frac{n-p}{F}\left(\operatorname{tr} T + (n-p-1)\,\frac{\Delta_1 F}{4F}\right).$$

Using (15), (16) and (18), we find the following relations for the local components C_{hijk} of the Weyl tensor C of $\overline{M} \times_F \widetilde{N}$ ([7]):

(19) $$C_{abcd} = \overline{R}_{abcd} - \frac{1}{n-2}\,(\overline{g}_{ad}\overline{S}_{bc} - \overline{g}_{ac}\overline{S}_{bd} + \overline{g}_{bc}\overline{S}_{ad} - \overline{g}_{bd}\overline{S}_{ac}) + \frac{n-p}{2(n-2)F}\,(\overline{g}_{ad}T_{bc} - \overline{g}_{ac}T_{bd} + \overline{g}_{bc}T_{ad} - \overline{g}_{bd}T_{ac}) + \frac{\kappa}{(n-2)(n-1)}\,\overline{G}_{abcd},$$

(20) $$C_{\alpha bc\beta} = -\frac{1}{n-2}\left(\frac{p-2}{2} T_{ab} + F\overline{S}_{ab}\right)\widetilde{g}_{\alpha\beta} - \frac{1}{n-2}\,\overline{g}_{ab}\widetilde{S}_{\alpha\beta} + \frac{1}{(n-2)(n-1)}\left(F\overline{\kappa} + \widetilde{\kappa} - \frac{(n-2p+1)\operatorname{tr} T}{2} + \frac{(p-1)(n-p-1)\Delta_1 F}{4F}\right)\overline{g}_{ab}\widetilde{g}_{\alpha\beta},$$

(21) $$C_{\alpha\beta\gamma\delta} = F\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{F}{n-2}\,(\widetilde{g}_{\alpha\delta}\widetilde{S}_{\beta\gamma} - \widetilde{g}_{\alpha\gamma}\widetilde{S}_{\beta\delta} + \widetilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta} - \widetilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma}) + FP\widetilde{G}_{\alpha\beta\gamma\delta},$$

(22) $$C_{abc\alpha} = C_{ab\alpha\beta} = C_{a\alpha\beta\gamma} = 0,$$

(23) $$P = \frac{1}{n-2}\left(\frac{F}{n-1} + \operatorname{tr} T + \frac{(n-2p)\Delta_1 F}{4F}\right).$$

We now consider the warped products $\overline{M} \times_F \widetilde{N}$ with $\dim \overline{M} = 1$ and $\dim \widetilde{N} = n-1 \geq 3$. Then

$$T_{11} = \overline{g}_{11}\overline{g}^{11} T_{11} = \operatorname{tr} T\, \overline{g}_{11},$$
(24)
$$H_{11} = \frac{1}{2}\, T_{11} + FL_R \overline{g}_{11} = \left(\frac{\operatorname{tr} T}{2} + FL_R\right)\overline{g}_{11},$$

where T_{11} is defined by (17), i.e.

$$T_{11} = \nabla_1 F_1 - \frac{1}{2F}\, F_1 F_1 = \partial_1 F_1 - F_1 \Gamma^1_{11} - \frac{1}{2F}\, F_1^2.$$

Using (18)–(24), we find

$$C_{\alpha 11\delta} = -\frac{1}{n-2}\Big(\widetilde{S}_{\alpha\delta} - \frac{\widetilde{\kappa}}{n-1}\,\widetilde{g}_{\alpha\delta}\Big)\overline{g}_{11}, \tag{25}$$

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = F(\widetilde{C}_{\alpha\beta\gamma\delta} + \frac{1}{(n-3)(n-2)}\Big(\widetilde{g}_{\alpha\delta}\Big(\widetilde{S}_{\beta\gamma} - \frac{\widetilde{\kappa}}{n-1}\,\widetilde{g}_{\beta\gamma}\Big) \\ + \widetilde{g}_{\beta\gamma}\Big(\widetilde{S}_{\alpha\delta} - \frac{\widetilde{\kappa}}{n-1}\,\widetilde{g}_{\alpha\delta}\Big) - \widetilde{g}_{\alpha\gamma}\Big(\widetilde{S}_{\beta\delta} - \frac{\widetilde{\kappa}}{n-1}\,\widetilde{g}_{\beta\delta}\Big) \\ - \widetilde{g}_{\beta\delta}\Big(\widetilde{S}_{\alpha\gamma} - \frac{\widetilde{\kappa}}{n-1}\,\widetilde{g}_{\alpha\gamma}\Big)\Big). \end{aligned} \tag{26}$$

In particular, if $n = 4$, (25) and (26) reduce to

$$C_{\alpha 11\delta} = -\frac{1}{2}\Big(\widetilde{S}_{\alpha\delta} - \frac{\widetilde{\kappa}}{3}\,\widetilde{g}_{\alpha\delta}\Big)\overline{g}_{11}, \tag{27}$$

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = \frac{F}{2}\Big(\widetilde{g}_{\alpha\delta}\Big(\widetilde{S}_{\beta\gamma} - \frac{\widetilde{\kappa}}{3}\,\widetilde{g}_{\beta\gamma}\Big) + \widetilde{g}_{\beta\gamma}\Big(\widetilde{S}_{\alpha\delta} - \frac{\widetilde{\kappa}}{3}\,\widetilde{g}_{\alpha\delta}\Big) \\ - \widetilde{g}_{\alpha\gamma}\Big(\widetilde{S}_{\beta\delta} - \frac{\widetilde{\kappa}}{3}\,\widetilde{g}_{\beta\delta}\Big) - \widetilde{g}_{\beta\delta}\Big(\widetilde{S}_{\alpha\gamma} - \frac{\widetilde{\kappa}}{3}\,\widetilde{g}_{\alpha\gamma}\Big)\Big), \end{aligned} \tag{28}$$

respectively. Further, from Lemma 4 of [7], it follows that (7) holds on $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$, where $\dim \overline{M} = 1$ and $\dim \widetilde{N} = n - 1 \geq 3$, if and only if

$$H_{11}\Big(\widetilde{R}_{\delta\alpha\beta\gamma} - \Big(\frac{\Delta_1 F}{4F} - \frac{\operatorname{tr} T}{2}\Big)\widetilde{G}_{\delta\alpha\beta\gamma}\Big) = 0, \tag{29}$$

$$(\widetilde{R}\cdot\widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu} = \Big(FL_R + \frac{\Delta_1 F}{4F}\Big)Q(\widetilde{g},\widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu} \tag{30}$$

on this set. By suitable contractions, (29) yields

$$H_{11}\Big(\widetilde{S}_{\alpha\beta} - (n-2)\Big(\frac{\Delta_1 F}{4F} - \frac{\operatorname{tr} T}{2}\Big)\widetilde{g}_{\alpha\beta}\Big) = 0,$$

$$H_{11}\Big(\widetilde{\kappa} - (n-2)(n-1)\Big(\frac{\Delta_1 F}{4F} - \frac{\operatorname{tr} T}{2}\Big)\Big) = 0.$$

Substituting the last relation into (29) we obtain

$$H_{11}\Big(\widetilde{R}_{\delta\alpha\beta\gamma} - \frac{\widetilde{\kappa}}{(n-2)(n-1)}\,\widetilde{G}_{\delta\alpha\beta\gamma}\Big) = 0. \tag{31}$$

We note that at every point $x \in U_C \cap U_S$ the tensor $\widetilde{R} - \frac{\widetilde{\kappa}}{(n-2)(n-1)}\widetilde{G}$ is non-zero. In fact, if $\widetilde{R} = \frac{\widetilde{\kappa}}{(n-2)(n-1)}\widetilde{G}$ at x, then, by making use of (19)–(21) we get $C = 0$ at x, a contradiction. Thus, from (31) it follows that $H_{11} = 0$ on $U_C \cap U_S$. This, by (24), yields

$$FL_R + \frac{\operatorname{tr} T}{2} = 0, \tag{32}$$

and using (30) we get

$$FL_R + \frac{\Delta_1 F}{4F} = C_1, \qquad C_1 = \text{const}, \tag{33}$$

on $U_C \cap U_S$. Evidently, (32) and (33) lead to

$$\frac{\Delta_1 F}{4F} - \frac{\operatorname{tr} T}{2} = C_1. \tag{34}$$

Thus we have

LEMMA 3.1. *The warped product* $\overline{M} \times_F \widetilde{N}$ *with* $\dim \overline{M} = 1$ *and* $\dim \widetilde{N} = n - 1 \geq 3$ *satisfies* (7) *on* $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$ *if and only if* (30), (33) *and* (34) *hold on this set.*

If $g_{11} = \overline{g}_{11} = \varepsilon = \pm 1$, then (34) yields

$$(F')^2 - 2F\left(\nabla_1 F_1 - \frac{1}{2F}(F')^2\right) = 4\varepsilon F C_1, \tag{35}$$

where $F_1 = F' = \partial F/\partial x^1$. Since $\nabla_1 F_1 = \partial F_1/\partial x^1 = F''$, (35) now becomes

$$FF'' - (F')^2 + 2\varepsilon C_1 F = 0. \tag{36}$$

We can easily check that the following functions are solutions of (36) (cf. [11, Remark 3.7]):

$$\begin{aligned}
&F(x^1) = \varepsilon C_1 \left(x^1 + \frac{\varepsilon c}{C_1}\right)^2, \qquad \varepsilon C_1 > 0,\\
&F(x^1) = \frac{c}{2}\left(\exp\left(\pm\frac{b}{2}\,x^1\right) - \frac{2\varepsilon C_1}{b^2 c}\exp\left(\mp\frac{b}{2}\,x^1\right)\right)^2, \qquad c > 0,\ b \neq 0,\\
&F(x^1) = \frac{2\varepsilon C_1}{c^2}\,(1 + \sin(cx^1 + b)), \qquad \varepsilon C_1 > 0,\ c \neq 0,
\end{aligned} \tag{37}$$

where b and c are constants and x^1 belongs to a suitable non-empty open interval of $\mathbb{R}$.

Now let $\overline{M} \times_F \widetilde{N}$ with $\dim \overline{M} = 1$ and $\dim \widetilde{N} = n - 1 \geq 3$ be a Roter type manifold. Thus (1) holds on $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$. In the local representation, (1) reads

$$\begin{aligned}
R_{hijk} = {} & \Phi(S_{hk}S_{ij} - S_{hj}S_{ik}) + \eta G_{hijk}\\
& + \mu(g_{hk}S_{ij} + g_{ij}S_{hk} - g_{hj}S_{ik} - g_{ik}S_{hj}),
\end{aligned} \tag{38}$$

where R_{hijk}, G_{hijk}, S_{hk} and g_{hk} are the local components of the tensors R, G, S and g, respectively. Since (7) holds on $U_C \cap U_S$, it follows that (34) is satisfied on this set. Now (15)–(16) and (18) become

$$R_{\alpha 11\beta} = -\frac{\operatorname{tr} T}{2}\,\overline{g}_{11}\widetilde{g}_{\alpha\beta}, \tag{39}$$

$$R_{\alpha\beta\gamma\delta} = F\Big(\widetilde{R}_{\alpha\beta\gamma\delta} - \Big(\frac{\operatorname{tr} T}{2} + C_1\Big)\widetilde{G}_{\alpha\beta\gamma\delta}\Big), \tag{40}$$

$$S_{11} = -\frac{(n-1)\operatorname{tr} T}{2F}\,\overline{g}_{11}, \tag{41}$$

$$S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \Big(\frac{(n-1)\operatorname{tr} T}{2} + (n-2)C_1\Big)\widetilde{g}_{\alpha\beta}, \tag{42}$$

respectively. Using (38)–(42) we can prove

LEMMA 3.2. *Let $U \subset U_C \cap U_S$ be a coordinate neighbourhood of $x \in U_C \cap U_S$ in the warped product $\overline{M} \times_F \widetilde{N}$ with* $\dim \overline{M} = 1$ *and* $\dim \widetilde{N} = n-1 \geq 3$. *Then* (38) *holds on U if and only if*

$$R_{1\alpha\beta 1} = \phi S_{11}S_{\alpha\beta} + \mu(g_{11}S_{\alpha\beta} + S_{11}g_{\alpha\beta}) + \eta g_{11}g_{\alpha\beta}, \tag{43}$$

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \phi(S_{\alpha\delta}S_{\beta\gamma} - S_{\alpha\gamma}S_{\beta\delta}) + \eta G_{\alpha\beta\gamma\delta} \\ &\quad + \mu(g_{\alpha\delta}S_{\beta\gamma} + g_{\beta\gamma}S_{\alpha\delta} - g_{\alpha\gamma}S_{\beta\delta} - g_{\beta\delta}S_{\alpha\gamma}) \end{aligned} \tag{44}$$

on this set.

In addition we have

LEMMA 3.3. *If the warped product $\overline{M} \times_F \widetilde{N}$ with* $\dim \overline{M} = 1$, $\overline{g}_{11} = \varepsilon = \pm 1$, *and* $\dim \widetilde{N} = n-1 \geq 3$ *satisfies* (1) *on $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$, then the following relations hold on this set*: (33), (34) *and*

$$\text{(a) } \mu = \frac{(n-1)\operatorname{tr} T}{2F}\,\phi, \qquad \text{(b) } \eta = \frac{\mu^2}{\phi} - \frac{\operatorname{tr} T}{2F}, \tag{45}$$

$$\widetilde{R} - C_1\widetilde{G} = \frac{\phi}{2F}\,(\widetilde{S} - (n-2)C_1\widetilde{g}) \wedge (\widetilde{S} - (n-2)C_1\widetilde{g}). \tag{46}$$

Proof. Let $U \subset U_C \cap U_S$ be a coordinate neighbourhood of $x \in U_C \cap U_S$. First of all we prove that (43) implies (45) on U. From (43), using (39), (41) and (42), we obtain

$$\begin{aligned} &\Big(\frac{(n-1)\operatorname{tr} T}{2F}\,\phi - \mu\Big)\widetilde{S}_{\alpha\beta} \\ &\qquad = \Big(\Big(\frac{\operatorname{tr} T}{2} + \frac{(n-2)\Delta_1 F}{4F}\Big)\Big(\frac{(n-1)\operatorname{tr} T}{2F}\,\phi - \mu\Big) + F\eta \\ &\qquad\quad + (1-(n-1)\mu)\,\frac{\operatorname{tr} T}{2}\Big)\widetilde{g}_{\alpha\beta}. \end{aligned} \tag{47}$$

We suppose that $\widetilde{S} - \frac{\widetilde{\kappa}}{n-1}\widetilde{g} = 0$ at x. Then (41) and (42) lead to

$$S_{11} = \varrho_1 g_{11}, \qquad S_{\alpha\beta} = \varrho_2 g_{\alpha\beta}, \tag{48}$$

for some $\varrho_1, \varrho_2 \in \mathbb{R}$. From (48) we have

$$S_{ij} = \varrho_2 g_{ij} + (\varrho_1 - \varrho_2)\varepsilon w_i w_j, \tag{49}$$

where $w_1 = 1$ and $w_2 = \cdots = w_n = 0$. Substituting (49) into (38), after some standard tensor calculations, we find $C_{hijk} = 0$, i.e. $C = 0$ at x, a contradiction. Therefore $\widetilde{S} - \frac{\widetilde{\kappa}}{n-1}\widetilde{g} \neq 0$ at x, and consequently, (47) implies (45). Applying (40) and (42) to (44), we obtain

$$\begin{aligned} \widetilde{R}_{\alpha\beta\gamma\delta} &= \frac{\phi}{F}\,(\widetilde{S}_{\alpha\delta}\widetilde{S}_{\beta\gamma} - \widetilde{S}_{\alpha\gamma}\widetilde{S}_{\beta\delta}) \\ &+ \left(\mu - \frac{\phi}{F}\left(\frac{\operatorname{tr} T}{2} + (n-2)\,\frac{\Delta_1 F}{4F}\right)\right)(\widetilde{g}_{\alpha\delta}\widetilde{S}_{\beta\gamma} + \widetilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta} - \widetilde{g}_{\alpha\gamma}\widetilde{S}_{\beta\delta} - \widetilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma}) \\ &+ \left(\eta F + \frac{\Delta_1 F}{4F} - 2\mu\left(\frac{\operatorname{tr} T}{2} + (n-2)\,\frac{\Delta_1 F}{4F}\right)\right. \\ &+ \left.\frac{\phi}{F}\left(\frac{\operatorname{tr} T}{2} + (n-2)\,\frac{\Delta_1 F}{4F}\right)^2\right)\widetilde{G}_{\alpha\beta\gamma\delta}. \end{aligned} \tag{50}$$

According to (34) and (45) we conclude that

$$\begin{aligned} \mu - \frac{\phi}{F}\left(\frac{\operatorname{tr} T}{2} + (n-2)\,\frac{\Delta_1 F}{4F}\right) &= (n-2)\left(\frac{\operatorname{tr} T}{2} - \frac{\Delta_1 F}{4F}\right)\frac{\phi}{F} \\ &= -(n-2)C_1\,\frac{\phi}{F}, \end{aligned} \tag{51}$$

$$\begin{aligned} \eta F + \frac{\Delta_1 F}{4F} - 2\mu\left(\frac{\operatorname{tr} T}{2} + (n-2)\,\frac{\Delta_1 F}{4F}\right) + \frac{\phi}{F}\left(\frac{\operatorname{tr} T}{2} + (n-2)\,\frac{\Delta_1 F}{4F}\right)^2 \\ = \left(\frac{\Delta_1 F}{4F} - \frac{\operatorname{tr} T}{2}\right)\left(1 + (n-2)^2\left(\frac{\Delta_1 F}{4F} - \frac{\operatorname{tr} T}{2}\right)\frac{\phi}{F}\right) \\ = C_1\left(1 + (n-2)^2 C_1\,\frac{\phi}{F}\right). \end{aligned} \tag{52}$$

Applying (51) and (52) to (50) we immediately get (46).

Remark 3.1. The relations (8) and (45) yield (32).

4. Generalized Cartan hypersurfaces. In this section we show that every generalized Cartan hypersurface is a Riemannian manifold satisfying (46). Thus, such manifolds are examples of fiber manifolds of dimension ≥ 3 satisfying (46).

Let $N^n_s(c)$, $n \geq 4$, be a semi-Riemannian space of constant curvature $c = \frac{\tau}{(n-1)n}$ with signature $(s, n-s)$, where τ is its scalar curvature. In addition let $\widetilde{M}$ be a hypersurface isometrically immersed in $N^n_s(c)$. The

Gauss equation of $\widetilde{M}$ in $N^n_s(c)$ reads (see e.g. [10] or [12])

$$\widetilde{R}_{hijk} = \varepsilon_1(H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\tau}{(n-1)n}\,\widetilde{G}_{hijk}, \qquad \varepsilon_1 = \pm 1, \tag{53}$$

where $\widetilde{R}_{hijk}$, $\widetilde{G}_{hijk}$ and H_{ij} denote the local components of the curvature tensor $\widetilde{R}$, the tensor $\widetilde{G}$ and the second fundamental tensor H of $\widetilde{M}$, respectively. Contracting (53) with $\widetilde{g}^{ij}$ and $\widetilde{g}^{kh}$, respectively, we obtain

$$\widetilde{S}_{hk} = \varepsilon_1(\operatorname{tr} H H_{hk} - H^2_{hk}) + \frac{(n-2)\tau}{(n-1)n}\,\widetilde{g}_{hk}, \tag{54}$$

$$\widetilde{\kappa} = \varepsilon_1((\operatorname{tr} H)^2 - \operatorname{tr}(H^2)) + \frac{(n-2)\tau}{n}, \tag{55}$$

where $H^2_{hk} = \widetilde{g}^{ij}H_{hi}H_{kj}$, $\operatorname{tr} H = \widetilde{g}^{hk}H_{hk}$, $\operatorname{tr}(H^2) = \widetilde{g}^{hk}H^2_{hk}$, $\widetilde{S}_{hk}$ are the local components of the Ricci tensor $\widetilde{S}$, and $\widetilde{\kappa}$ is the scalar curvature of $\widetilde{M}$. We recall that the following condition of pseudosymmetry type is fulfilled on $\widetilde{M}$ (see e.g. [10] or [12]):

$$\widetilde{R}\cdot\widetilde{R} - Q(\widetilde{S},\widetilde{R}) = -\frac{(n-3)\tau}{(n-1)n}\,Q(\widetilde{g},\widetilde{C}), \tag{56}$$

where $\widetilde{C}$ is the Weyl conformal tensor of $\widetilde{M}$. By making use of (3), (56) turns into

$$\widetilde{R}\cdot\widetilde{R} - Q(\widetilde{S},\widetilde{R}) = -\frac{(n-3)\tau}{(n-1)n}\,Q(\widetilde{g},\widetilde{R}) + \frac{\tau}{(n-1)n}\,Q(\widetilde{g},\widetilde{g}\wedge\widetilde{S}),$$

and from (4) we get $Q(\widetilde{g},\widetilde{g}\wedge\widetilde{S}) = -Q(\widetilde{S},\widetilde{G})$. Applying this to the relation above, we find

$$\widetilde{R}\cdot\widetilde{R} = Q\Big(\widetilde{S},\widetilde{R} - \frac{\tau}{(n-1)n}\,\widetilde{G}\Big) - \frac{(n-3)\tau}{(n-1)n}\,Q\Big(\widetilde{g},\widetilde{R} - \frac{\tau}{(n-1)n}\,\widetilde{G}\Big). \tag{57}$$

In addition, we assume that

$$\widetilde{R}\cdot\widetilde{R} = \frac{\tau}{(n-1)n}\,Q(\widetilde{g},\widetilde{R}) \tag{58}$$

on $U_{\widetilde{S}} \subset U_{\widetilde{R}} \subset \widetilde{M}$. Comparing the right hand sides of (57) and (58) we obtain

$$Q\Big(\widetilde{S} - \frac{(n-2)\tau}{(n-1)n}\,\widetilde{g},\widetilde{R} - \frac{\tau}{(n-1)n}\,\widetilde{G}\Big) = 0. \tag{59}$$

If we set $C_1 = \frac{\tau}{(n-1)n}$, then (59) becomes

$$Q(\widetilde{S} - (n-2)C_1\widetilde{g}, \widetilde{R} - C_1\widetilde{G}) = 0. \tag{60}$$

Further, we assume that

$$\operatorname{rank}(\widetilde{S} - (n-2)C_1\widetilde{g}) > 1 \tag{61}$$

on $U_{\widetilde{S}}$. Now from (60), in view of Proposition 4.1 of [4], it follows that

$$\widetilde{R} - C_1\widetilde{G} = \frac{\widetilde{\psi}}{2}\,(\widetilde{S} - (n-2)C_1\widetilde{g}) \wedge (\widetilde{S} - (n-2)C_1\widetilde{g}), \tag{62}$$

where $\widetilde{\Psi}$ only takes positive or negative values on $U_{\widetilde{S}}$. Thus we have

PROPOSITION 4.1. *Let $\widetilde{M}$ be a hypersurface in $N_s^n(c)$ with $n \geq 4$. If* (58) *and* (61) *hold on $U_{\widetilde{S}} \subset \widetilde{M}$ then* (62) *is satisfied on this set.*

Let $N^2(c_1)$ be a minimal surface with non-zero constant curvature c_1 in the standard unit n-sphere $S^n(1)$ of $\mathbb{E}^{n+1}$, $n \geq 4$. We denote by $\widetilde{M}$ the tubular hypersurface $T_{\pi/2}(N^2(c_1))$ with radius $\pi/2$ around $N^2(c_1)$. Such a hypersurface is called a *generalized Cartan hypersurface* ([3, Section 6]). Clearly, $\widetilde{M}$ is an $(n-1)$-dimensional hypersurface in $S^n(1)$, $\varepsilon_1 = 1$, and $C_1 = \frac{\tau}{(n-1)n} = 1$ on $\widetilde{M}$. It is known that the second fundamental tensor H of $\widetilde{M}$ has three distinct eigenvalues (i.e. principal curvatures): $\lambda_1 = \lambda$, $\lambda_2 = -\lambda$, $\lambda_3 = \cdots = \lambda_{n-1} = 0$, and $\lambda \neq 0$ at every point. Therefore the tensor H^2 has two distinct eigenvalues at every point of $\widetilde{M}$: $\mu_1 = \mu_2 = \lambda^2$, $\mu_3 = \cdots = \mu_{n-1} = 0$, $\operatorname{tr} H = 0$ and $\operatorname{rank} H = 2$, i.e. the type number of $\widetilde{M}$ is 2. The last fact implies (58) on $\widetilde{M}$, i.e. $\widetilde{R}\cdot\widetilde{R} = Q(\widetilde{g},\widetilde{R})$ on $\widetilde{M}$ (see e.g. [12, Section 5]). Evidently, $\widetilde{M}$ is a pseudosymmetric manifold of constant type. The Ricci tensor $\widetilde{S}$ and the scalar curvature $\widetilde{\kappa}$ of $\widetilde{M}$, by making use of (54), (55), and the relations above, can be expressed by

$$\widetilde{S} = -H^2 + (n-2)\widetilde{g}, \tag{63}$$

$$\widetilde{\kappa} = -\operatorname{tr}(H^2) + (n-2)(n-1) = -2\lambda^2 + (n-2)(n-1). \tag{64}$$

Now, we consider the case where the hypersurface $\widetilde{M}$ is of dimension ≥ 4, i.e. the ambient space is of dimension $n \geq 5$. We suppose that (5) holds at a point of $\widetilde{M}$. Comparing the right hand sides of (5) and (63) we get $H^2 = (n-2-\alpha)\widetilde{g} - \beta w \otimes w$. It follows that $n-2-\alpha$ is an eigenvalue of H^2 of multiplicity $n-2$, a contradiction. Thus, a relation of the form (5) cannot be satisfied for any point of $\widetilde{M}$, and (61) holds on $\widetilde{M}$. Finally, in view of Proposition 4.1, every generalized Cartan hypersurface satisfies (62). At every point of such a hypersurface there are three distinct principal curvatures and therefore its Weyl conformal curvature tensor $\widetilde{C}$ is non-zero everywhere. We note that (25) and (26) imply that every warped product of a line or a circle and a manifold of dimension $n-1 \geq 4$, isometric to an open part of a generalized Cartan hypersurface, is a non-conformally flat manifold.

Now, let $\widetilde{M}$ be a 3-dimensional generalized Cartan hypersurface. Then (63) and (64) turn into

$$\widetilde{S} - \frac{\widetilde{\kappa}}{3}\,\widetilde{g} = -H^2 + \frac{\operatorname{tr}(H^2)}{3}\,\widetilde{g} \neq 0,$$
$$\widetilde{\kappa} = -\operatorname{tr}(H^2) + 6 = 2(3 - \lambda^2),$$

respectively. It follows from our considerations that the Ricci tensor $\widetilde{S}$ of $\widetilde{M}$ has two distinct eigenvalues $\varrho_1 = 2$ and $\varrho_2 = \varrho_3 = 2 - \lambda^2 = \widetilde{\kappa}/2 - 1$ at every point. Therefore $\operatorname{rank}(\widetilde{S} - (2-\lambda^2)\widetilde{g}) = 1$ on $\widetilde{M}$. For every point of $\widetilde{M}$, the last relation is equivalent to (cf. (6))

$$(\widetilde{S} - (2-\lambda^2)\widetilde{g}) \wedge (\widetilde{S} - (2-\lambda^2)\widetilde{g}) = 0,$$

which yields

$$-\frac{1}{\lambda^2}\left(\frac{1}{2}\,\widetilde{S}\wedge\widetilde{S} - (2-\lambda^2)\widetilde{g}\wedge\widetilde{S} + (2-\lambda^2)^2\widetilde{G}\right) = 0. \tag{65}$$

Furthermore, $\widetilde{C} = 0$, which by (3), gives $\widetilde{R} = \widetilde{g}\wedge\widetilde{S} - (\widetilde{\kappa}/2)\widetilde{G}$. The last relation, by making use of (65), turns into

$$\widetilde{R} - \widetilde{G} = -\frac{1}{2\lambda^2}\,(\widetilde{S} - 2\widetilde{g}) \wedge (\widetilde{S} - 2\widetilde{g}),$$

i.e. (10) with $\beta = C_1 = 1$, $\widetilde{\kappa}/2 - C_1 = 2 - \lambda^2 = \alpha$ and $\phi = (\alpha - 2C_1)^{-1} = -\lambda^{-2}$. Finally, we note that (27) and (28) imply that every warped product of a line or a circle and a 3-dimensional manifold isometric to an open part of generalized Cartan hypersurface is a non-conformally flat manifold. Thus we have

THEOREM 4.1.

(i) *For every generalized Cartan hypersurface $\widetilde{M}$ of dimension ≥ 4, the relation (62) with $C_1 = 1$ holds on $U_{\widetilde{S}} \cap U_{\widetilde{C}} = \widetilde{M}$.*
(ii) *For every 3-dimensional generalized Cartan hypersurface $\widetilde{M}$ the relation (62) with $C_1 = 1$ holds on $U_{\widetilde{S}} = \widetilde{M}$.*
(iii) *Every warped product of a 1-dimensional manifold and an $(n-1)$-dimensional manifold, $n \geq 4$, isometric to an open part of a generalized Cartan hypersurface is a non-conformally flat manifold.*

We finish this section with another example of a hypersurface satisfying (62). Let $\widetilde{M}$ be a hypersurface in $N_s^n(c)$, $n \geq 4$, satisfying

$$H^2 = \alpha H + \beta g \tag{66}$$

on $U_{\widetilde{S}} \subset \widetilde{M}$, where α and β are some functions on $U_{\widetilde{S}}$. Using (53)–(55) and (66) we obtain (cf. [15, Proposition 3.3])

$$\widetilde{R} - C_1\widetilde{G} = \varepsilon(\operatorname{tr} H - \alpha)^{-2}\Big(\frac{1}{2}\,\widetilde{S}\wedge\widetilde{S} - ((n-2)C_1 - \varepsilon\beta)\widetilde{g}\wedge\widetilde{S} + ((n-2)C_1 - \varepsilon\beta)^2\widetilde{G}\Big), \tag{67}$$

where $C_1 = \frac{\tau}{(n-1)n}$ and τ is the scalar curvature of the ambient space. Clearly, if $\beta = 0$ on $U_{\widetilde{S}}$, then (67) reduces to

$$\widetilde{R} - C_1\widetilde{G} = \frac{\varepsilon}{2}\,(\operatorname{tr} H - \alpha)^{-2}(\widetilde{S} - (n-2)C_1\widetilde{g}) \wedge (\widetilde{S} - (n-2)C_1\widetilde{g}). \tag{68}$$

Thus we have

THEOREM 4.2. *If $\widetilde{M}$ is a hypersurface in $N^n_s(c)$, $n \geq 4$, satisfying*

$$H^2 = \alpha H \tag{69}$$

on $U_{\widetilde{S}} \subset \widetilde{M}$ for some function α on $U_{\widetilde{S}}$, then (68) *holds on this set.*

An example of a hypersurface in a semi-Euclidean space $\mathbb{E}^n_s$, $n \geq 4$, satisfying (69) is given in [15, Example 3.1]. In addition, the hypersurface $\widetilde{M} = S^p \times \mathbb{E}^{n-1-p}$ in $\mathbb{E}^n$, $2 \leq p \leq n-2$, also satisfies (69).

5. Main results

THEOREM 5.1. *Let $\overline{M} \times_F \widetilde{N}$ be the warped product of a line or a circle $(\overline{M}, \overline{g})$, with $\overline{g}_{11} = \varepsilon = \pm 1$, and an $(n-1)$-dimensional semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n - 1 \geq 3$, satisfying*

$$\widetilde{R} - C_1\widetilde{G} = \frac{\widetilde{\phi}}{2}\,(\widetilde{S} - (n-2)C_1\widetilde{g}) \wedge (\widetilde{S} - (n-2)C_1\widetilde{g}) \tag{70}$$

on $U_{\widetilde{S}} \subset \widetilde{N}$, where $\widetilde{\phi}$ is some function on $U_{\widetilde{S}} \subset \widetilde{N}$ and C_1 is a constant, with F defined by one of the three equalities in (37). *Then* (1) *holds on $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$.*

Proof. It follows from our assumptions that (34)–(36) hold on $U_C \cap U_S$. Further, we set $L_R = -\frac{\operatorname{tr} T}{2F}$. Thus (33) is satisfied. Now (15)–(16) turn into (39)–(42). Next we set $\phi = F\widetilde{\phi}$. Thus (70) turns into (46). We now define the functions μ and η by (45). It is easy to verify that (43) and (44) are satisfied. Thus, in view of Lemma 3.2, we have (38), i.e. (1), which completes the proof.

Theorem 5.1, together with Proposition 4.1, leads to

THEOREM 5.2. *Let $(\overline{M}, \overline{g})$ be a line or a circle, with $\overline{g}_{11} = \varepsilon = \pm 1$, and let $(\widetilde{N}, \widetilde{g})$ with $\dim \widetilde{N} = n - 1 \geq 3$ be a semi-Riemannian manifold isometric to an open part of a hypersurface $\widetilde{M}$ in an n-dimensional space of constant curvature $N^n_s(c)$, $n \geq 4$, satisfying* (70) *and $C_1 = \frac{\tau}{(n-1)n}$ on $U_{\widetilde{S}} \subset \widetilde{M}$. Then* *the warped product* $\overline{M} \times_F \widetilde{N}$ *with* F *defined by one of the three equalities in* (37) *satisfies* (1) *on* $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$.

Now Theorem 4.1, together with Theorem 5.2, implies

THEOREM 5.3. *Let* $(\overline{M}, \overline{g})$ *be a line or a circle, with* $\overline{g}_{11} = \varepsilon = \pm 1$, *and let* $(\widetilde{N}, \widetilde{g})$ *with* $\dim \widetilde{N} = n - 1 \geq 3$ *be a Riemannian manifold isometric to an open part of a generalized Cartan hypersurface* $\widetilde{M}$ *in* $S^n(1)$, $n \geq 4$. *Then the warped product* $\overline{M} \times_F \widetilde{N}$ *with* F *defined by one of the three equalities in* (37) *satisfies* (1) *on* $U_C \cap U_S \subset \overline{M} \times_F \widetilde{N}$.

We finish our paper with the following remarks:

REMARK 5.1. Our investigations on semi-Riemannian manifolds (M, g), $n \geq 3$, satisfying (1) on $U_S \subset M$ lead to a particular subclass of manifolds consisting of all manifolds (M, g), $n \geq 3$, for which (10) holds on $U_S \subset M$.

REMARK 5.2. Consider the warped product $\overline{M} \times_F \widetilde{N}$ of a line or a circle $(\overline{M}, \overline{g})$, with $\overline{g}_{11} = \pm 1$, the warping function F and an $(n-1)$-dimensional semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n - 1 \geq 3$, locally isometric to an open part of a hypersurface in $N^n_s(c)$. Thus (56) holds on $\widetilde{N}$. Moreover, let F satisfy (37) with $C_1 = \frac{\tau}{(n-1)n}$. Then (34) reads

$$\frac{\operatorname{tr} T}{2} - \frac{\Delta_1 F}{4F} = -\frac{\tau}{(n-1)n}.$$

In addition we set $L = \frac{n-2}{2} \frac{\operatorname{tr} T}{F}$. Using the last two equations, (56) becomes

$$\widetilde{R} \cdot \widetilde{R} - Q(\widetilde{S}, \widetilde{R}) = (n-3)\left(\frac{LF}{n-2} - \frac{\Delta_1 F}{4F}\right) Q(\widetilde{g}, \widetilde{C}).$$

Now, in view of Theorem 4.2 of [5], we see that $\overline{M} \times_F \widetilde{N}$ satisfies (9).

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